Plan of Lectures

1. Time domain semiclassical approach to dynamical tunneling
2. Complex dynamics in one variable
3. Complex dynamics in two variables
4. **How to apply general theory of complex dynamics to tunneling problems**

Some important properties derived from the convergent theorem

Theorem (Bedford-Smillie)

1. **For any unstable periodic orbit p,** $\overline{W^s(p)} = J^+, \ \overline{W^u(p)} = J^-$

2. μ **satisfies the mixing property and is hyperbolic measure, where** $\operatorname{supp}\mu = J^*$

3. $\overline{\{\text{Unstable periodic points}\}} = J^*$

Fundamental working hypothesis

1. Vacant interior conjecture ($J^{\pm} = K^{\pm}$ and $J = K$)
2. $J^{*} = J$

Note : $J^{*} \subset J$ **for generic cases and** $J^{*} = J$ **for hyperbolic cases.**

“Dynamics” connecting KAM curves

- { KAM curves (either real or complex) } $\subset K$ (=Filled Julia set)
- $K = J = J^*$ ($\Leftarrow$ working hypothesis)

“KAM curves are subsets of the Julia set J^*”

- μ is mixing and ergodic (supp $\mu = J^*$)

“KAM curves are no more dynamical barriers in $\mathbb{C}^2$”

How to apply general theory to tunneling problems ?

Quantum propagator

$$K(a,b) = \langle b|\hat{U}^n|a\rangle = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_j dq_j \prod_j dp_j \exp\left[\frac{i}{\hbar}S(\{q_j\},\{p_j\})\right]$$

$|a\rangle$: **initial state** $|b\rangle$: **final state**

Semiclassical propagator ($\Leftarrow$ saddle point evaluation of $K(a,b)$)

$$K^{sc}(a,b) = \sum_\gamma A_n^{(\gamma)}(a,b)\exp\left\{\frac{\mathrm{i}}{\hbar}S_n^{(\gamma)}(a,b)\right\}$$

$\mathcal{A}_a = \{\,(p,q)\in\mathbb{C}^2 \mid A(p,q) = a\,\}$: **initial manifold**

$\mathcal{B}_b = \{\,(p,q)\in\mathbb{C}^2 \mid B(p,q) = b\,\}$: **final manifold**

Step 1 : Incorporate the boundary conditions

A set of classical orbits contributing to $K^{sc}(a, b)$

$$\mathcal{M}_n^{a,b} \equiv \{ (p, q) \in \mathbb{C}^2 \mid A(p_n, q_n) = a \text{ and } B(p_n, q_n) = b \}$$

where $(p_n, q_n) = P^n(p, q)$.

Instead of $\mathcal{M}_n^{a,b}$ we consider the sequence of hyperplanes

$$\mathcal{M}_n^{*,b} \equiv \{ (p, q) \in \mathbb{C}^2 \mid B(p_n, q_n) = b \}$$

We further introduce "limit" of $\mathcal{M}_n^{*,b}$ (in the Hausdorff topology) as

$$\mathcal{M}_\infty^b \equiv \lim_{n\to\infty} \mathcal{M}_n^{*,b} \quad \text{and} \quad \mathcal{M}_\infty \equiv \bigcup_{\beta\in\mathbb{R}} \mathcal{M}_\infty^b$$

Step 2 : Define the "tunneling orbits"

Semiclassical sum

$$K^{sc}(a,b) = \sum_{\gamma} A_n^{(\gamma)}(a,b)\exp\left\{\frac{i}{\hbar}S_n^{(\gamma)}(a,b)\right\}$$

The behavior of $\mathrm{Im}\, S_n^{(\gamma)}$ as $n \to \infty$

$\mathrm{Im}\, S_n^{(\gamma)} \to +\infty$: **negligible amplitude**

$\mathrm{Im}\, S_n^{(\gamma)} \to -\infty$: **unphysical explosion**

$\Rightarrow$ **should be removed by the Stokes phenomenon**

Therefore, it is reasonable to define the *tunneling orbits* as

$$C_{\mathrm{Laputa}} \equiv \{ (q,p) \in \mathcal{M}_\infty \mid \mathrm{Im}\, S_n(q,p) \text{ converges absolutely at } (q,p) \}$$

boundary conditions — *necessary condition for tunneling orbits*

Tunneling orbits and Julia sets

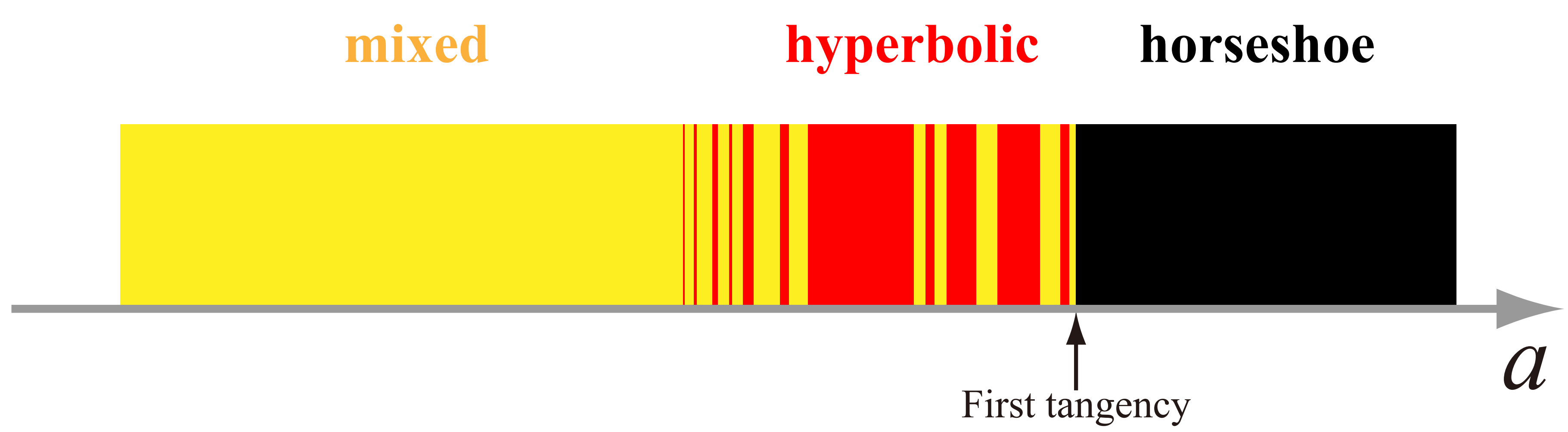
Theorem For the Hénon map P,

(i) If P is hyperbolic and $h_{\mathrm{top}}(P|_{\mathbb{R}^2}) = \log 2$, then $C_{\mathrm{Laputa}} = J^+$

(ii) If P is hyperbolic and $h_{\mathrm{top}}(P|_{\mathbb{R}^2}) > 0$, then $\overline{C_{\mathrm{Laputa}}} = J^+$

(iii) If $h_{\mathrm{top}}(P|_{\mathbb{R}^2}) > 0$, then $J^+ \subset \overline{C_{\mathrm{Laputa}}} \subset K^+$

Here $h_{\mathrm{top}}(P|_{\mathbb{R}^2})$ is topological entropy confined on $\mathbb{R}^2$.



Remark 1

$J^+ \subset \overline{C_{\text{Laputa}}} \subset K^+$ in the genetic case (i), therefore if the vacant interior conjecture (*i.e.* $J^\pm = K^\pm,\ J = K$) is true, then

$$\overline{C_{\text{Laputa}}} = J^+$$

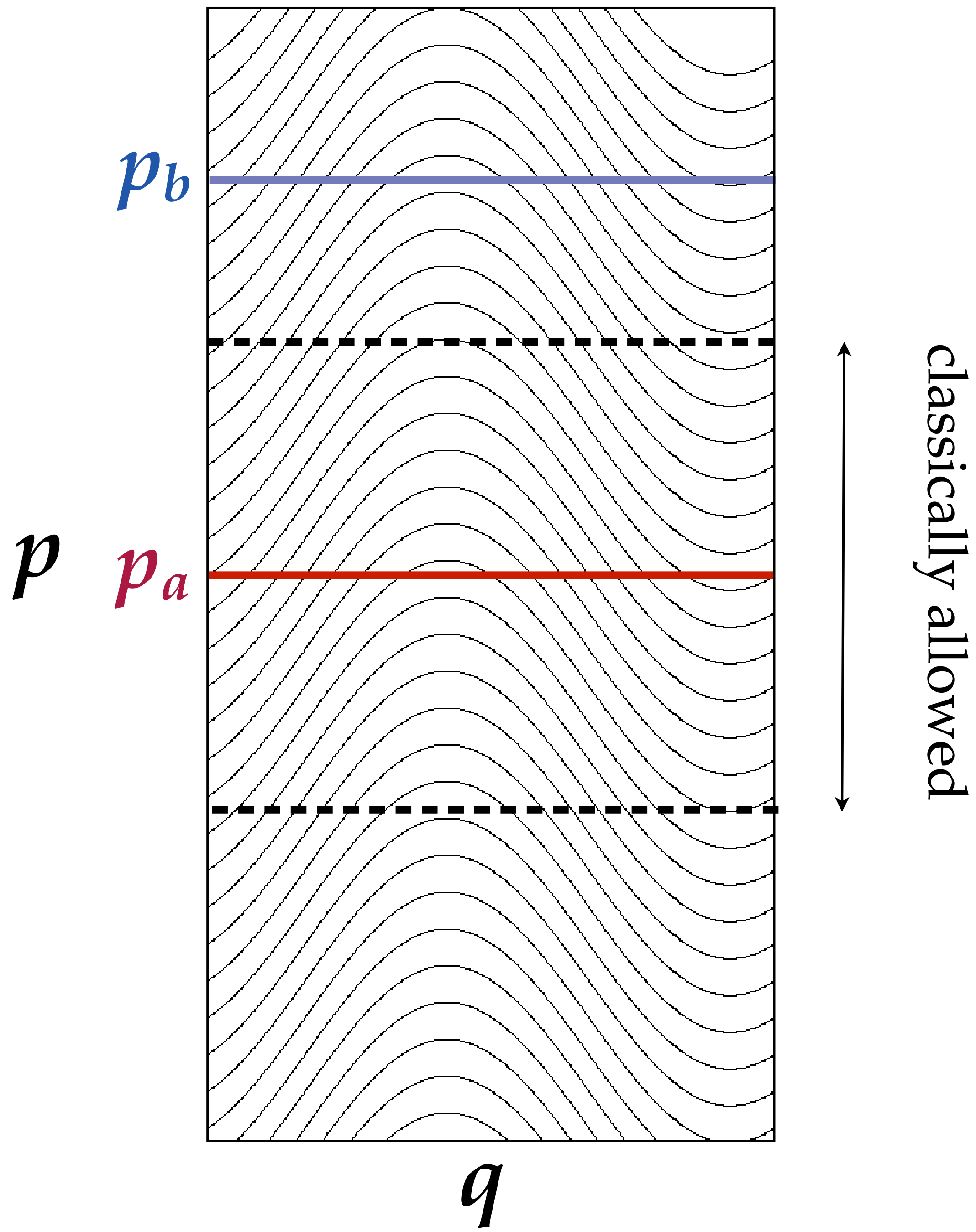
holds even in the generic case (iii).

Remark 2

Note that $\overline{C_{\text{Laputa}}} = J^+$ holds in hyperbolic (or generic) cases, whereas $C_{\text{Laputa}} = J^+$ in the horseshoe situation. There indeed exist exponentially many orbits contained in $J^+ \backslash C_{\text{Laputa}}$ in hyperbolic (or generic) cases. They itinerates in the complex space and do not have convergent imaginary action.

A completely integrable model

$$F : \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} p + K \sin q \\ q + \omega \end{pmatrix}$$



$\mathcal{M}_n^{a,b} = A_a \cap F^{-n}(B_b) = \emptyset$ for $\forall n \in \mathbb{Z}$

if B_b is outside the classically allowed region.

where

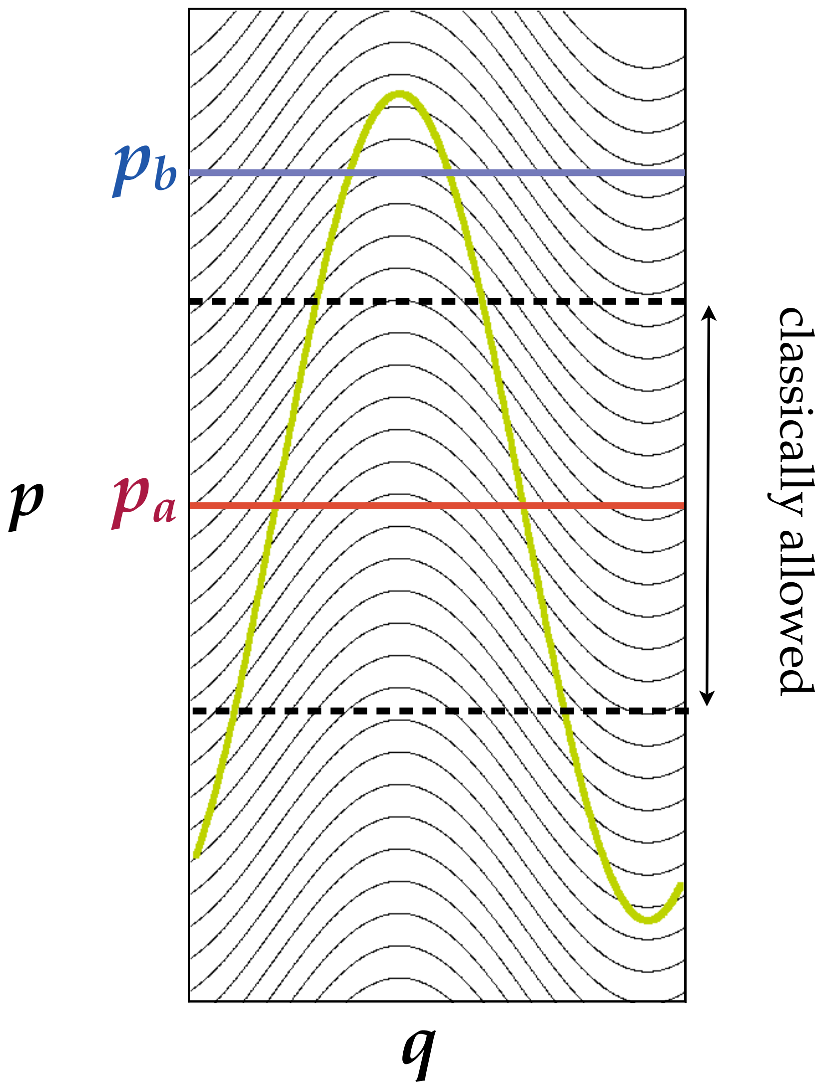
$A_a = \{ (p, q) \in \mathbb{R}^2 \,|\, p = p_a \}$

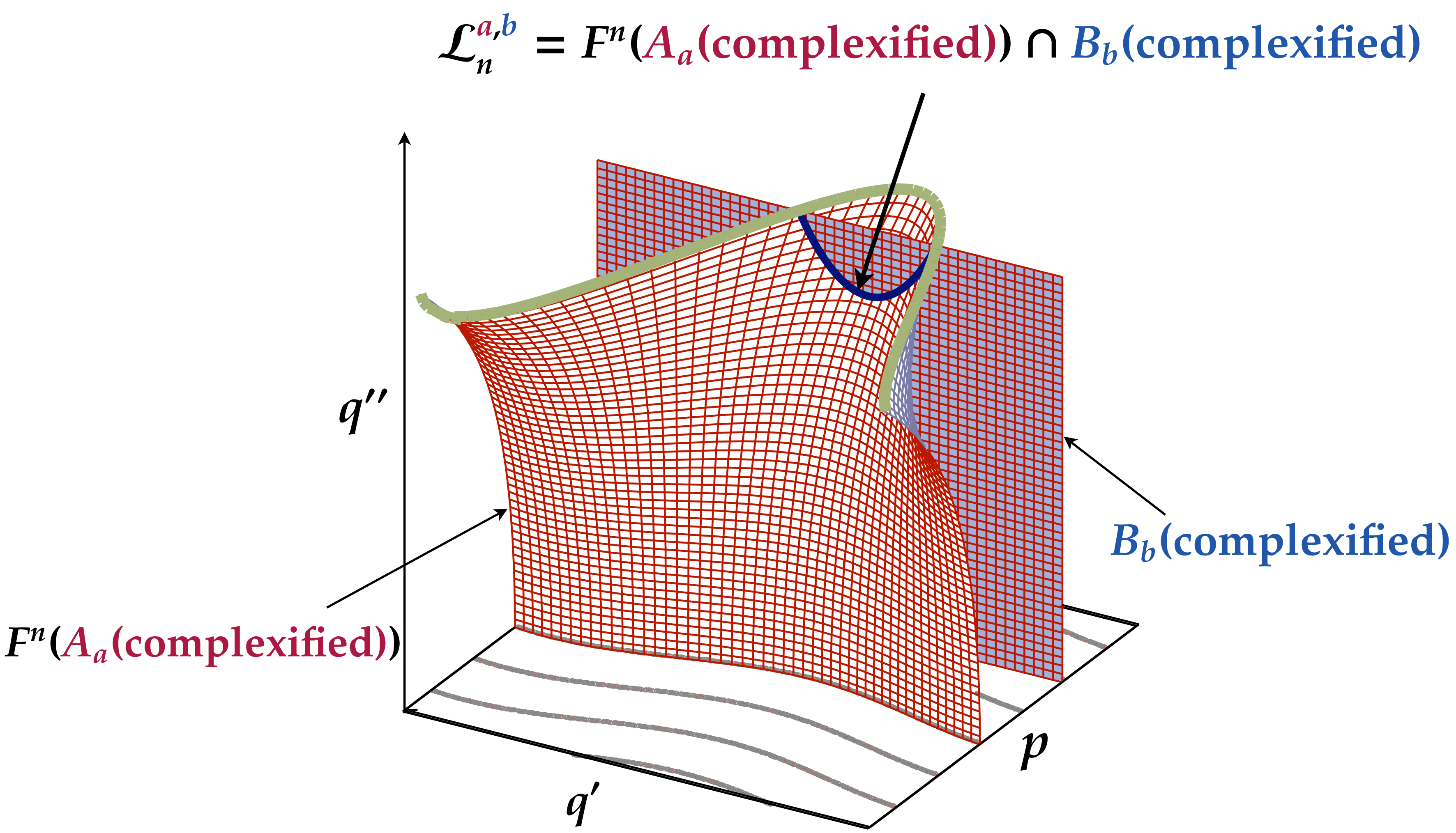
$B_b = \{ (p, q) \in \mathbb{R}^2 \,|\, p = p_b \}$

Tunneling transport on complexified KAM cuvres

$$K_n^{sc}(p_a, p_b) = \sum_{\gamma} A_n^{(\gamma)}(p_a, p_b) \exp\left[\frac{i}{\hbar} S_n^{(\gamma)}(p_a, p_b)\right] \neq 0$$

even if $\mathcal{L}_n^{a,b} \equiv F^n(A_a) \cap B_b = \emptyset$ on $\mathbb{R}^2$

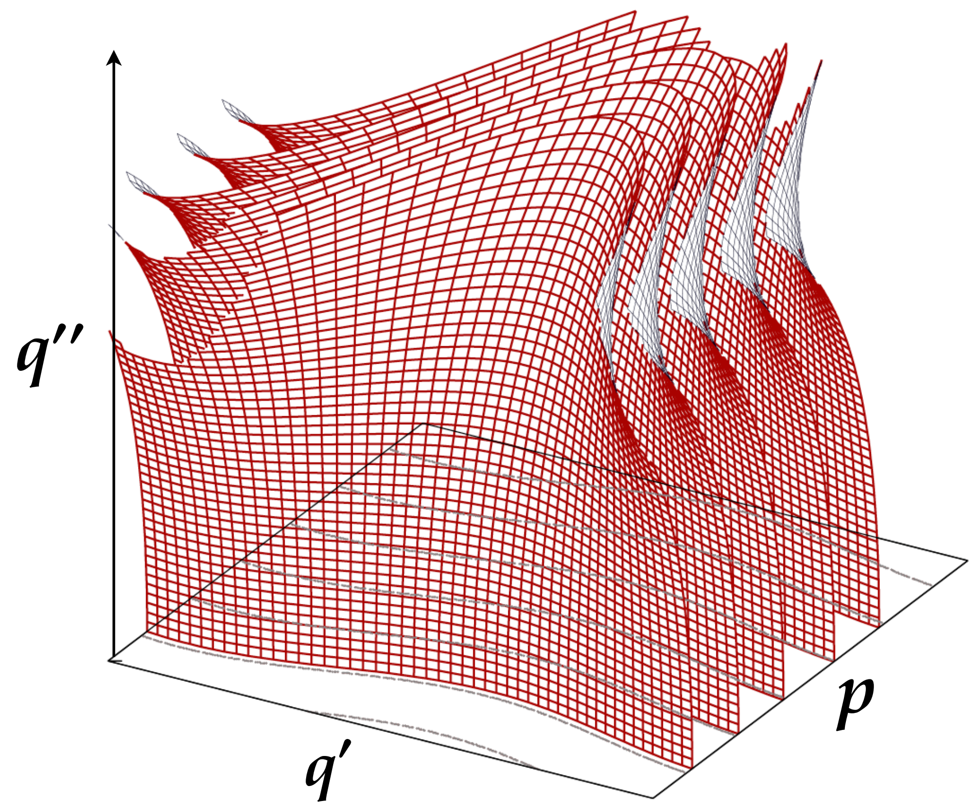




What if we take KAM curves as initial and final states ?

The transition from one invariant curve to another invariant curve

$$A_a = \{ (p, q) \in \mathbb{R}^2 \mid I(p, q) = I_a \} \qquad B_b = \{ (p, q) \in \mathbb{R}^2 \mid I(p, q) = I_b \}$$



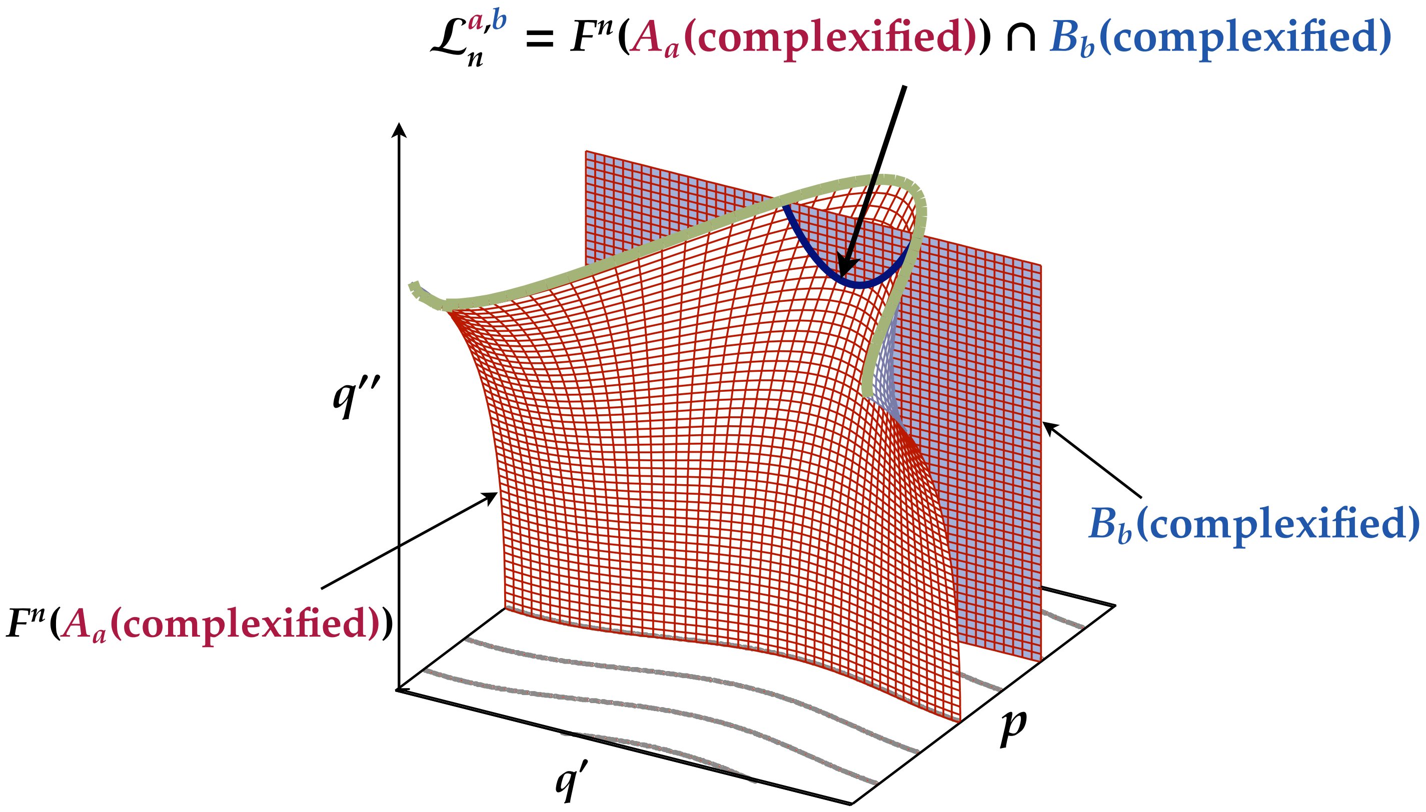
No contributions in the semiclassical propagator :

$$K^{sc}(I_a, I_b) = \sum_{\gamma} A_n^{(\gamma)}(I_a, I_b) \exp\left\{ \frac{\mathrm{i}}{\hbar} S_n^{(\gamma)}(I_a, I_b) \right\} = 0$$

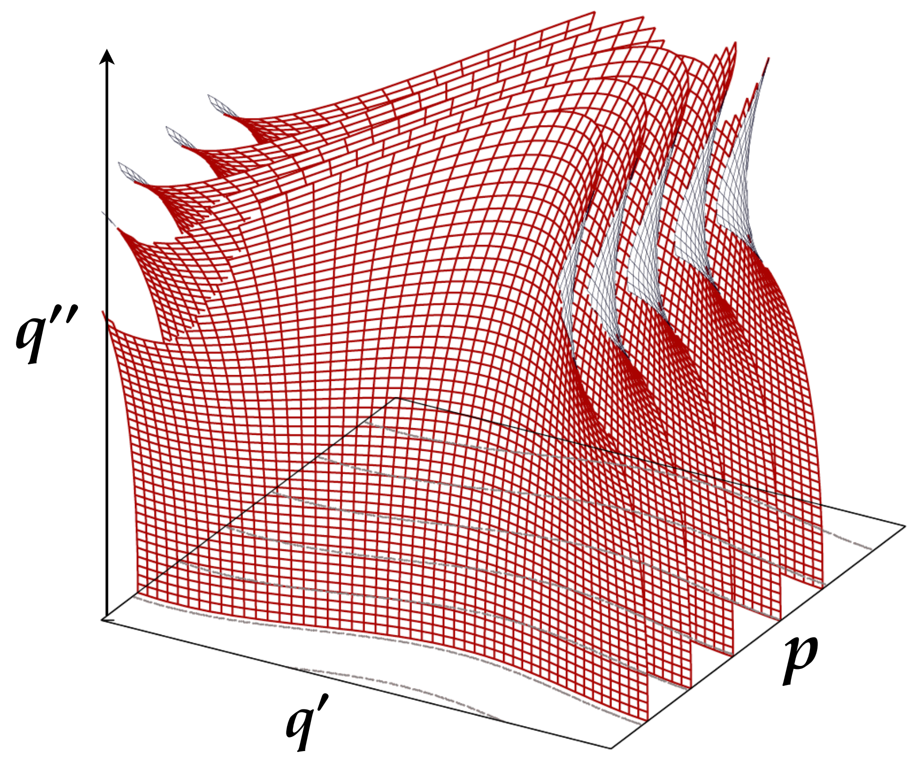
since $I(p, q)$ is invariant in the whole $\mathbb{C}^2$ plane.

Initial and final state dependency

$K^{sc}(p_a, p_b) \neq 0$



$K^{sc}(I_a, I_b) = 0$



Tunneling in the integrable model *is not* driven by the (complex) dynamics

What about in the non-integrable system?

Analyticity of complexified KAM curves

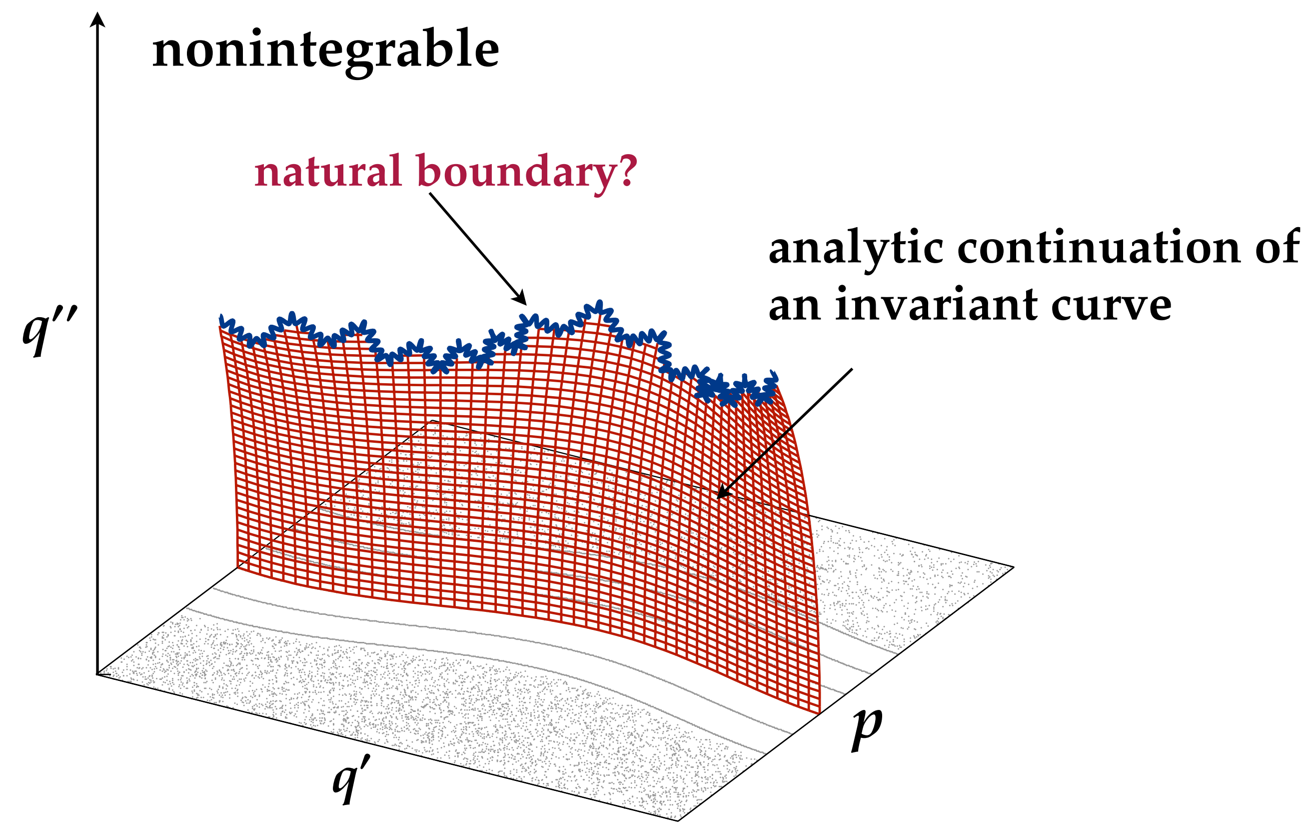
The rotation on the KAM curve C_ω is expressed as a constant rotation in a sutable coordinate θ:

$$\sigma : \theta \mapsto \theta + 2\pi\omega \pmod{2\pi}$$

In order to have such a coordinate θ, the conjugation function φ satisfying

$$\begin{array}{ccc} F : C_\omega & \longmapsto & C_\omega \\ \varphi \downarrow & & \downarrow \varphi \\ \sigma : T^1 & \longmapsto & T^1 \end{array}$$

has to be analytic with respect to θ.



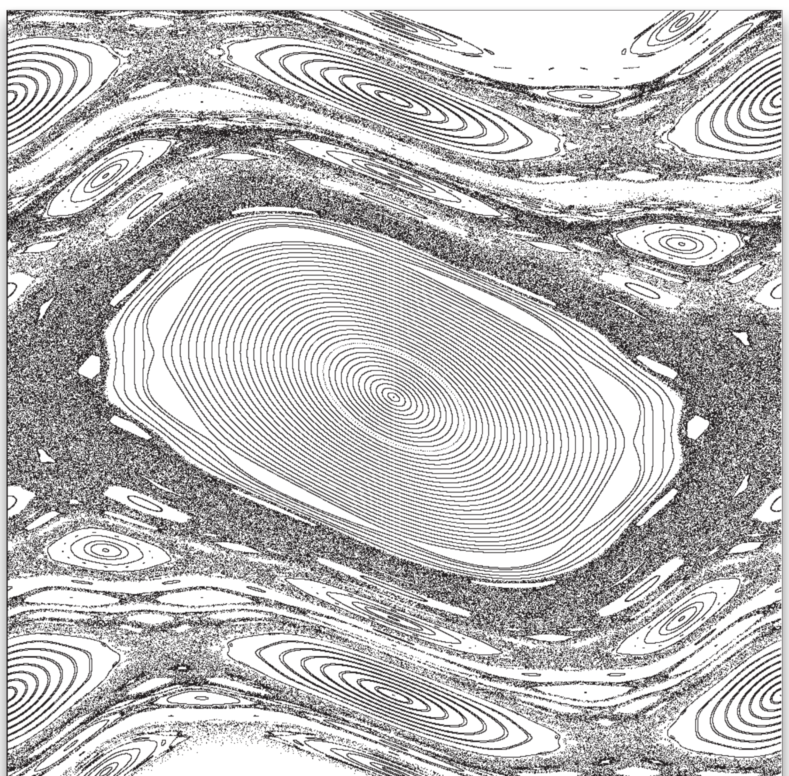
For given ω, assume

$$\varphi(\theta,\omega) = \sum_n a_n(\omega) e^{in\theta}$$

KAM curve can be complexified up to where ?

⇒ Natural boundary (Percival, Greene, Berretti, Marmi, Gentile ⋯)

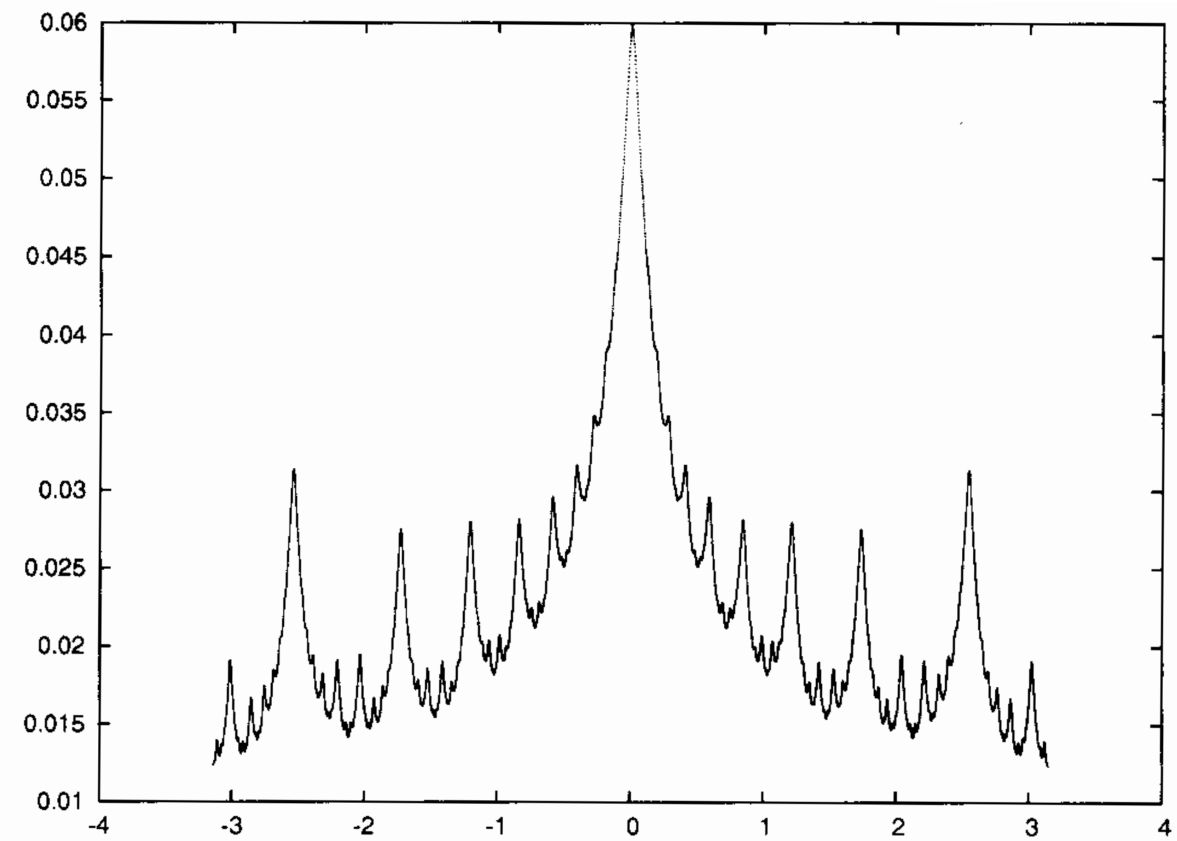
Natural boundary for an analytic map (standard map)



p

q'

Natural boundary for Standard map
$(V(q) = K \sin q)$



q''

q'

The natural boundary and the Julia set in 1-dimensional maps

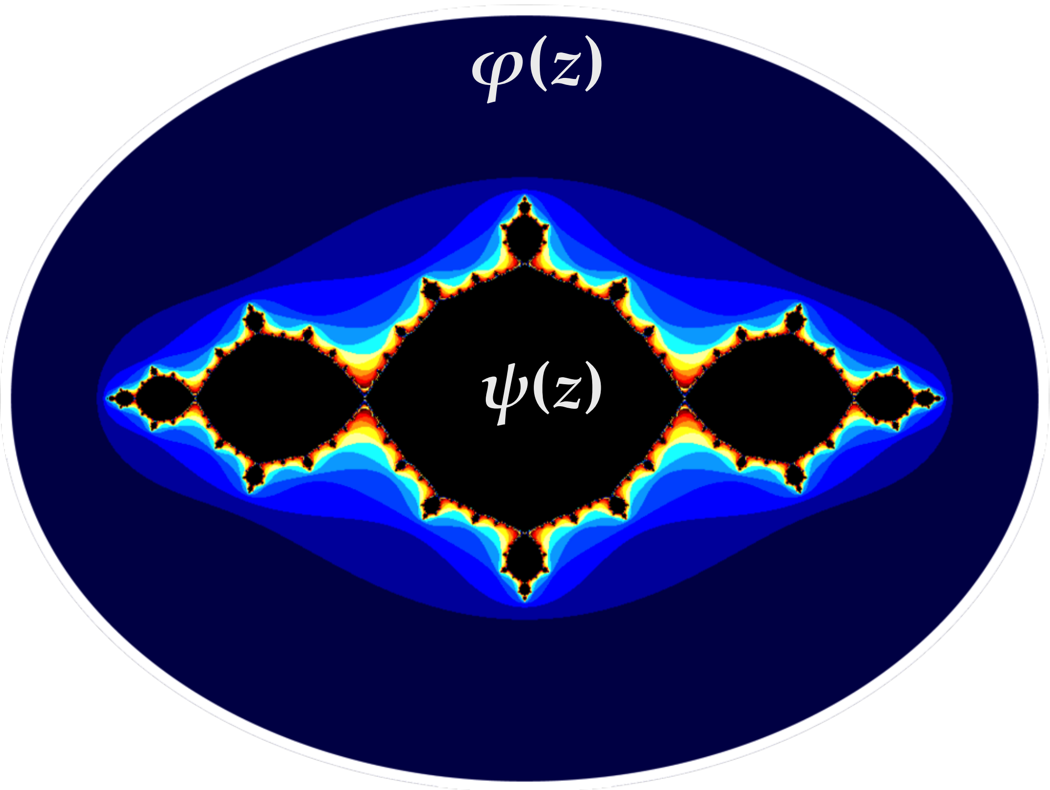
Theorem (Milnor, Costin-Krustal , ...)

The domain of analyticity of $\psi(z)$ is K_P, and $J_P = \partial K_P$ is a singularity barrier (= natural boundary) of $\psi(z)$.

Theorem (Costin-Krustal , ...)

The domain of analyticity of the Böttcher function $\varphi(z)$ is K_P and $J_P = \partial K_P$ is a singularity barrier (= natural boundary) of $\varphi(z)$.

Natural boundaries of ψ or φ = the Julia set



The natural boundary and the Julia set in 2-dimensional maps

“Vacant interior conjecture”

The filled Julia sets of the area-preserving map have no interior points :

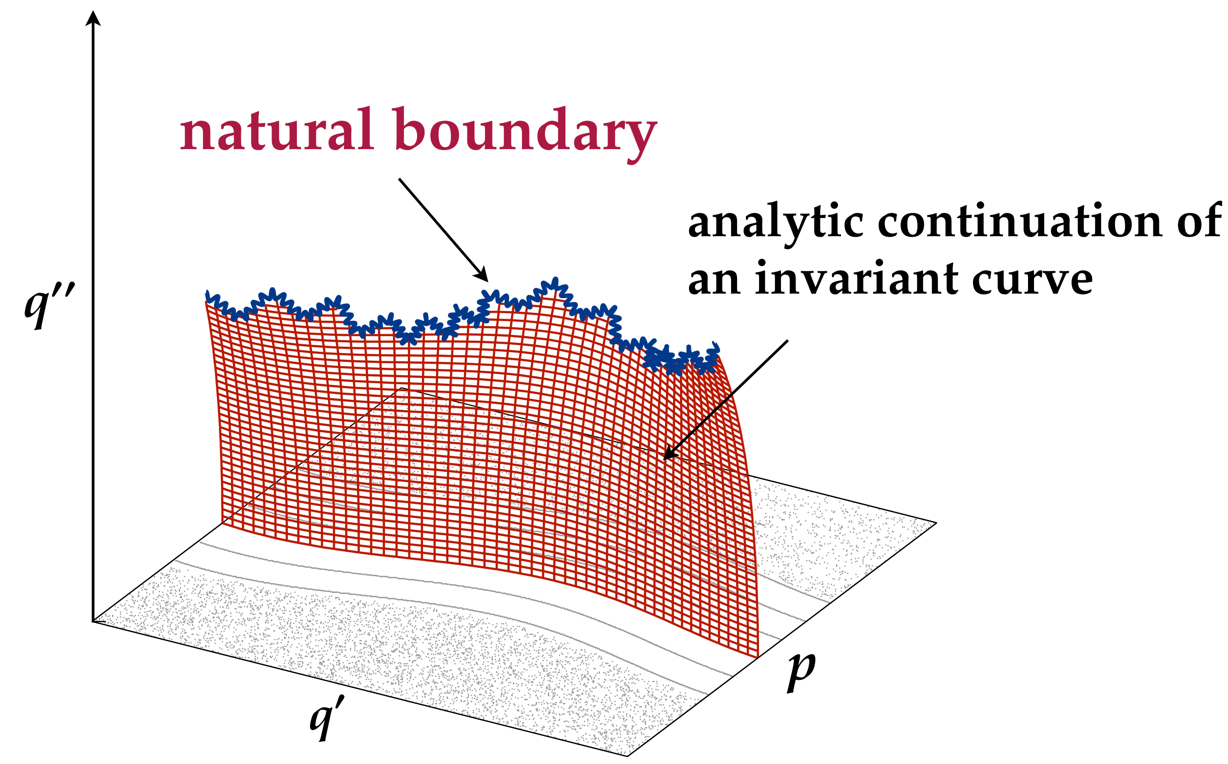
$$J^{\pm} = K^{\pm} \quad \text{hence} \quad J = K$$

If the vacant interior conjecture is true, then

$$\{\text{ KAM curves (either real or complex) }\} \subset \text{the Julia set } J$$

Therefore, we may expect that

$$\text{Natural boundaries of } \varphi \;\subset\; \text{the Julia set}$$

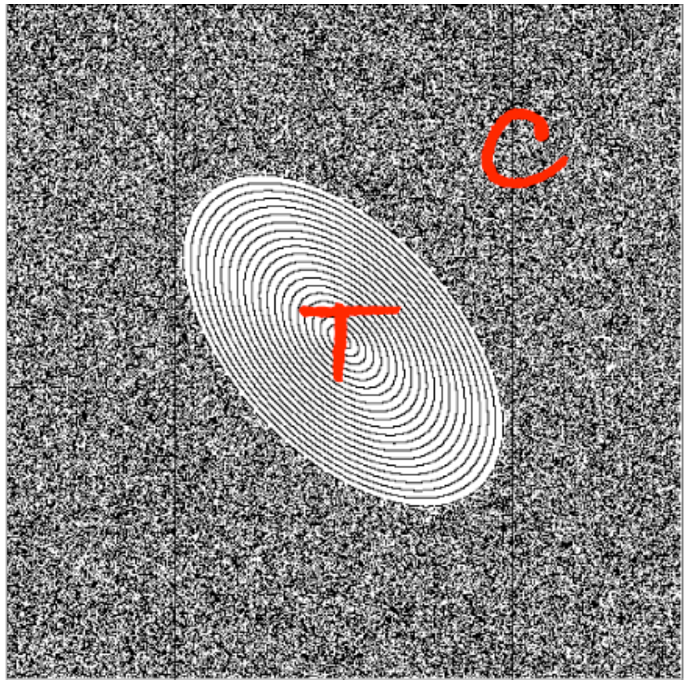


The most dominant complex orbits for the tunneling transport

The general theory tells us that

1. KAM curves are no more dynamical barriers in $\mathbb{C}^2$
2. The orbits with convergent imaginary action are dense in the Julia set J^+

"Which are the most dominant complex orbits controlling the tunneling transition from $\mathcal{T}$ to $\mathcal{C}$?"

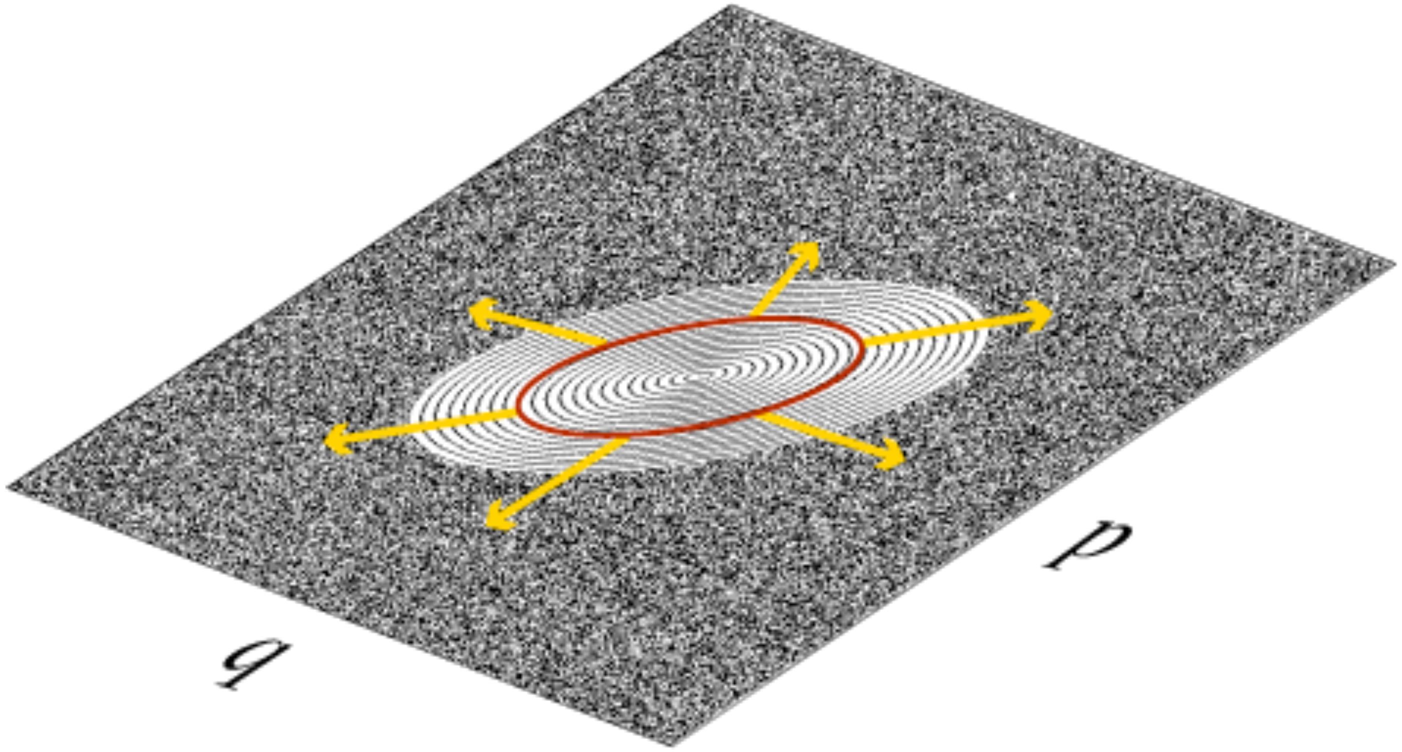


Going out from $\mathcal{T}$ to $\mathcal{C}$ directly : the most dominant paths ?

Direct paths are optimal since they gain no imaginary action Im S_n

"KAM curves are no more dynamical barriers in $\mathbb{C}^2$"

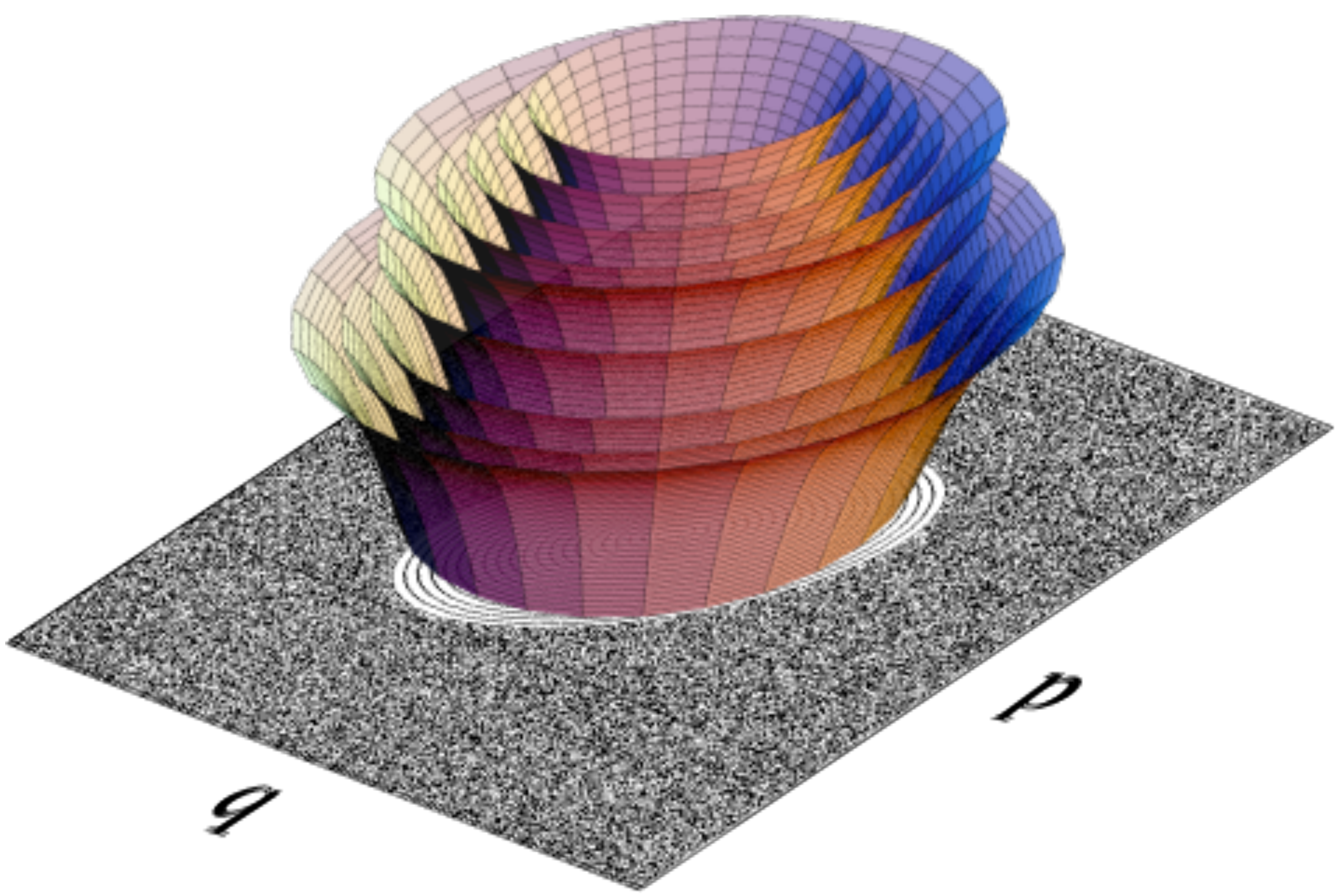
implies that for arbitrary neighborhoods $U(z_1)$ and $U(z_2)$ of any two points z_1 and z_2 in {KAM curves (either real or complex)}, there exists n such that $U(z_1) \cap P^n(U(z_2)) \neq \emptyset$.



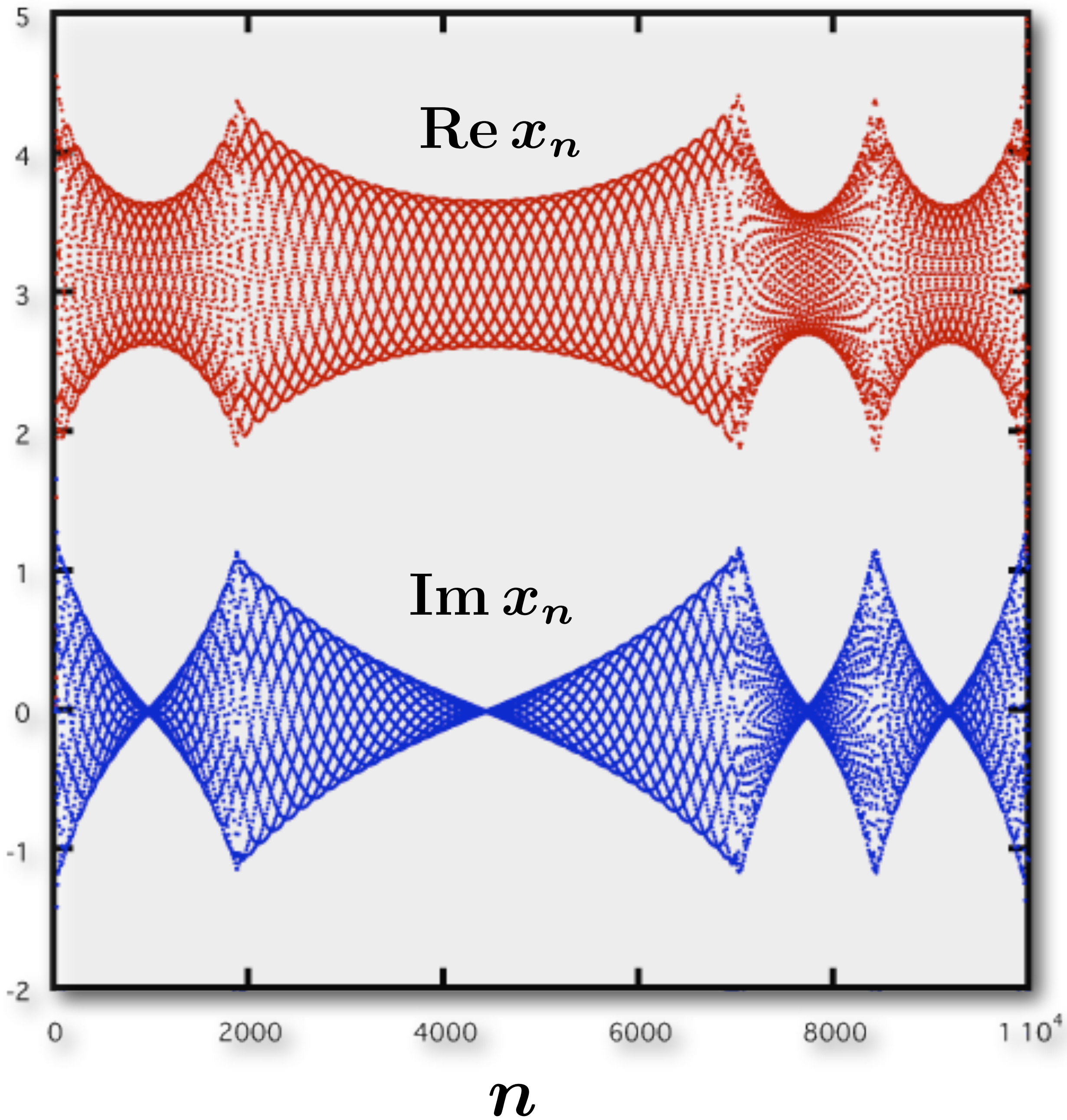
This is not the case since KAM curves are extended to the complex plane.

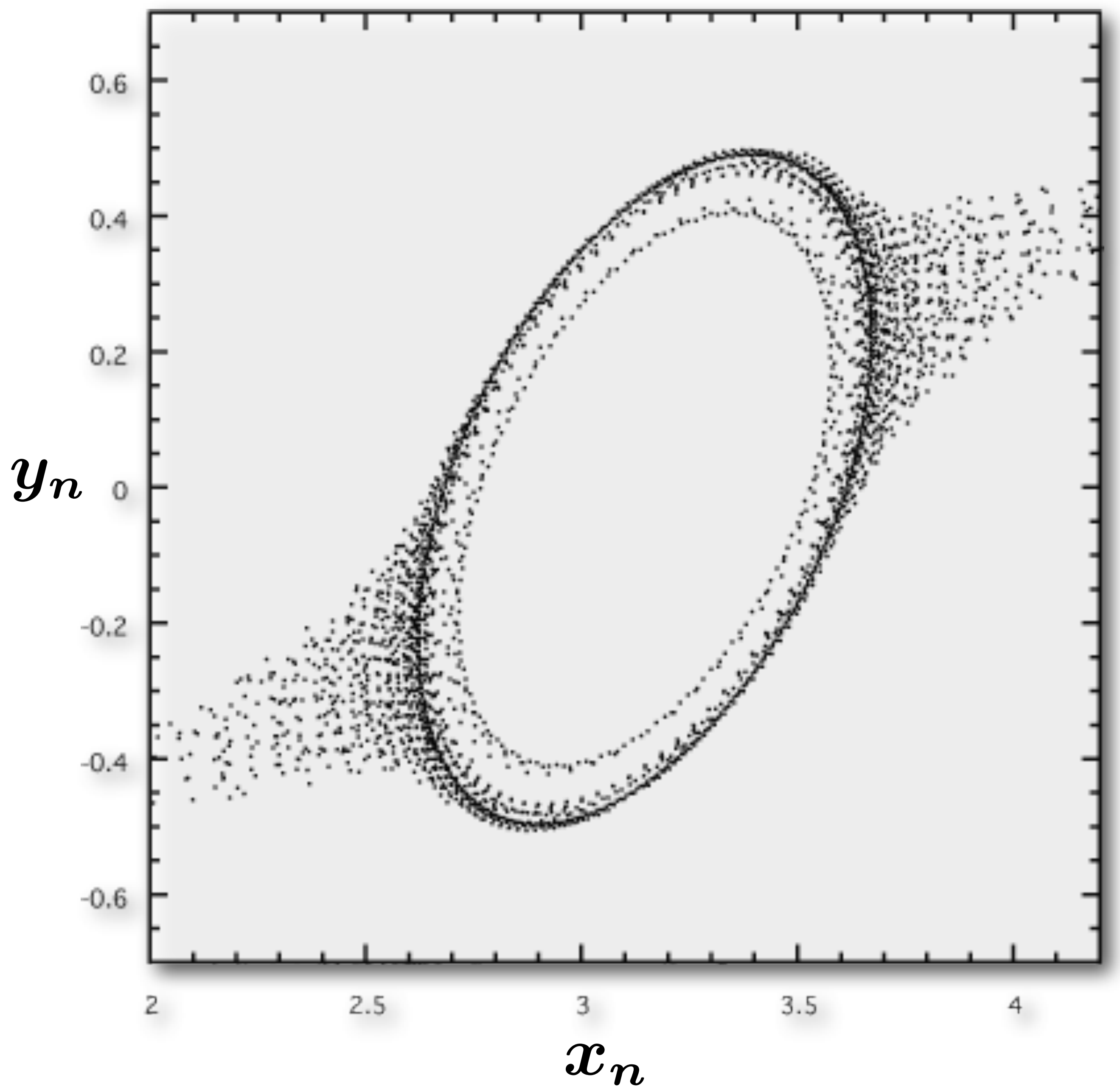
Recall the dimension counting of the complexified KAM curves

"The (Hausdorff) dimension of rotational domains associated with the convergent conjugating function $\varphi(\theta,\omega) = \sum_n a_n(\omega)e^{in\theta}$ is $(3+\alpha)$."



An orbit itinerating among different complex KAM curves





The most dominant tunneling orbits

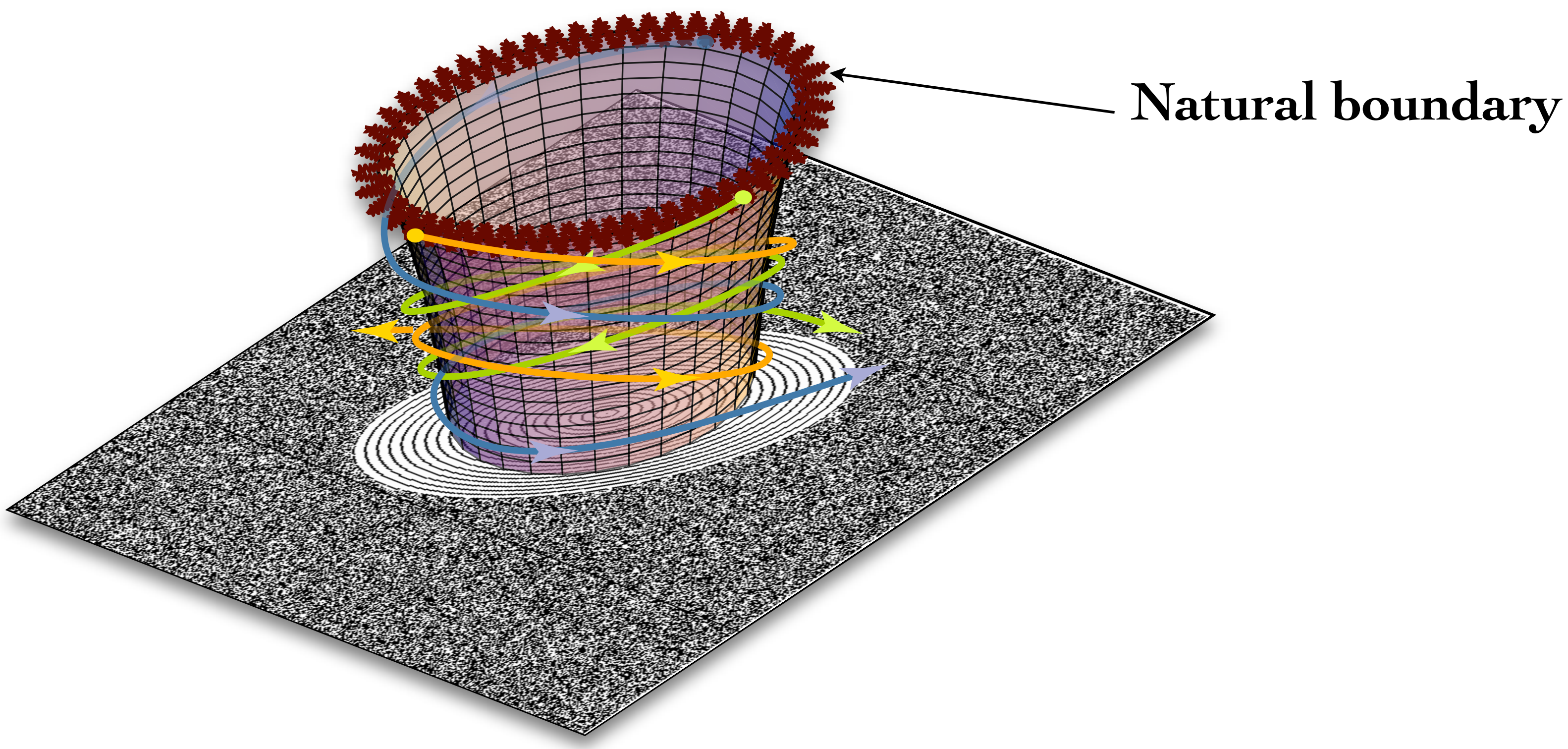
Semiclassical propagator

$$K_n^{sc}(a,b) = \sum_{\gamma} A_n^{(\gamma)}(a,b) \exp\left\{\frac{i}{\hbar} S_n^{(\gamma)}(a,b)\right\}$$

— How to save $\mathrm{Im} S_n^{(\gamma)}(a,b)$ —

1. **Start at the edge of complexfied KAM curves.**
   minimize the initial imaginary depth

2. **Go down to the real plane as fast as possible.**
   minimize the imaginary action gained in the itinerary



1. The existence of optimal orbits

There exist orbits which start at the natural boundary of KAM curve and tend to the real plane

(Proof)

1. Since {natural boundaries} $\subset J^*$ (hypothesis) and
   $\overline{\{\text{unstable periodic orbits}\}} = J^*$ (Bedford-Smillie),
   there exists an unstable periodic orbit $P \in$ the natural boundary of a given KAM curve.

2. For any neighborhood $U(P)$, there exist an unstable periodic orbit $P' \in \mathbb{R}^2$ such that $U(P) \cap W^s(P') \neq \emptyset$.
   This is due to $\overline{W^s(P)} = J^+$ (Bedford-Smillie).

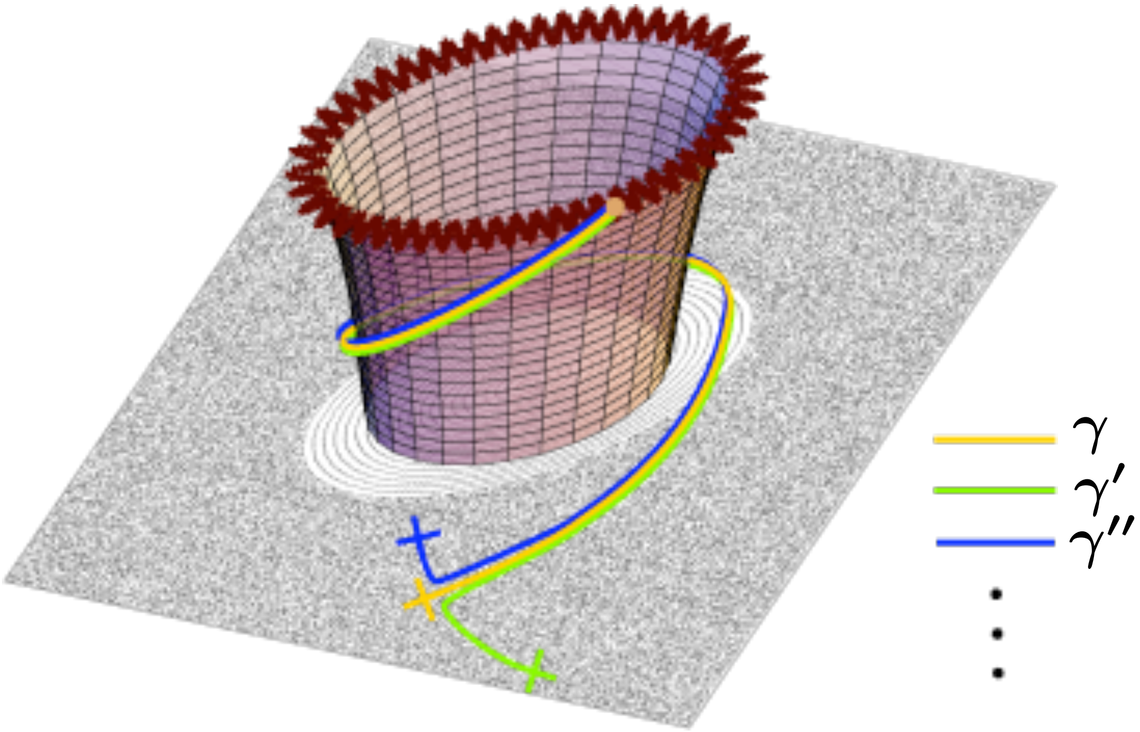
2. Optimal orbits are exponentially many

There exist exponentially many optimal orbits with comparable imaginary action.

(Proof)

1. **Take an optimal path γ, which starts at the natural boundary of a KAM curve and tends to an unstable periodic orbit on the real plane.**

2. **Since $\gamma \in J^+$ (hypothesis) and $\overline{W^s(P)} = J^+$ (Bedford-Smille), exponentially many $\gamma' \sim \gamma$ exist.**



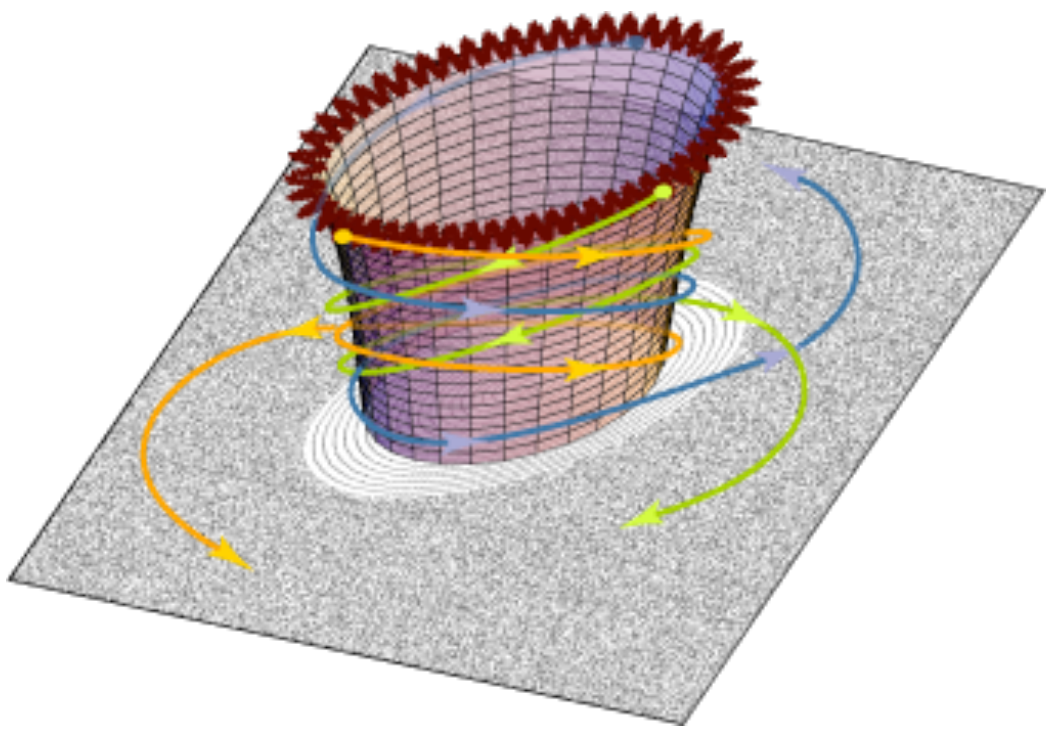
3. After landing the real plane

Optimal orbits follow almost real dynamics after reaching the real plane.

(Proof)

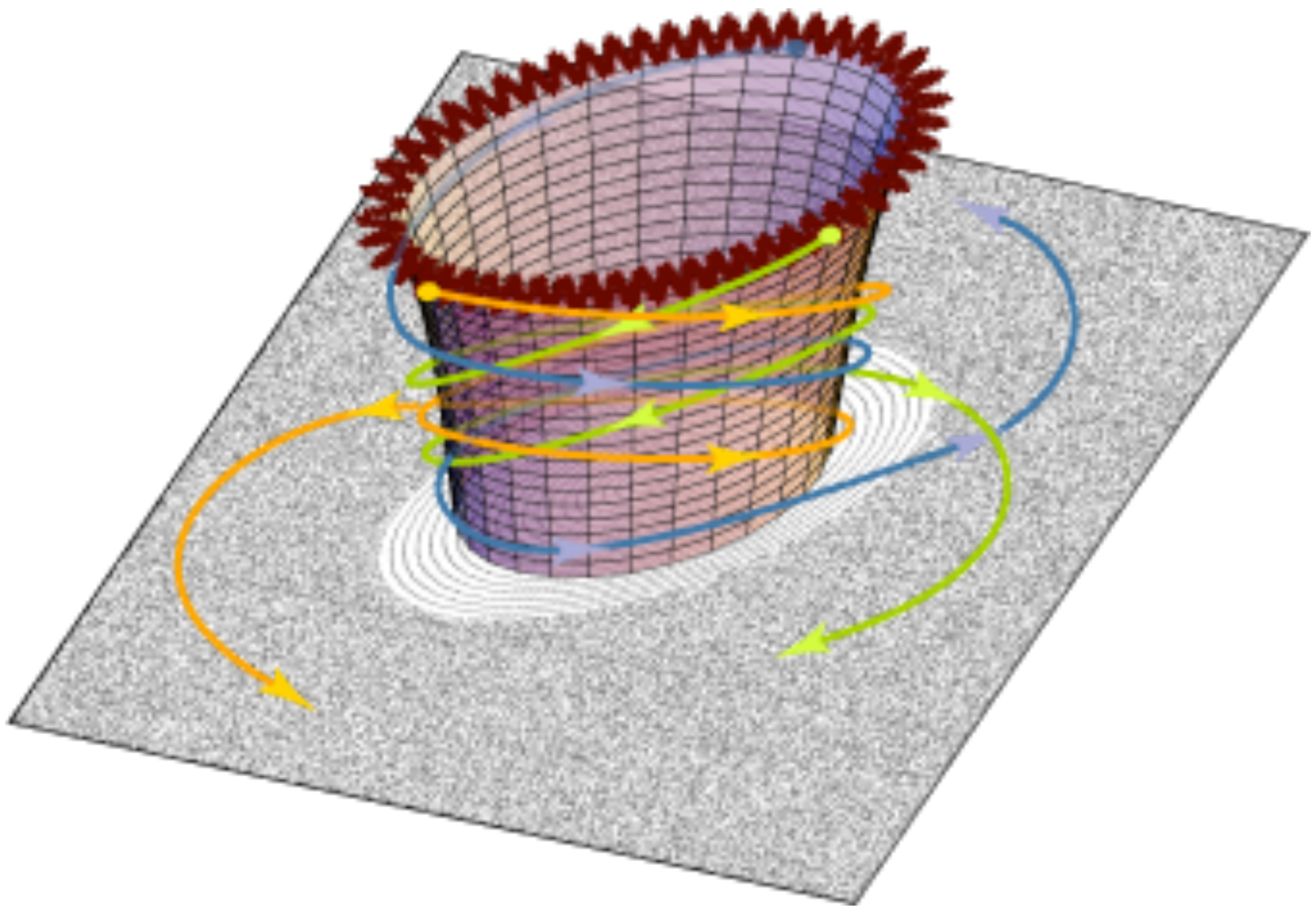
Use the fact that the orbit staring in the neighborhood of P visit the neighborhood of any other unstable periodic orbits P_i ($i = 1, 2, \cdots$).

In order to prove this, we use some theorems (Bedford-Lyubich-Smillie, Katok) and, Lambda Lemma.



The most dominant tunneling orbits

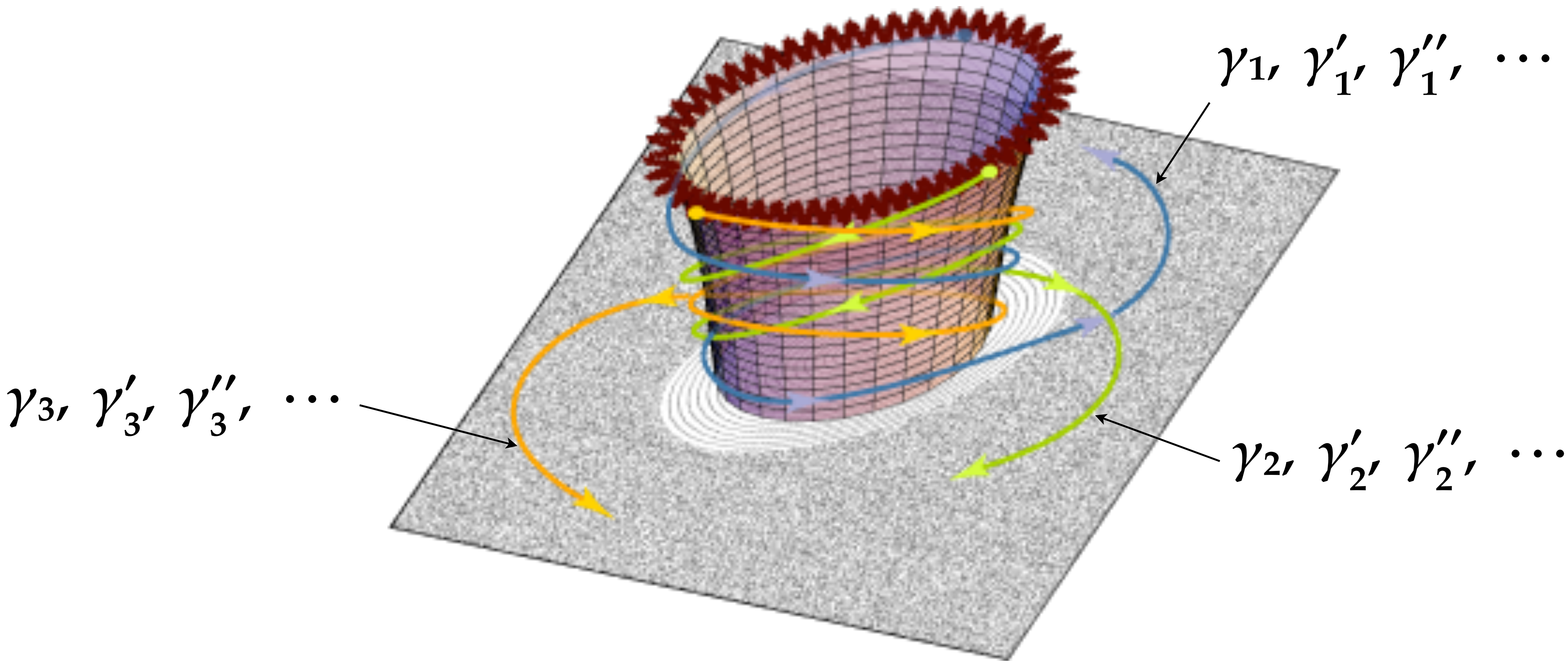
1. Go down from the torus $\mathcal{T}$ to chaotic $\mathcal{C}$ regions along the stable manifolds
2. Attracted by real chaos $\mathcal{C}$ (unstable periodic orbits on $\mathbb{R}^2$) exponentially
3. Move as if they are real orbits after reaching $\mathcal{C}$

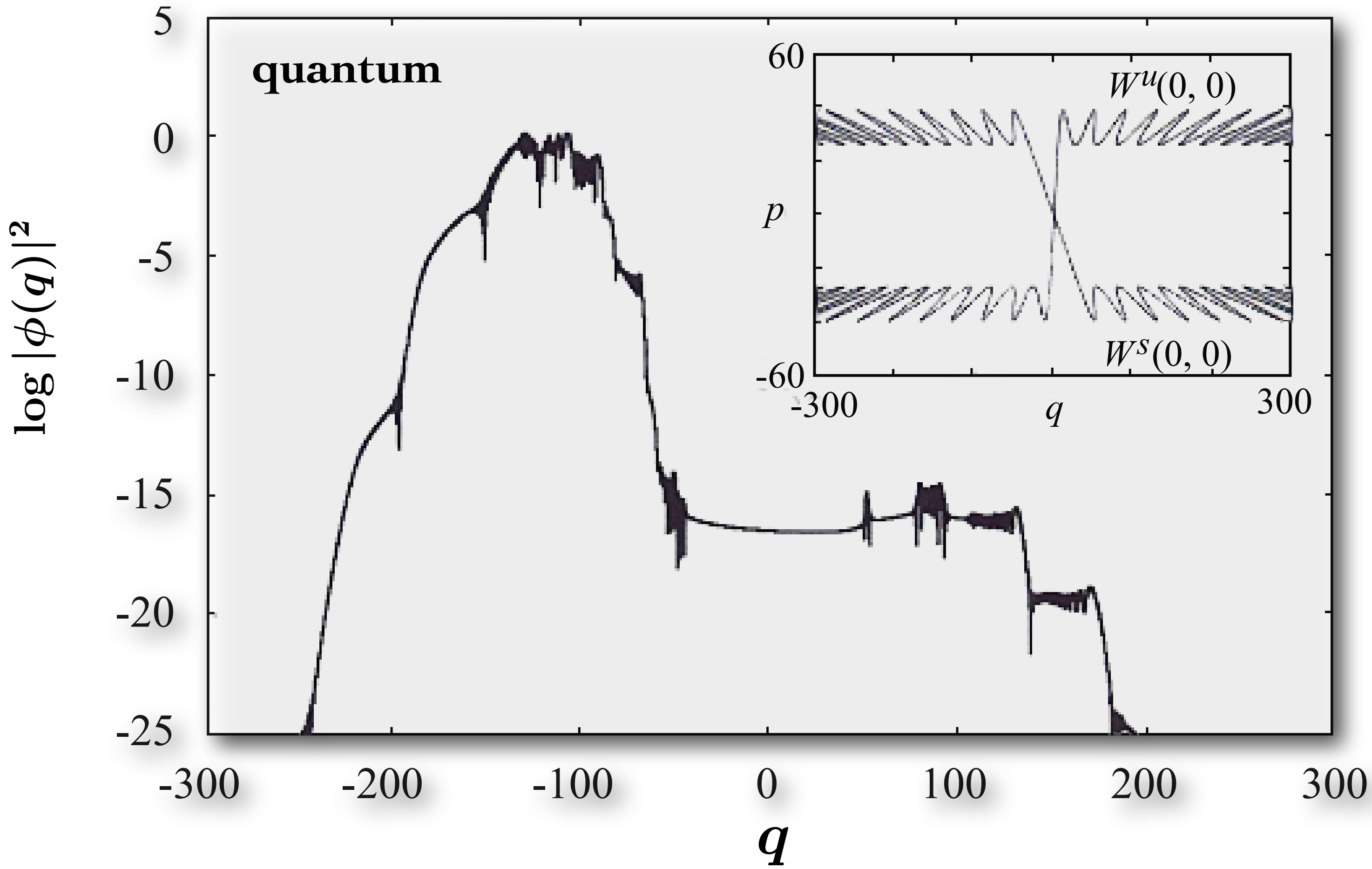


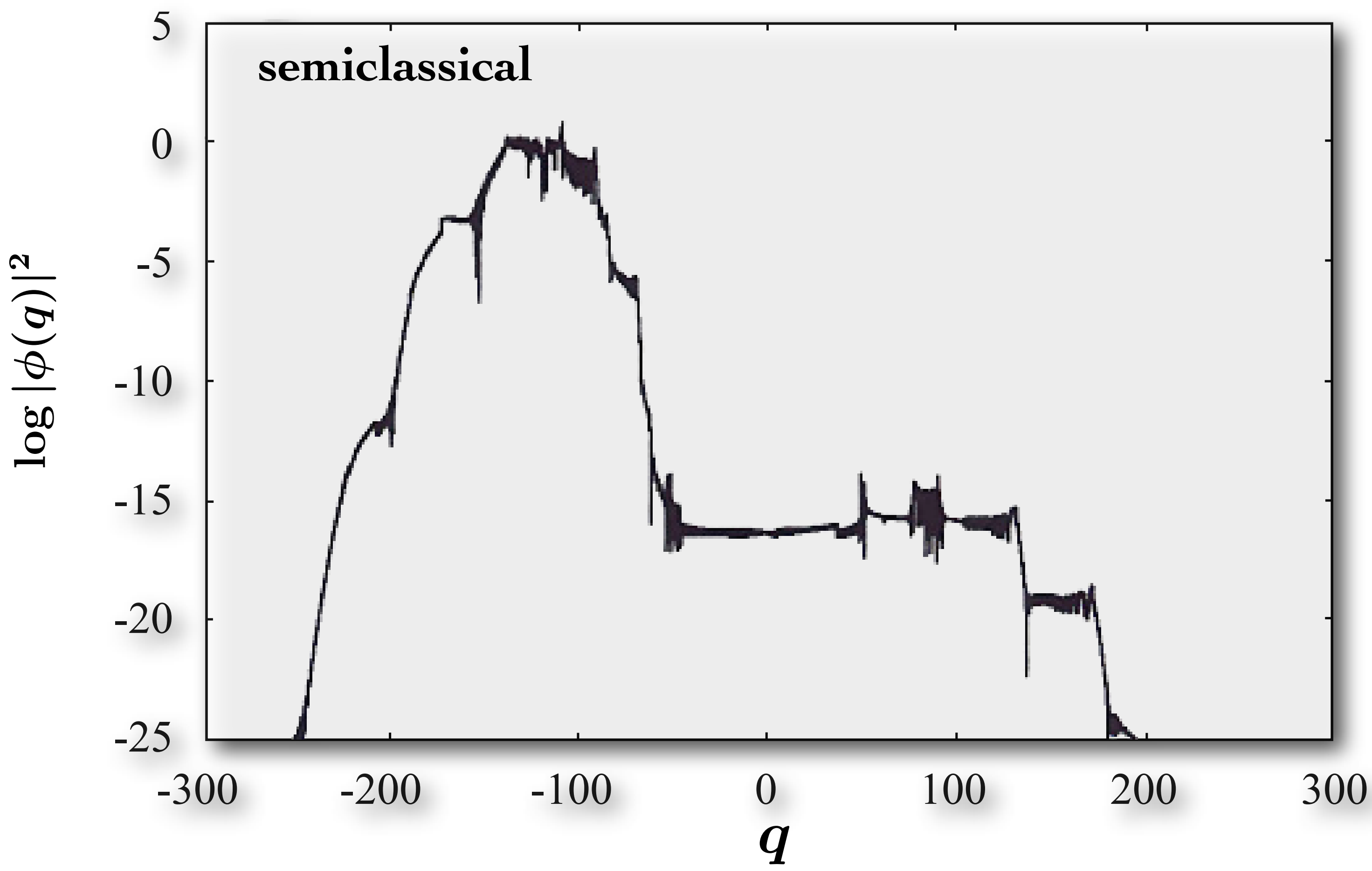
The variety of the optimal orbits

Each optimal orbit is accompanied by *a family* of optimal orbits with comparable imaginary action.

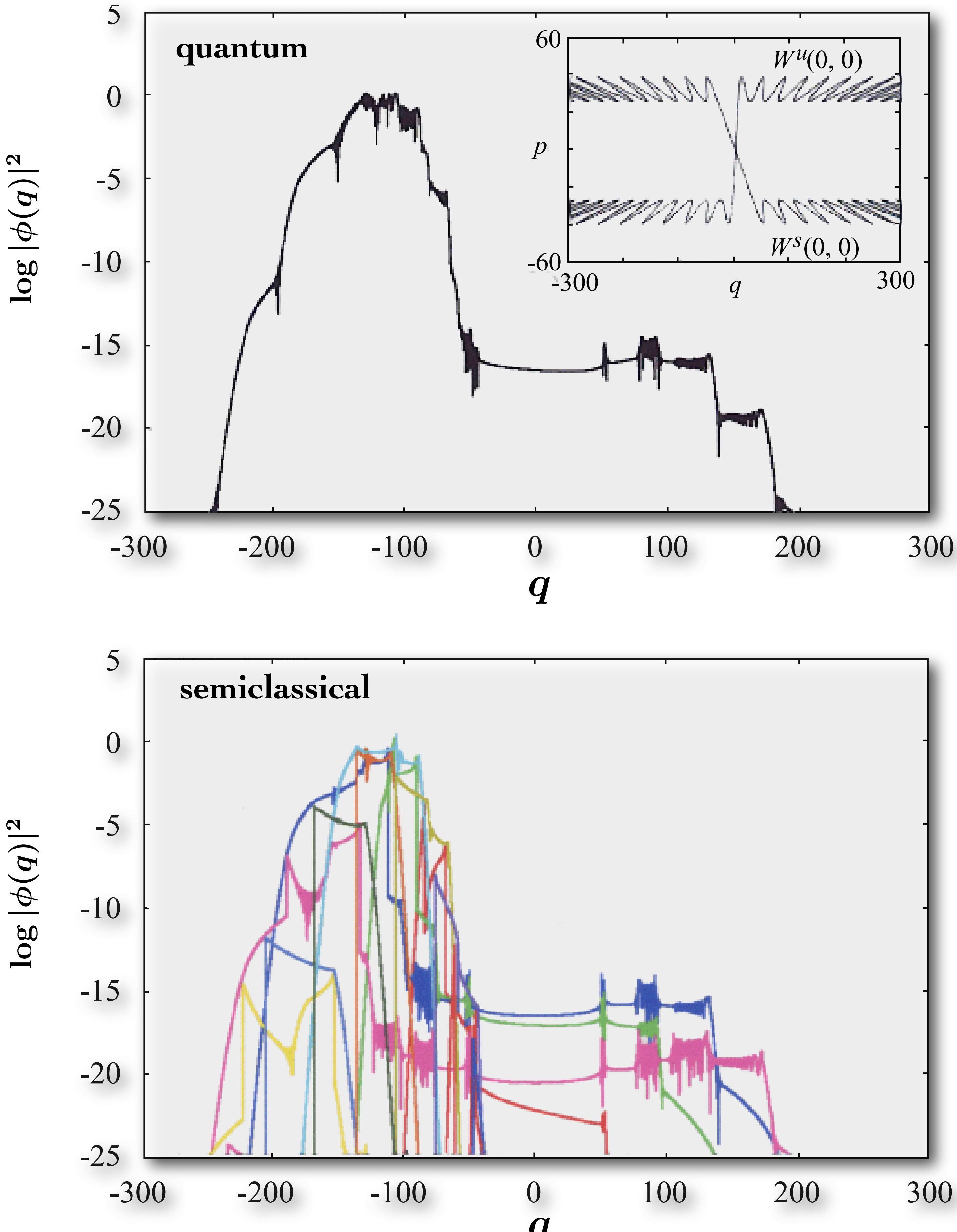
'Hard' question : A single family or (infinitely) many families ?







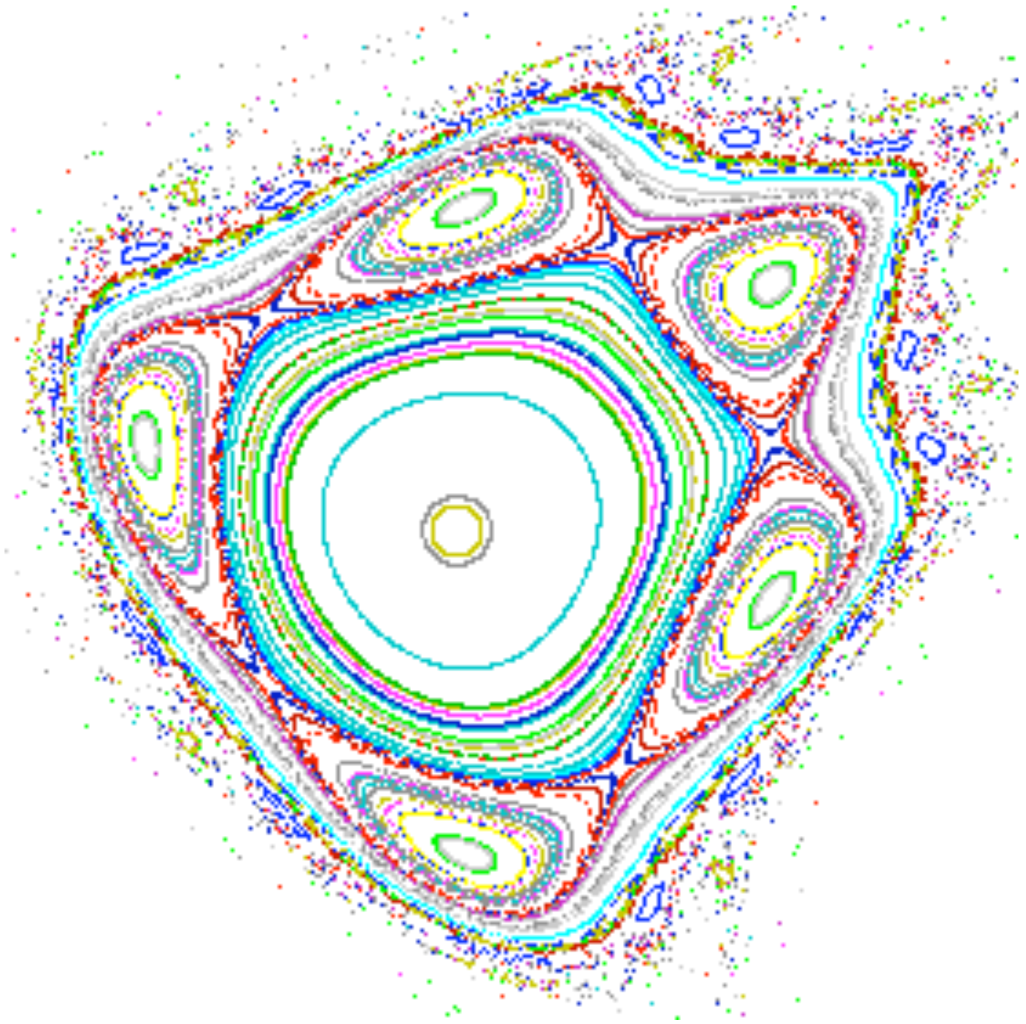
Onishi et al 2003



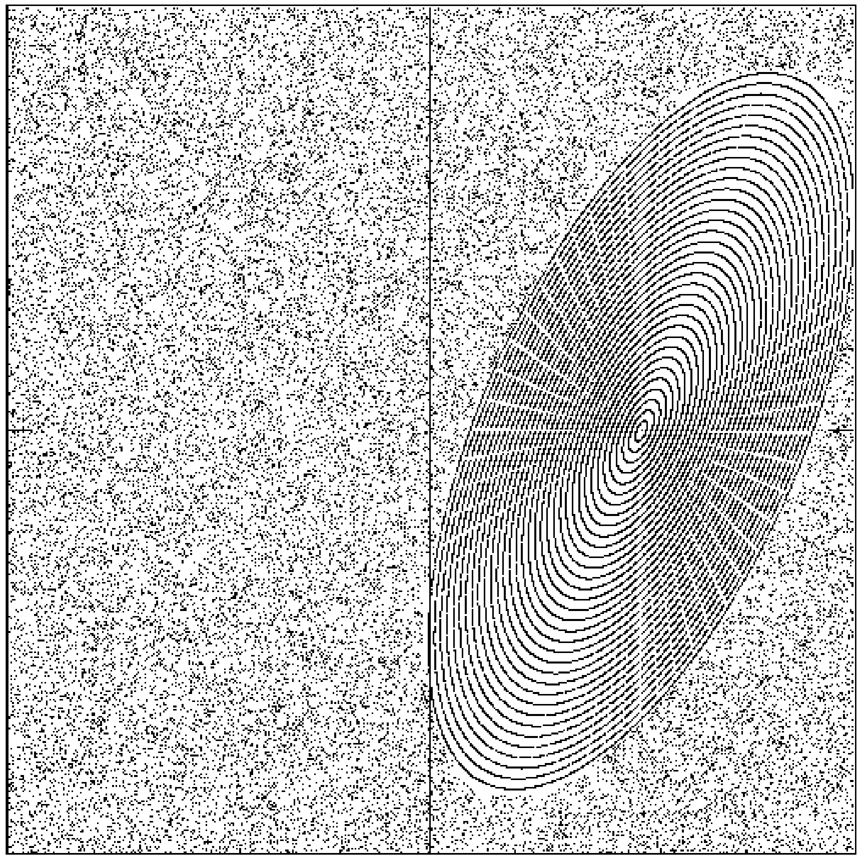
Onishi et al 2003

Some examples validating the scenario

Hénon map



Piecewise linear map



Piecewise linear map

$F_\alpha : \mathbb{T}^2 \mapsto \mathbb{T}^2$

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} p - V'(q) \\ q + T'(p') \end{pmatrix}$$

where $\mathbb{T}^2$ is a 2-dimensional torus with coordinates (p, q) mod 1, and

$$V'(q) = \begin{cases} -\alpha q - \dfrac{1}{2} & (\,-\tfrac{1}{2} < q < 0\,) \\ +\alpha q - \dfrac{1}{2} & (\,0 < q < +\tfrac{1}{2}\,) \end{cases}$$

$$T'(p) = p$$

‘Smoothing’ of F_α

$F_{\alpha,\beta} : \mathbb{R}^2 \mapsto \mathbb{R}^2$

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} p - V'(q) \\ q + T'(p') \end{pmatrix}$$

$$V'(q) = \sum_{n=-\infty}^{\infty} \Big[V_-(q)\Big\{\theta_\beta(q-n) - \theta_\beta(q-n-\frac{1}{2})\Big\} + V_+(q)\Big\{\theta_\beta(q-n+\frac{1}{2}) - \theta_\beta(q-n)\Big\}\Big]$$

$$T'(p) = \sum_{n=-\infty}^{\infty} p\Big\{\theta_\beta(p-n-\frac{1}{2}) - \theta_\beta(p-n+\frac{1}{2})\Big\}$$

where

$$\theta_\beta(x) \equiv \frac{1}{2}\Big[1 + \tanh(\beta x)\Big]$$

and

$$V_-(q) = -\alpha p - \frac{1}{2}, \quad V_+(q) = +\alpha p - \frac{1}{2}$$

‘Complexification’ of $F_{\alpha,\beta}$

$$F_{\alpha,\beta} : \mathbb{C}^2 \mapsto \mathbb{C}^2 \quad \overset{\beta \to \infty}{\Longrightarrow} \quad \mathcal{F}_\alpha : \mathbb{T}^2 \times \mathbb{C} \mapsto \mathbb{T}^2 \times \mathbb{C}$$

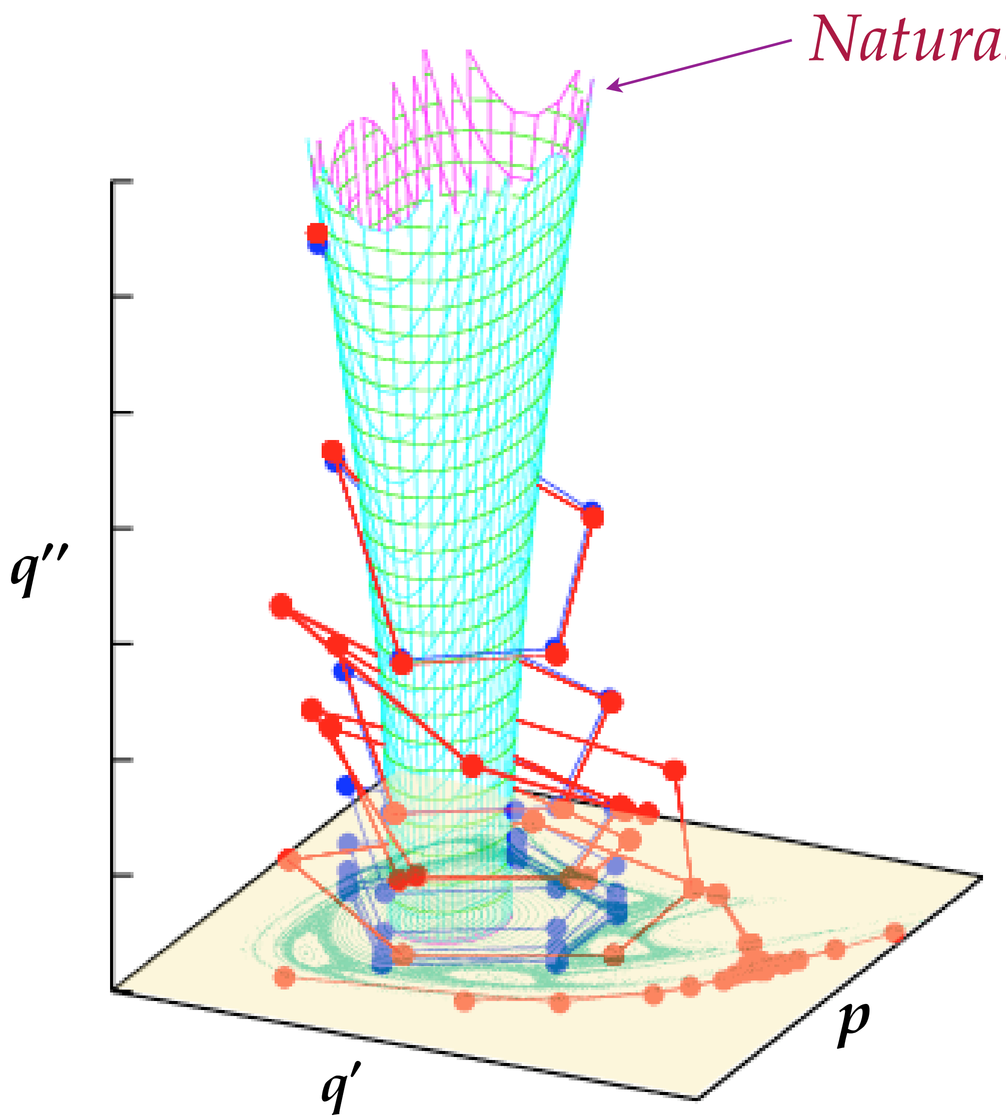
$$\mathcal{F}_\alpha : \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} p - V'(q) \\ q + T'(p') \end{pmatrix}$$

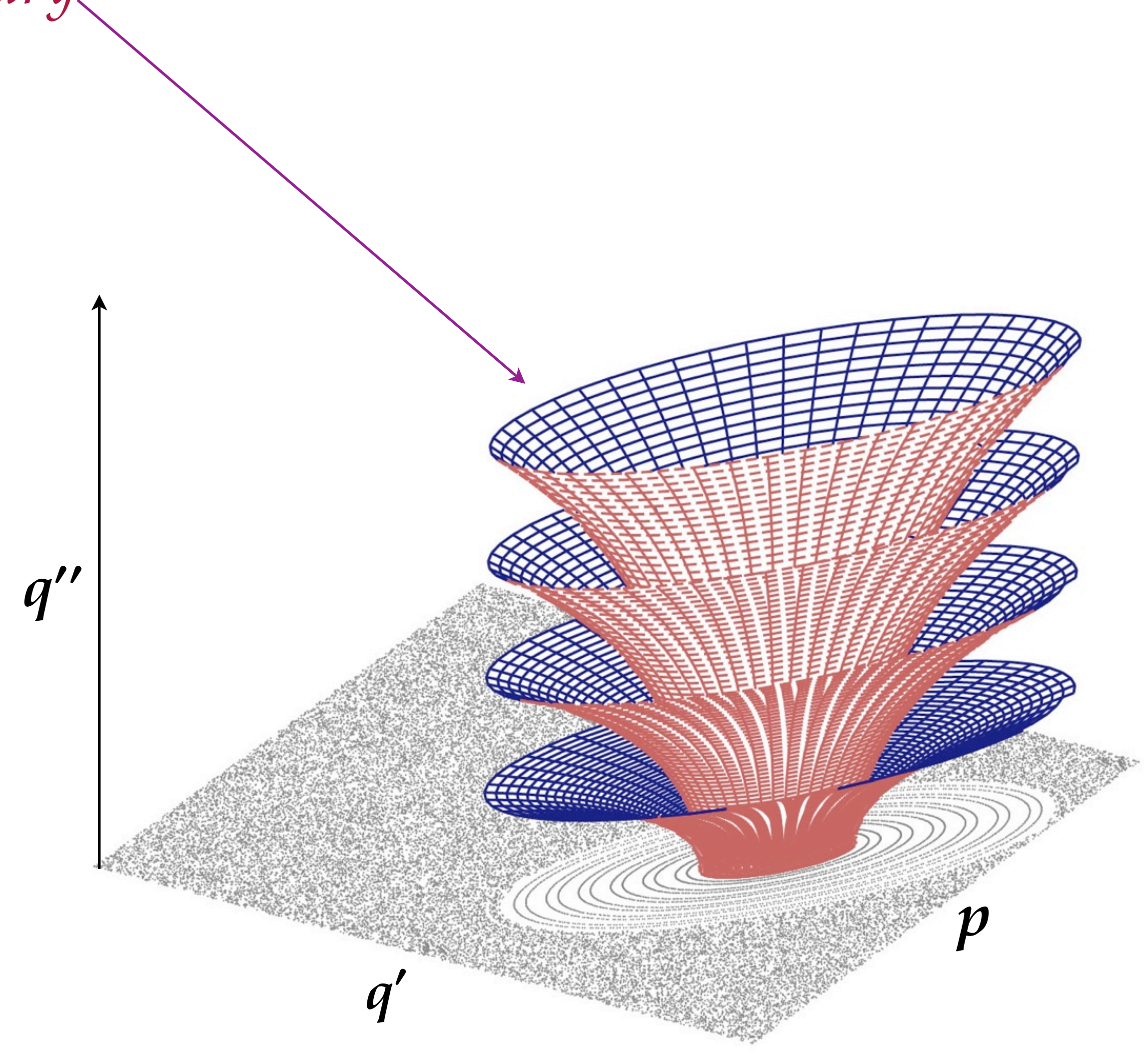
$$V'(q) = \begin{cases} -\alpha q - \dfrac{1}{2} & (-\frac{1}{2} < \mathrm{Re}\, q < 0) \\ +\alpha q - \dfrac{1}{2} & (0 < \mathrm{Re}\, q < +\frac{1}{2}) \end{cases}$$

$$T'(p) = p$$

Hénon map

piecewise linear map





Questions we asked were

* **essential differences between one- and multi-dimensions ?**
* **dynamically disconnected regions are connected, why and how ?**
* **evaluate or even define the tunneling probability in multi-dimensional systems, is it possible ?**