2. Complex dynamics in one variable

1-dimensinol polynomial maps and the Julia set

Consider 1-dimensinol polynomial maps with degree d

$$P : z \mapsto P(z)$$

where

$$P(z) = z^d + a_1 z^{d-1} + \cdots + a_d \qquad (d \geq 2)$$

Classify the orbits according to the behavior of $n \to \infty$

$$F_P = \{ z \in \mathbb{C} \mid \lim_{n\to\infty} P^n(z) = \infty \} \quad : \quad \text{Fatou set}$$

$$K_P = \{ z \in \mathbb{C} \mid \lim_{n\to\infty} P^n(z) \text{ is bounded} \} \quad : \quad \text{Filled Julia set}$$

$$K_P = \mathbb{C} - F_P$$

In particular

$$J_P = \partial K_P \quad : \quad \text{Julia set}$$

The dynamics around $z = 0$ or $z = \infty$

Suppose

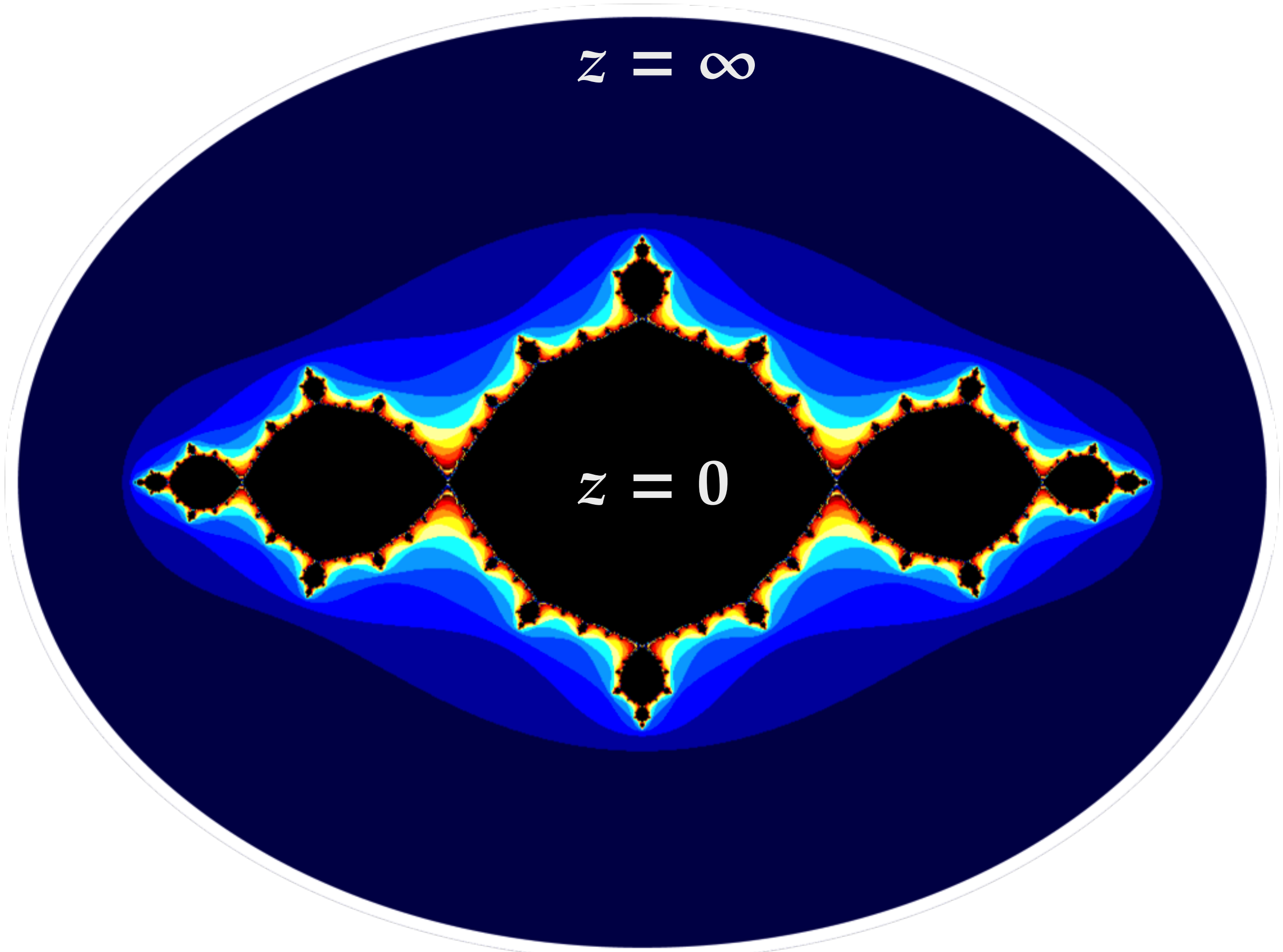
$$P : z \mapsto a_1 z + a_2 z^2 + \cdots a_d z^d \qquad (a_1 \neq 0)$$

Then, $z = 0$ and $z = \infty$ are attracting fixed points.

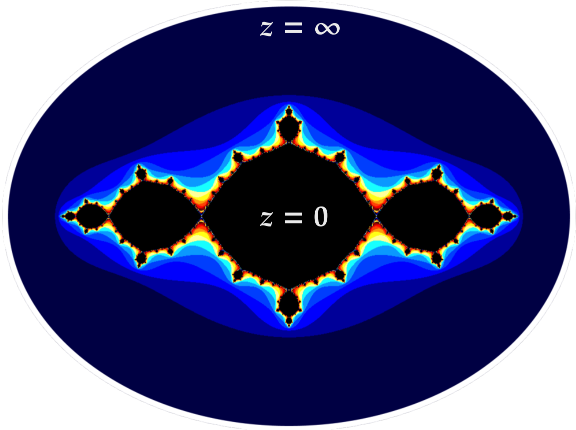
The dynamics around $z = 0$ and $z = \infty$ are rather simple.

$P(z) \sim z$ **around** $z = 0$

$P(z) \sim z^d$ **around** $z = \infty$



The behavior around $z = 0$



Theorem (Koenigs) $F(z)$ is holomorphic near $z = 0$ and has the Taylor expansion

$$F(z) = \lambda z + c_2 z^2 + \cdots \qquad (0 < |\lambda| < 1)$$

Then there exists a conformal map $\psi : U \to \mathbb{C}$ which satisfies the functional equation (Schröder equation)

$$\psi(F(z)) = \lambda \psi(z) \quad (z \in U)$$

where U is a neighborhood of $z = 0$.

Note : If $|\lambda| > 1$, then one can show the same assertion by considering the inverse function.

(Proof)

Step 1

Obtain a formal solution for $\psi(z)$ by assuming

$$\psi(z) = \sum_{\ell=0}^{\infty} a_\ell z^\ell$$

The coefficients $\{a_\ell\}$ are expressed as

$$a_\ell = \frac{K_\ell(c_2, \cdots, c_\ell, a_2, \cdots, a_{\ell-1})}{\lambda^\ell - \lambda}$$

where $K_\ell(c_2, \cdots, c_\ell, c_2, \cdots, a_{\ell-1})$ are a polynomial function of $(c_2, \cdots, c_\ell, a_2, \cdots, a_{\ell-1})$.

Note : if $|\lambda| = 1$, even the formal solution cannot be constructed.

Step 2

Prove the convergency of $\psi(z) = \sum_{\ell=0}^{\infty} a_\ell z^\ell$.

The behavior around $z = 0$

— Linearization around a neutral fixed point —

Theorem (Siegel-Moser) For

$$F(z) = \lambda z + c_2 z^2 + \cdots \quad (\lambda = e^{2\pi i \alpha}, \ \ \alpha : \text{irrational})$$

suppose that there exist $a, b > 0$ such that $\left|\alpha - \frac{p}{q}\right| < \frac{a}{q^b}$ for all $p, q \in \mathbb{Z}$.
Then there is a nbd U of $z = 0$ on which $F(z)$ is analytically conjugate to the irrational rotation, that is, $z \mapsto \lambda z$.

More specifically,

Theorem (Bryuno-Yoccoz) For quadratic maps

$$F(z) = \lambda z + c_2 z^2 \quad (\lambda = e^{2\pi i \alpha}, \ \ \alpha : \text{irrational})$$

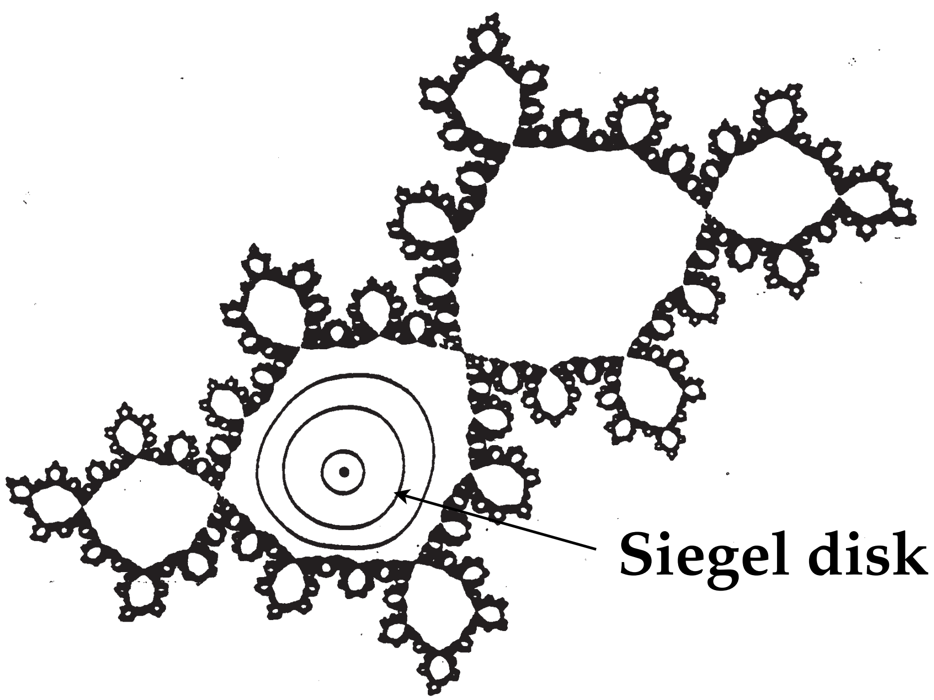
Rotational domains

— Siegel disk $\mathcal{D}$ —

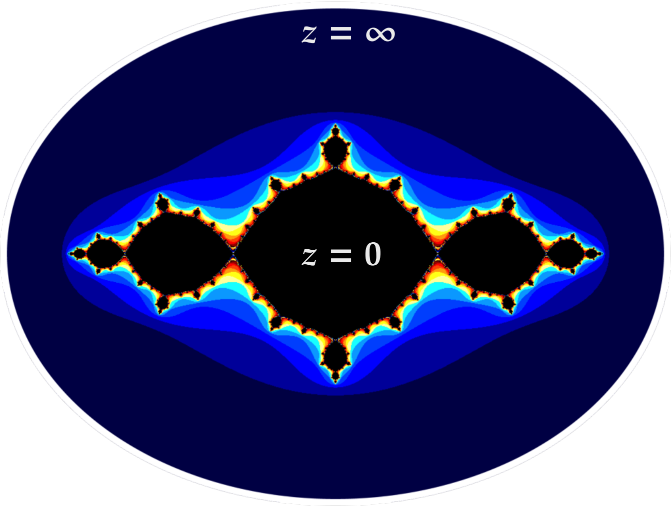
Theorem (Siegel) Around a neutral fixed point $z = 0$,

$$F(z) = \lambda z + c_2 z^2 + \cdots \quad (\lambda = e^{2\pi i\alpha},\ \alpha : \text{Diophantine number})$$

has a region $\mathcal{D}$ which is conjugate to an irrational rotation.
Such a region $\mathcal{D}$ is called the Siegel disk .



Note : If there exists a Siegel disk $\mathcal{D}$, then $\mathrm{Area}(K_P) > 0$,
where $\mathrm{Area}(\cdot)$ denotes 2-dimensinoal area in $\mathbb{C}$.



The behavior around $z = \infty$

Theorem (Böttcher) For a sufficiently large R, there exists a conformal map $\varphi(z)$ of $V = \{\, |z| > R \,\}$ into $\mathbb{C}$ which has the form

$$\varphi(z) = z + b_0 + \frac{b_1}{z} + \cdots$$

and satisfies

$$\varphi\big(P(z)\big) = \big\{\varphi(z)\big\}^d$$

$\varphi(z)$ is called the **Böttcher function**

(Proof)

Step 1

Consider

$$\psi(z) = \log \frac{P(z)}{z^d}$$

Step 2

$$P(z) = z^d \exp \psi(z)$$

$$\begin{aligned} P^2(z) = P\Big(P(z)\Big) &= P(z^d \exp \psi(z)) \\ &= (z^d \exp \psi(z))^d \exp\Big(z^d \exp \psi(z)\Big) \\ &= z^{d^2} \exp\Big(d\psi(z) + \psi(P(z))\Big) \end{aligned}$$

Inductively, we have

$$P^n(z) = z^{d^n} \exp\Big(d^{n-1}\psi(z) + d^{n-2}\psi(P(z)) + \cdots + \psi(P^{n-1}(z))\Big)$$

Step 3

$$\varphi_n(z) = \left(P^n(z)\right)^{d^{-n}} = z\exp\left(\frac{1}{d}\psi(z) + \frac{1}{d^2}\psi(P(z)) + \cdots + \frac{1}{d^n}\psi(P^{n-1}(z))\right)$$

$\displaystyle\sum_{j=1}^{\infty} \frac{1}{d^j}\psi(P^{j-1}(z))$ is uniformly convergent, hence

$$\varphi(z) = \lim_{n\to\infty} \varphi_n(z) = z\exp\left(\frac{1}{d}\psi(z) + \frac{1}{d^2}\psi(P(z)) + \cdots\right)$$

does so, and satisfies the desired functional relation: $\varphi\left(P(z)\right) = \left\{\varphi(z)\right\}^d$.

Check: $\quad \text{lhs} = \varphi\left(P(z)\right) = P(z)\exp\left(\frac{1}{d}\psi(P(z)) + \frac{1}{d^2}\psi(P^2(z)) + \cdots\right)$

$$\begin{aligned}\text{rhs} = \left\{\varphi(z)\right\}^d &= z^d\exp\left(\frac{1}{d}\psi(P(z)) + \frac{1}{d^2}\psi(P^2(z)) + \cdots\right)^d \\ &= z^d\exp\left(\psi(z) + \frac{1}{d}\psi(P(z)) + \cdots\right) \\ &= P(z)\exp\left(\frac{1}{d}\psi(P(z)) + \frac{1}{d^2}\psi(P^2(z)) + \cdots\right)\end{aligned}$$

Green function

We define the Green function as

$$G(z) \equiv \log|\varphi(z)|$$

where $\varphi(z)$ is the Böttcher function.

$G(z)$ can be extended to the Fatou set F_P as the harmonic function, that is

$$\Delta G(z) = 0$$

For $K_P = \mathbb{C} - F_P$, we define

$$G(z) = 0$$

Then one can prove that $G(z)$ is continuous and subharmonic in $\mathbb{C}$.

Note : Subharmonic function $f(z)$ is a function satisfying

$$f(z) \leq \frac{1}{2\pi}\int_0^{2\pi} f(z + re^{i\theta})d\theta$$

Potential theoretic approach

We define the *Green function* by

$$G(z) \equiv \log|\varphi(z)|$$

where $\varphi(z)$ is the Böttcher function.

$G(z)$ can be extended to the Fatou set F_P as the harmonic function :

$$\Delta G(z) = 0$$

For $K_P = \mathbb{C} - F_P$, we define

$$G(z) = 0$$

More explicit expression for the Green function

$$G(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ \left| P^n(z) \right|$$

where $\log^+ t \equiv \max\{ \log t,\ 0 \}$.

Check:

Recall $\varphi_n(z) = \left(P^n(z)\right)^{d^{-n}}$.   Take "log" and $n \to \infty$ in both sides

Remark :

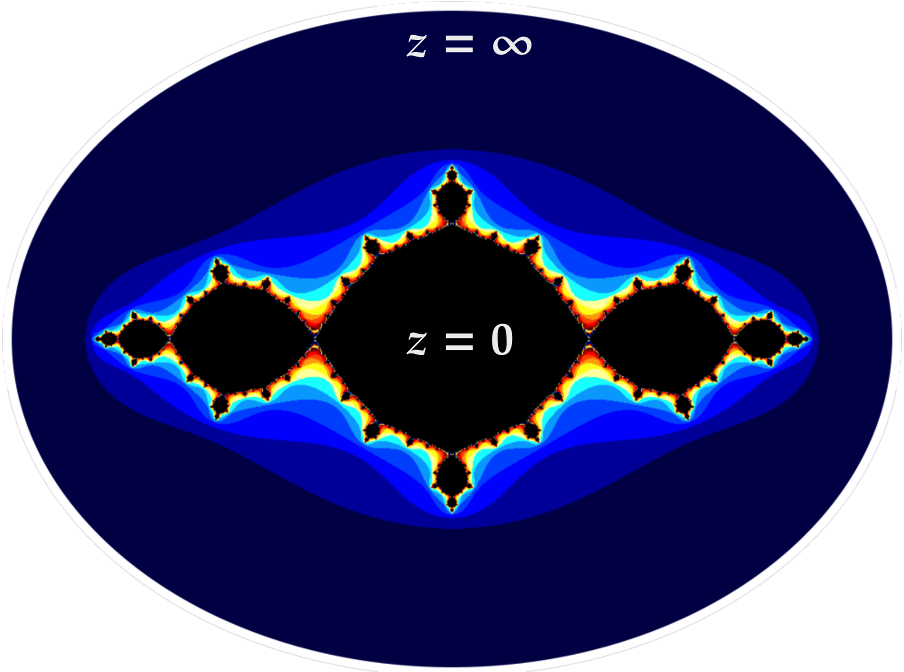
Instead of using the Böttcher function, we can also introduce the Green function through the functional equation:

$$G(z) = \frac{1}{d^n} G\left(P^n(z)\right)$$

Invariant measure induced by the Green function

We here introduce $\mu(z)$ through the "Poisson equation"

$$\mu(z) \equiv \frac{1}{2\pi}\Delta G(z)$$



where $G(z)$ is the Green function

$$G(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ \left| P^n(z) \right|$$

Theorem (Brolin, 1965)

1. $\mu_n(z) = \frac{1}{d^n} \sum_{z_0 \in P^{-n}(a)} \delta(z - z_0) \to \mu(z)$ for arbitrary $z = a$

2. supp $\mu(z) = J_P$

3. the map P preserves the measure μ, and is strongly mixing

(**Proof**)

1. $\mu_n(z) = \frac{1}{d^n} \sum_{z_0 \in P^{-n}(a)} \delta(z - z_0) \to \mu(z)$ **for arbitrary** $z = a$

Step 1

$g(z) = \log|z|$ is a fundamental solution for $\frac{1}{2\pi}\Delta g(z) = \delta(z)$

Therefore, $\frac{1}{2\pi}\Delta \log|P^n(z) - a| = \sum_{P^n(z_0)=a} \delta(z - z_0)$

Step 2

prove $\lim_{n\to\infty} \frac{1}{d^n} \log|P^n(z) - a| = G(z)$, where $G(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+\left|P^n(z)\right|$

(in case $z \in F_P$, $P_n(z) \to \infty$, thus $|P^n(z) - a| \sim |P^n(z)|$.

also in case $z \in K_p$, $P_n(z)$ is bounded, so $\lim_{n\to\infty} \frac{1}{d^n} \log|P^n(z) - a| = 0$)

Step 3

apply "$\frac{1}{2\pi}\Delta$" to both sides

2. supp $\mu(z) = J_P$

Step 1 (supp $\mu \subset J_P$)

The Green function $G(z) = \log|\varphi(z)| = \lim_{n\to\infty} \frac{1}{d^n} \log^+\left| P^n(z) \right|$ is harmonic on F_P (Fatou set), which implies $\mu = 0$ on F_P.
Recall $G = 0$ on K_P (definition of G). Thus, supp $\mu \subset J_P$ follows.

Step 2 (supp $\mu \supset J_P$)

Suppose that there exists a point $z \in J_P$ and its neighborhood U such that supp $\mu \cap U = \emptyset$. This implies $\Delta G = 0$ on U (that is, G is harmonic on U).

On the other hand, $G \equiv 0$ in $(U \cap K_P)$ (by definition of G) and $G \geq 0$ on $\mathbb{C}$, thus $G \equiv 0$ on the whole U due to the principle of minimum values (since G is harmonic). This contradicts that $G > 0$ on $U \cap F_P$ (G is positive on F_P).

3. The map P preserves the measure μ, and is strongly mixing.

In order to prove P is mixing, we have to show

$$\lim_{n\to\infty}\int_{J_P} f(P^n(z))g(z)d\mu(z) = \int_{J_P} f(z)d\mu(z)\cdot\int_{J_P} g(z)d\mu(z)$$

Step 1

Consider the mass distribution $\{\mu_n(\cdot, w)\}$ produced by a starting point w. If we allow w to be a function of n, we get a sequence $\{\mu_n(\cdot, w_n)\}$. $\mu_n(z) \to \mu(z)$ (statement 1.) implies that $\mu_n(\cdot, w_n) \to \mu(\cdot)$.

Step 2

Let $\{Q_j\}_{j=1}^{k}$ be a finite number of boxes which cover J_P, then we can prove $\mu_n(Q_j, w_n) \to \mu(Q_j)$ $(1 \le j \le k)$.

Step 3

For any function $g(z)$ which are constant on each box Q_j, then from the result of step 2 we have

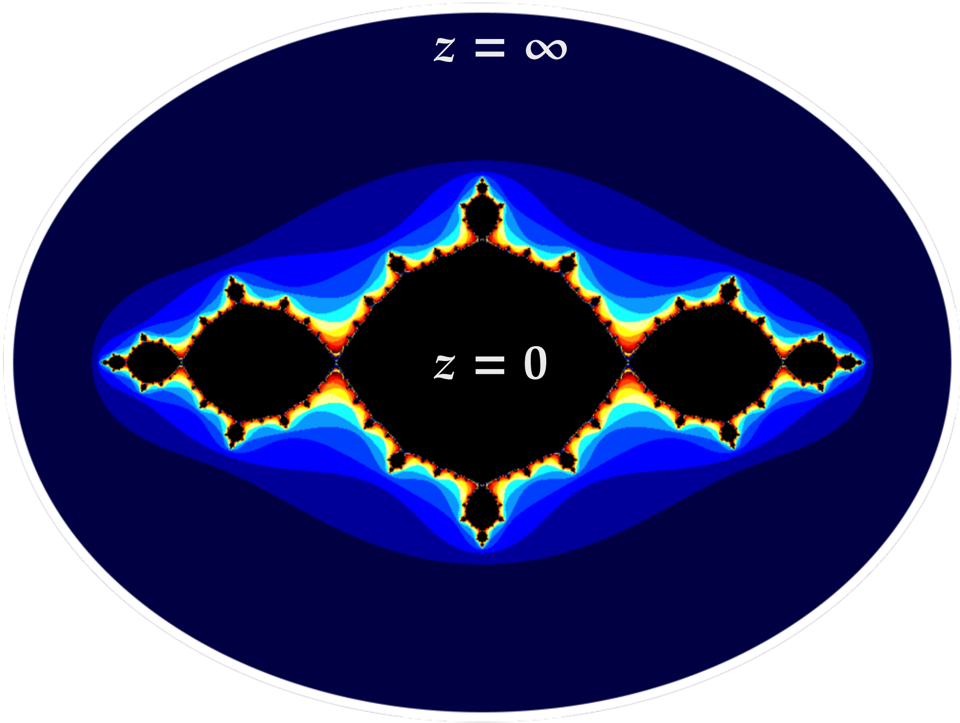
$$\lim_{n\to\infty}\sum_{\nu=1}^{d^\nu}\frac{1}{d^n}g(\zeta_{-n}^{(\nu)}) = \int_{J_P} g(z)d\mu(z)$$

where $\zeta \in J_P$ and $\{\zeta_{-n}^{(\nu)}\}$ are preimages of ζ of order n.

Step 4

For any function $f(z)$, $g(z)$ which is constant on each box Q_j,

$$\begin{aligned}
\lim_{n\to\infty}\int_{J_P} f(P^n(z))g(z)d\mu(z) &= \lim_{n\to\infty}\lim_{m\to\infty}\sum\frac{1}{d^{n+m}}f(\zeta_{-m}^{(\nu)})g(\zeta_{-(m+n)}^{(\nu)})\\
&= \lim_{n\to\infty}\lim_{m\to\infty}\sum\frac{1}{d^m}f(\zeta_{-m}^{(\nu)})\cdot\sum_{\zeta_{-m}^{(\nu)}\ \text{fixed}}\frac{1}{d^n}g(\zeta_{-(m+n)}^{(\nu)})\\
&= \int_{J_P} f(z)d\mu(z)\cdot\int_{J_P} g(z)d\mu(z)
\end{aligned}$$



The border of analyticity

Theorem (Milnor, Costin-Krustal , ...)

The domain of analyticity of $\psi(z)$ is K_P, and $J_P = \partial K_P$ is a singularity barrier (= natural boundary) of $\psi(z)$.

Theorem (Costin-Krustal , ...)

The domain of analyticity of the Böttcher function $\varphi(z)$ is F_P and $J_P = \partial K_P$ is a singularity barrier (= natural boundary) of $\varphi(z)$.

Note : An example of the function with a natural boundary

$$f(z) = \sum_{k=0}^{\infty} z^{2^k}$$

The radius of convergence : $r = 1$

$$\begin{aligned} f\Big(\exp(im\pi/2^n)\Big) &= \exp(im\pi/2^n) + \exp(im\pi/2^{n-1}) + \cdots \exp(im\pi) \\ &\quad + \exp(i2m\pi) + \exp(i2^2 m\pi) + \cdots + \exp(i2m^{\ell}\pi) + \cdots \\ &= \exp(im\pi/2^n) + \exp(im\pi/2^{n-1}) + \cdots \exp(im\pi) \\ &\quad +1 + 1 + \cdots + 1 + \cdots \\ &= \infty \end{aligned}$$

Note that $z = \exp(im\pi/2^n)$ $(n = 0, 1, 2, \cdots ;\ m = 0, 1, 2, \cdots)$ **are dense on** $|z| = 1$**, therefore** $f(z)$ **cannot be analytically continued beyond** $|z| = 1$**.**

(Proof for $\psi(z)$)

Since $\psi(F(z)) = \lambda\psi(z)$ we have

(1) $$\psi(F^n(z)) = \lambda^n\psi(z)$$

Recall that $J_P = \partial K_P = \overline{\{\text{ repelling fixed points }\}}$.

Assume z_0 is a repelling fixed point of $F(z)$ of period n, and is a point of analyticity of $\psi(z)$.

- **The relation (1) implies $\psi(z_0) = 0$, since $|\lambda| < 1$.**
- **$(F^n)'(z_0)\psi'(z_0) = \lambda^n\psi'(z_0)$. but since $|(F^n)'(z_0)| > 1$ and $|\lambda| < 1$, this implies $\psi'(z_0) = 0$.**
- **Inductively, we have $\psi^{(m)}(z_0) = 0$ for all m.**
- **Since we have assumed that $\psi(z)$ is analytic, this entails $\psi(z) \equiv 0$.**

Note:

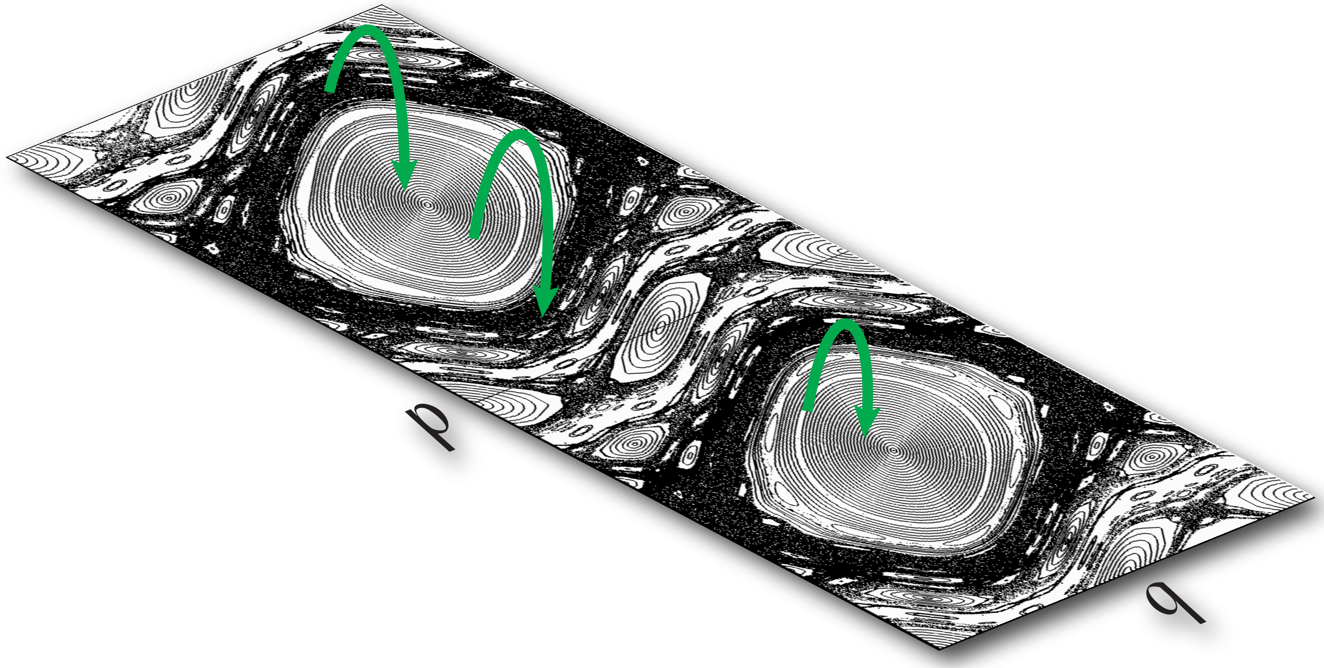
What if $\lambda = e^{2\pi i\alpha}$ where α is a Diophantine number ?

3. Complex dynamics in two variables

2-dimensinal area-preserving maps

$$F: \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} p - V'(q) \\ q + H'(p') \end{pmatrix}$$

- Standard map : $H(p) = \dfrac{p^2}{2}, \quad V(q) = K \cos q$
- Kicked Harper map : $H(p) = K \cos q, \quad V(q) = K \cos q$
- Cubic potential map : $H(p) = \dfrac{p^2}{2}, \quad V(q) = -\dfrac{q^3}{3} - cq$

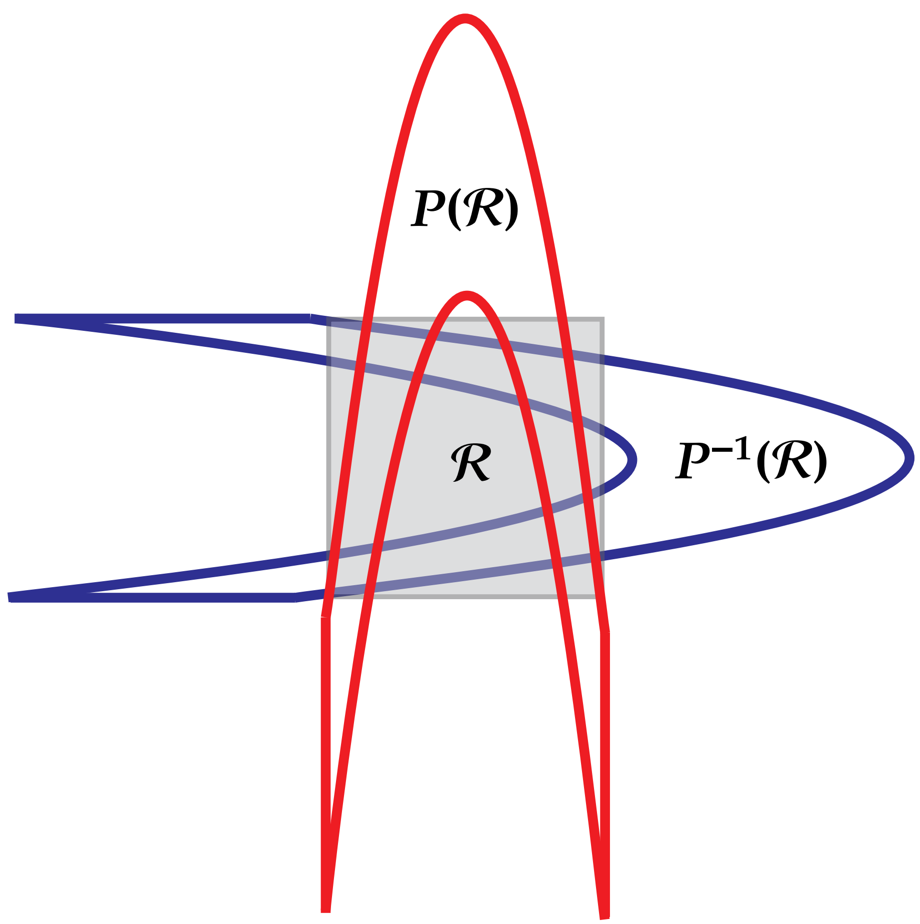


“The cubic potential map” F is transformed into the *Hénon map* by an affine transformation $(p, q) = (y - x, y - 1)$,

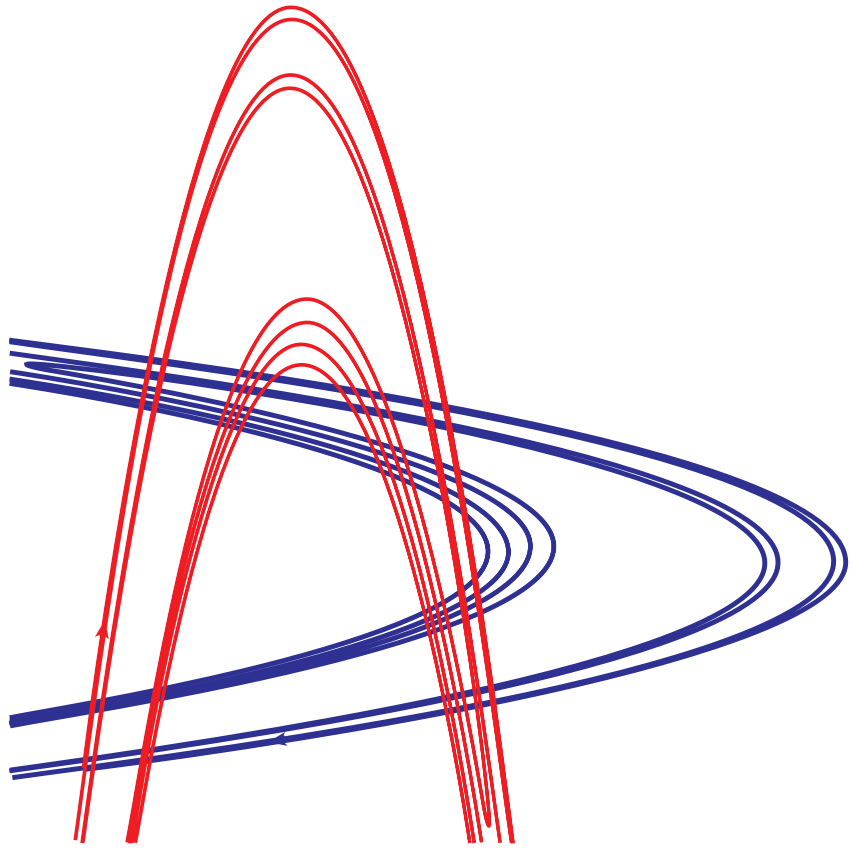
$$P : \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ y^2 - x + a \end{pmatrix}$$

($a = 1 - c$: nonlinear parameter)

$n = 1$



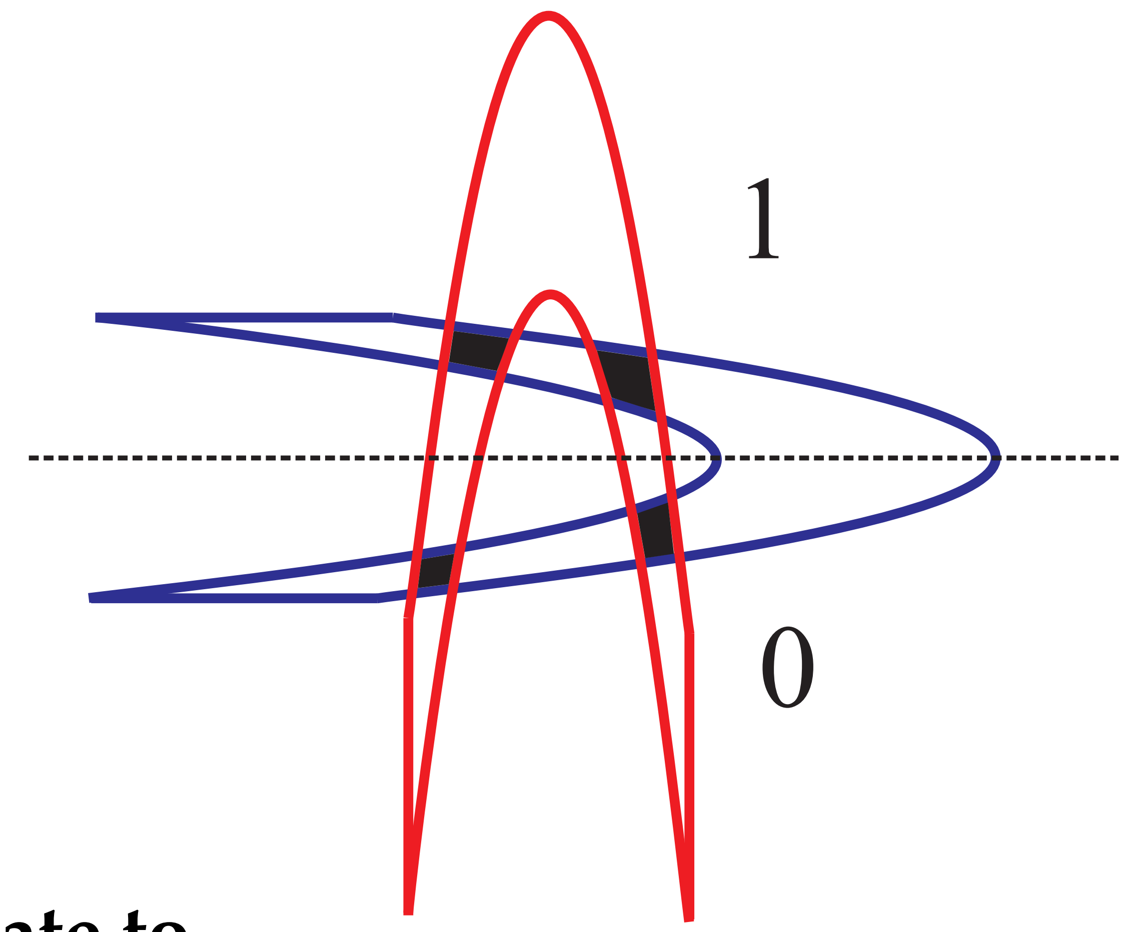
$n \to \infty$



Horseshoe case $(a > a_f)$

We have a generating partition which admits the symbolic dynamics with the binary coding $\{0, 1\}$

$$
\begin{array}{ccc}
P : K & \longmapsto & K \\
\varphi \downarrow & & \downarrow \varphi \\
\sigma : \Sigma & \longmapsto & \Sigma
\end{array}
$$



where $\Sigma = \{0, 1\}^{\mathbb{Z}}$

Then the Hénon map P is conjugate to

$$\sigma(\cdots s_{-1}s_0.s_1s_2\cdots) = (\cdots s_0s_1.s_2s_3\cdots)$$

Parameter dependence

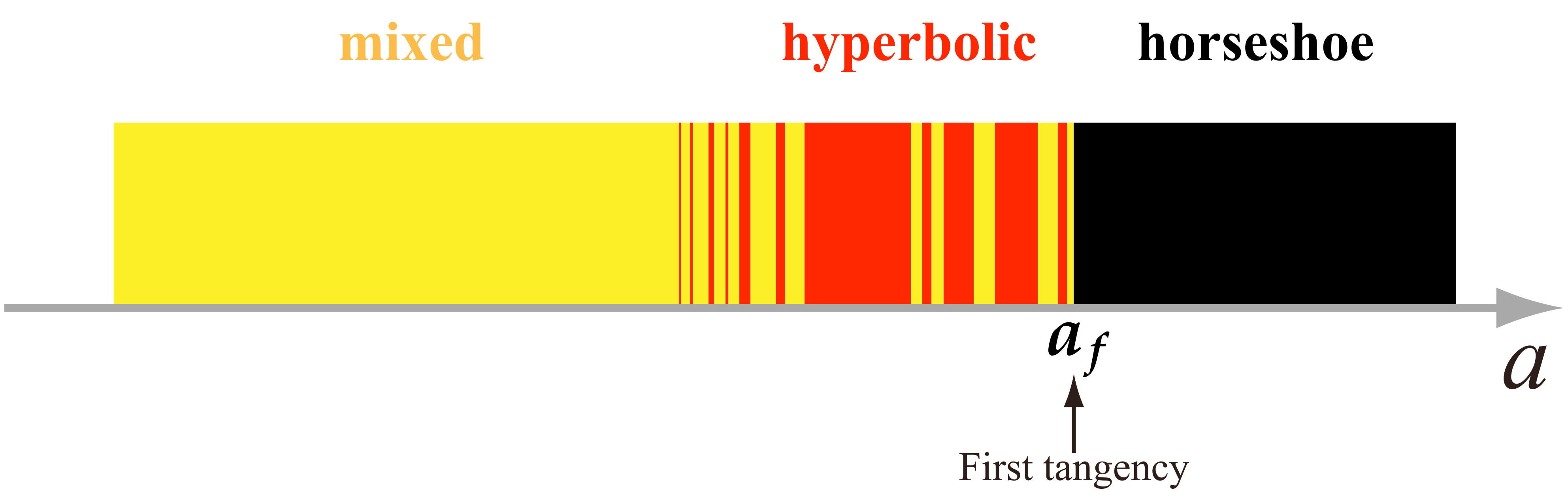
Hénon map :

$$P : \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ y^2 - x + a \end{pmatrix}$$

(a : nonlinear parameter)

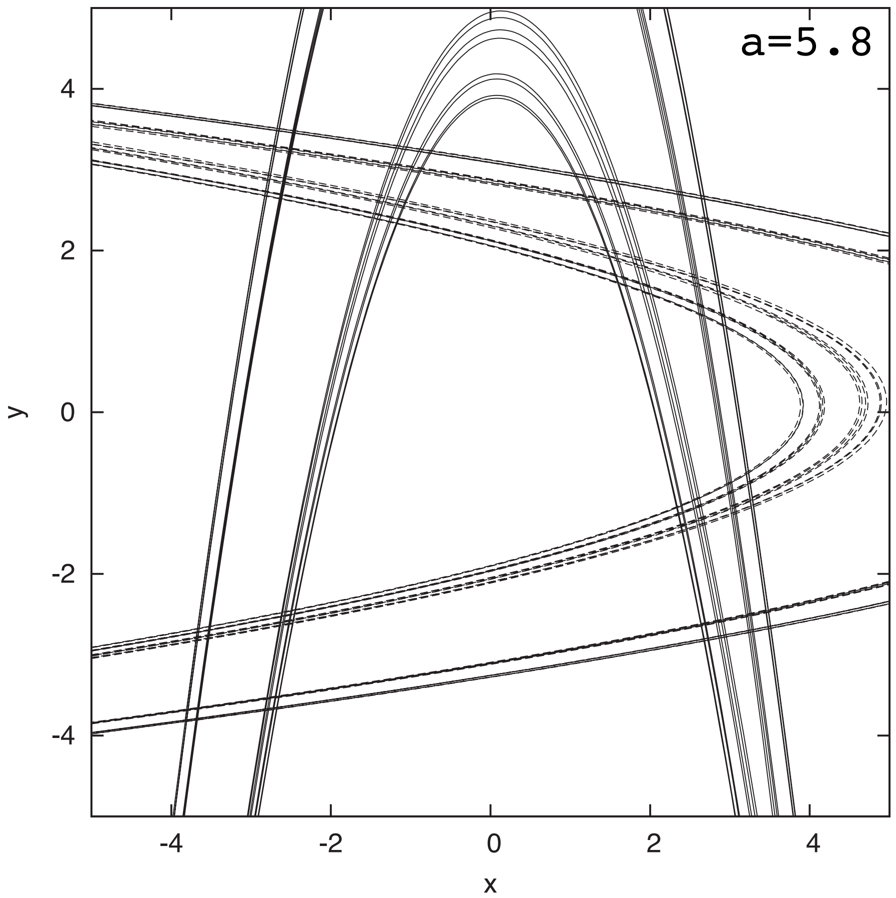
Depending on a, the Hénon map shows different dynamical behaviors:

1. **horseshoe**
2. **hyperbolic (but not horseshoe)**
3. **mixed**

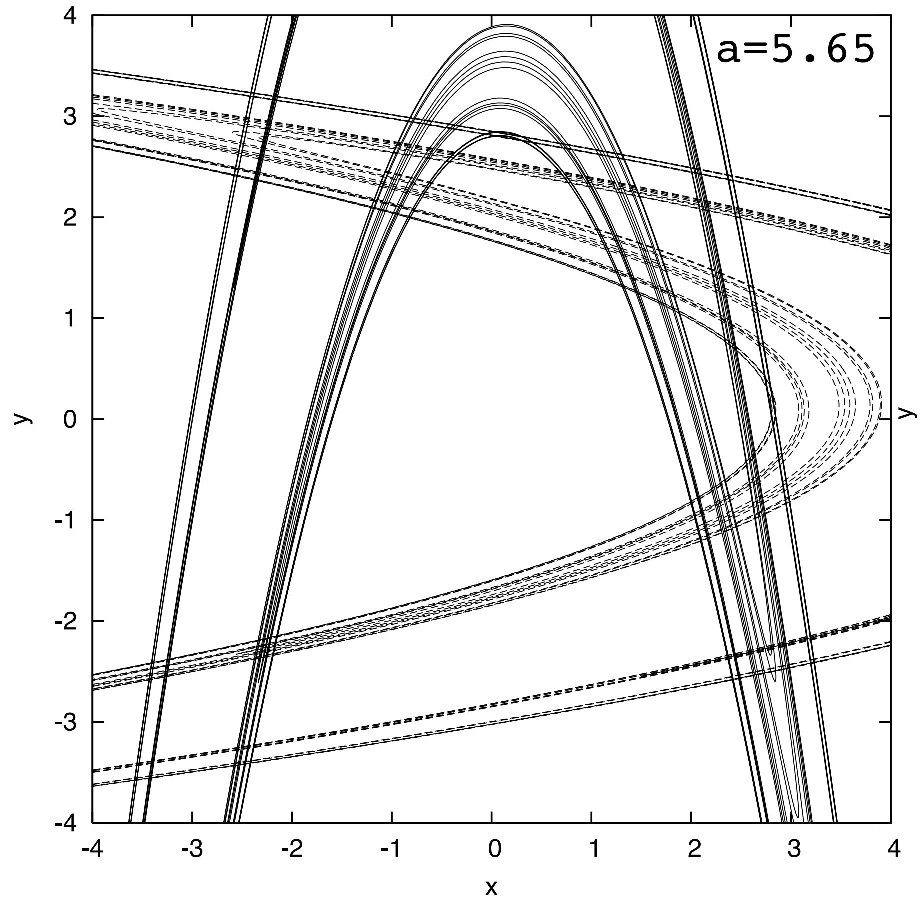


As a decreases,

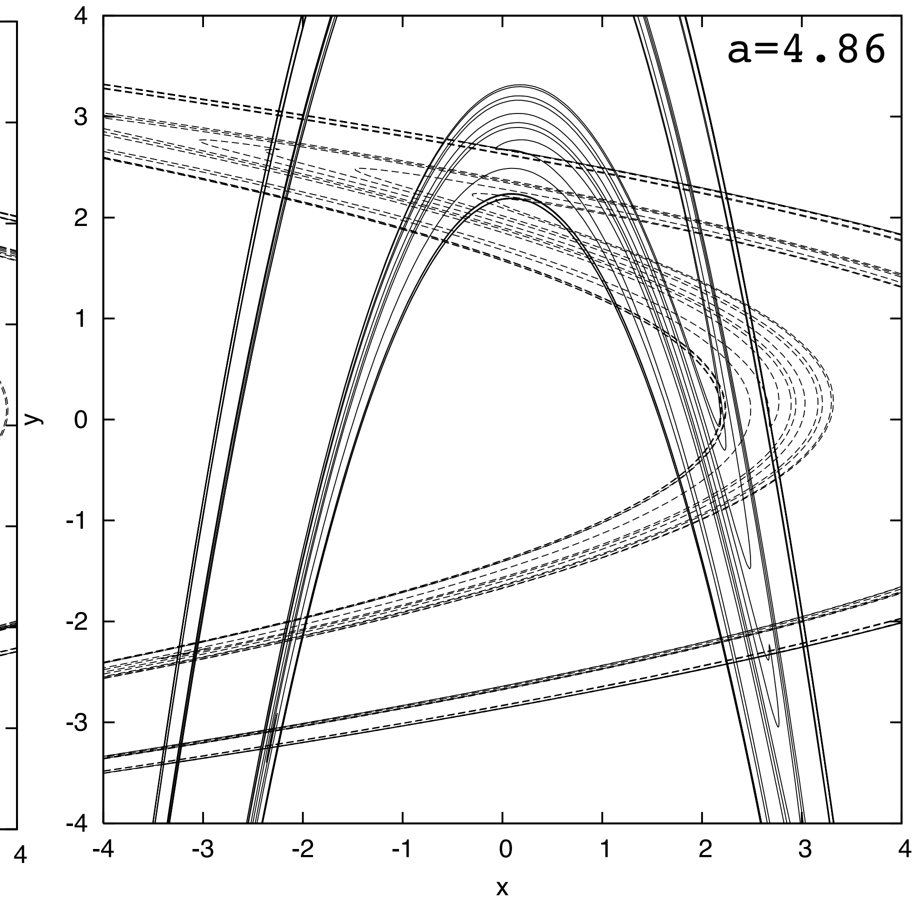
complete horseshoe

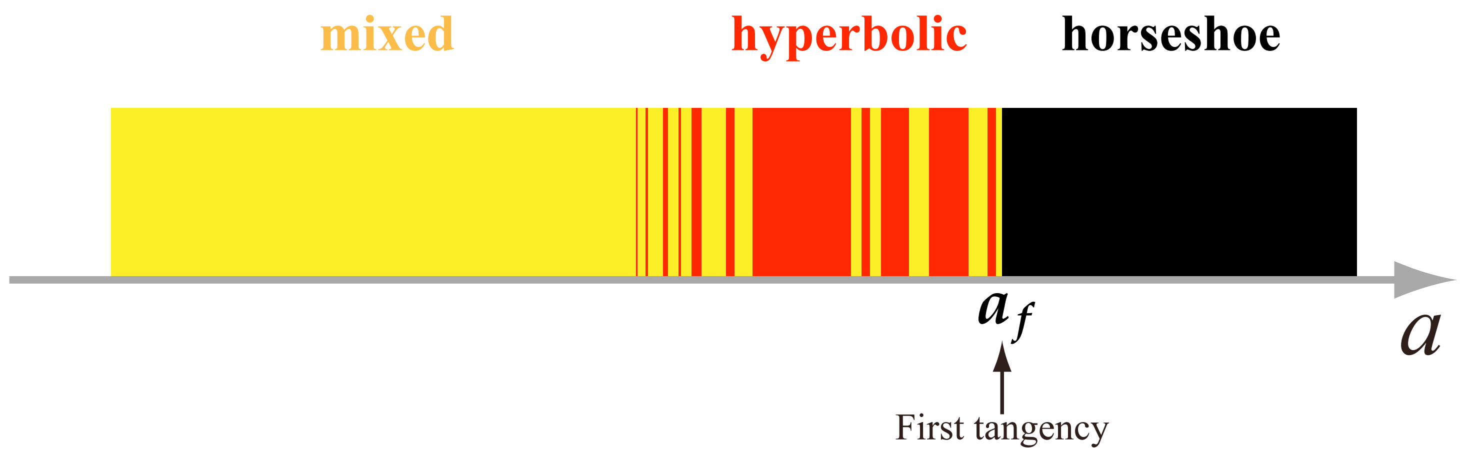


horseshoe broken



horseshoe broken

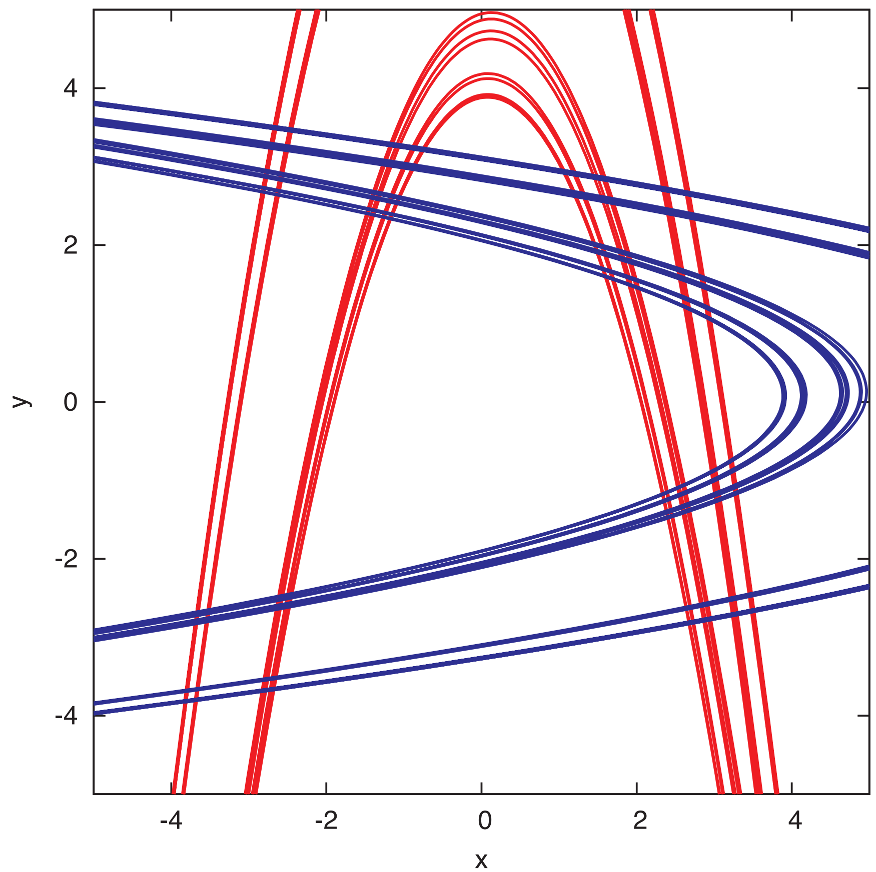




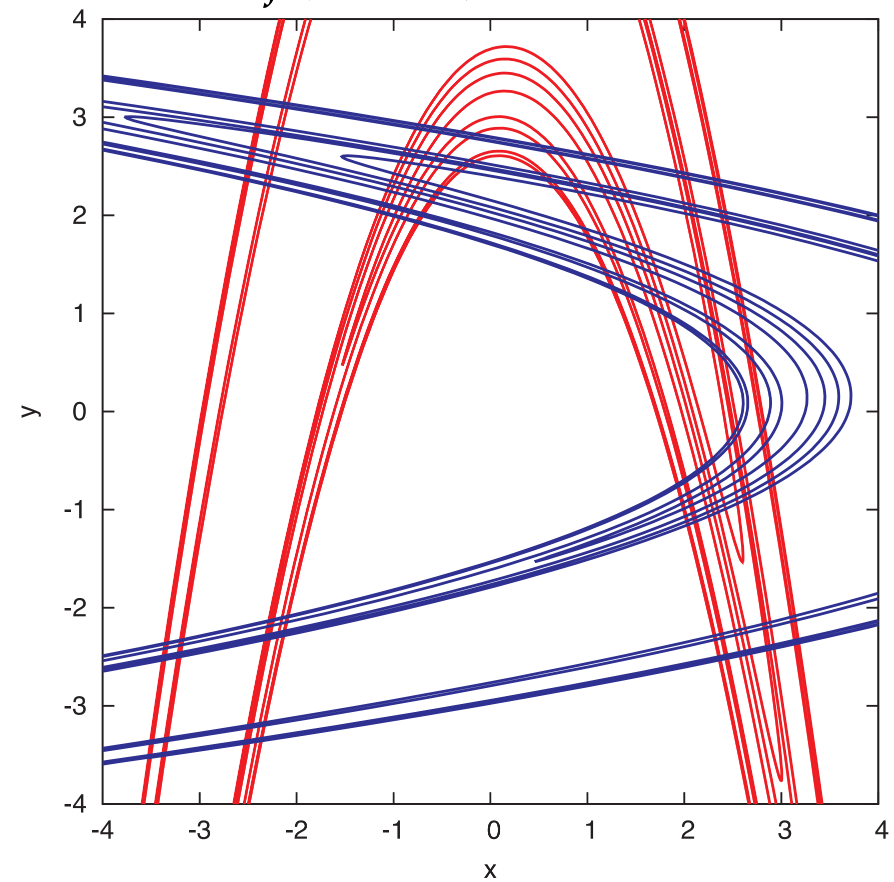
Hyperbolic case (not horseshoe)

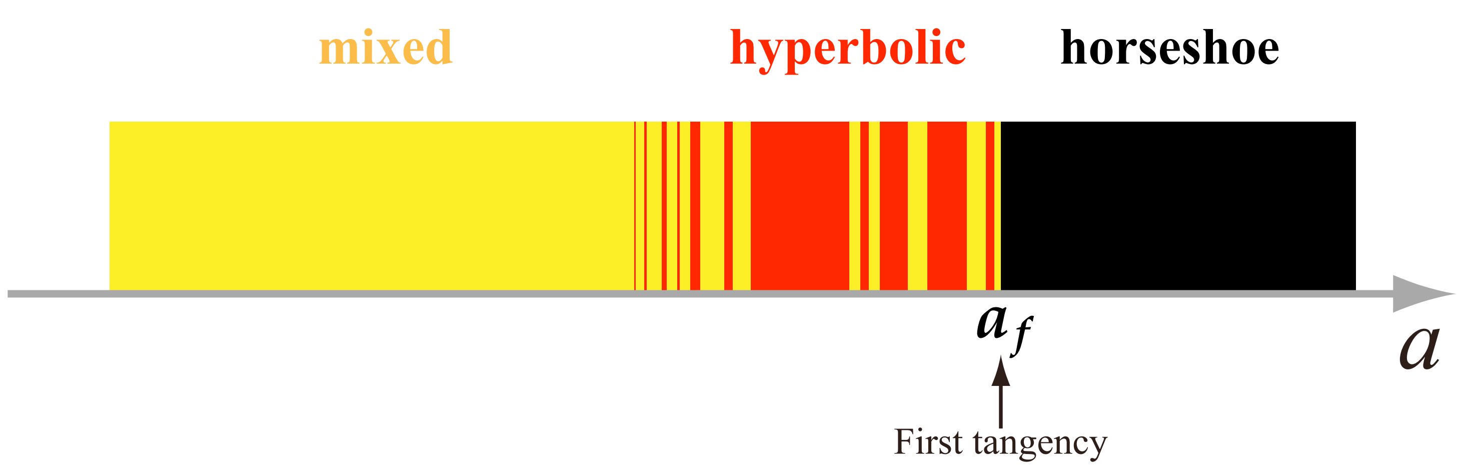
Even after the horseshoe is broken, the dynamics is still hyperbolic, if all the stable and unstable manifolds intersect transversally.

$a > a_f$(Before)

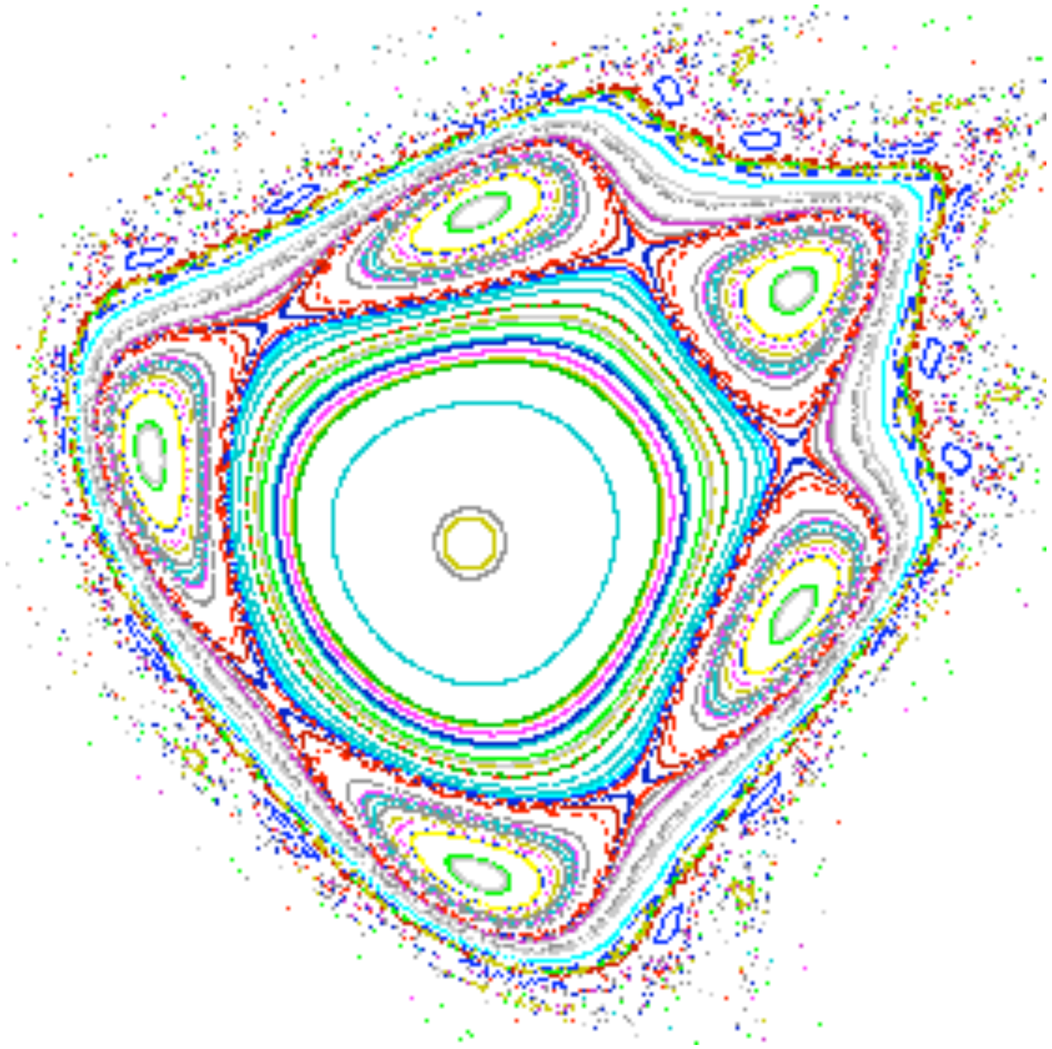


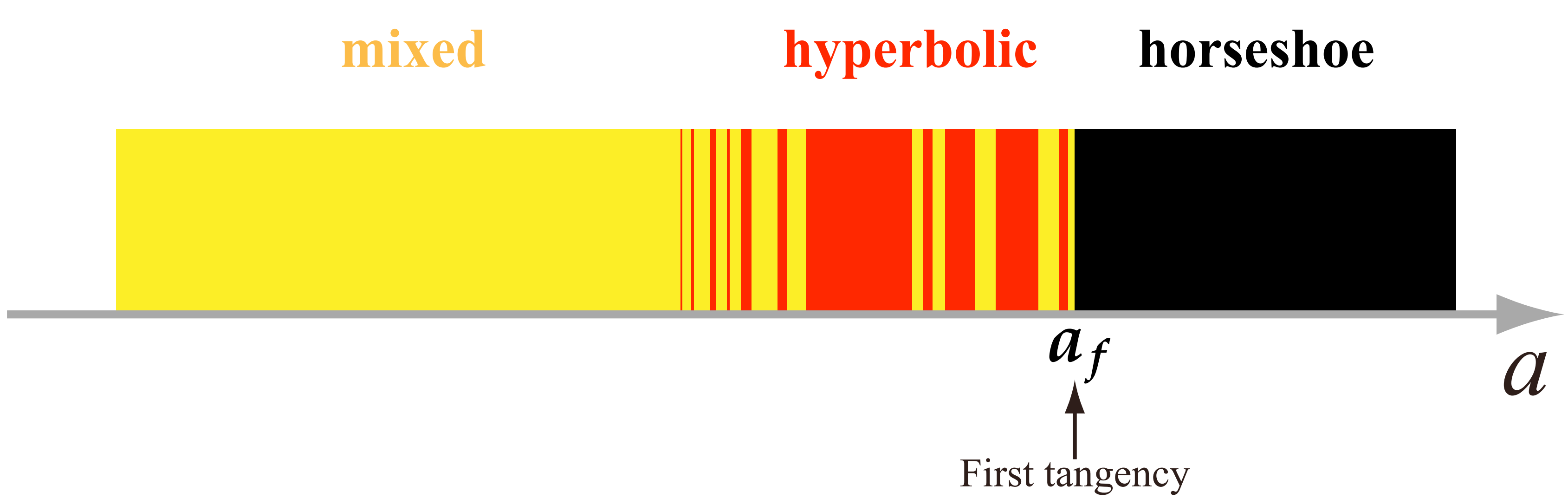
$a < a_f$(After)





Mixed phase space ($a \sim 1$)





Julia sets for 2-dimensional maps

Classify the orbits according to the behavior of $n \to \infty$

$$F^{\pm} = \{\, (x, y) \in \mathbb{C}^2 \;\mid\; \lim_{n\to\infty} P^{\pm n}(x, y) \to \infty \;(\, n \to \infty \,)\, \}$$

$$K^{\pm} = \{\, (x, y) \in \mathbb{C}^2 \;\mid\; \lim_{n\to\infty} P^{\pm n}(x, y) \text{ is bounded in } \mathbb{C}^2 \,\}$$

In particular

$K = K^+ \cap K^-$	:	**filled Julia set**
$J^{\pm} = \partial K^{\pm}$	:	**forward (resp. backward) Julia set**
$J = J^+ \cap J^-$	:	**Julia set**

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Green function

Define

$$G^{\pm}(x,y) \equiv \lim_{n\to+\infty} \frac{1}{2^n} \log^+ \left| P^{\pm n}(x,y) \right|$$

where $\log^+ t \equiv \max\{ \log t,\ 0 \}$.

$G^{\pm}(x,y)$ **is continuous and** ***plurisubharmonic*** **on** $\mathbb{C}^2$. **Thus, by identifying** $(x,y) = (z_1, z_2)$**, we can apply the** dd^c**-operator (= complex Laplacian) :**

$$dd^c u \equiv 2i \sum_{j,k=1}^{2} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k$$

to $G^{\pm}(x,y)$ **in the sense of distribution so as to get the (1, 1)-*currents* :**

$$\mu^{\pm} \equiv \frac{1}{2\pi} dd^c G^{\pm}$$

~ ***Poisson equation*** **(**$\mu^{\pm}$**: charge distribution)**

Remark 1 *Plurisubharmonic function*

"Generalization of the *subharmonic function* to several variables cases"

Recall that subharmonic function $f(z)$ is a function satisfying

$$f(z) \leq \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta$$

Definition **Upper semi-continuous function $f(z_1, z_2)$ is *plurisubharmonic* if $f(z_1, z_2)$ is subharmonic or identically equal to $-\infty$ on *any* complex 1-dimensional line.**

Remark 2 *Currents*

"Differential forms whose coefficients are given by distributions"

Definition Denote the set of differential (p, q)-forms whose coefficients are contained in C_0^∞ (distributions) by $\mathcal{D}^{p,q}(\Omega)$:

$$\mathcal{D}^{p,q}(\Omega) \equiv \left\{ \sum_{|I|=p,|J|=q} u_{I,J}\, dz_{i_1} \wedge \cdots dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge dz_{j_q} \;\middle|\; u_{I,J} \in C_0^\infty(\Omega) \right\}$$

Then, linear functionals on $\mathcal{D}^{2-p,2-q}(\Omega)$ (= elements of the dual space of $\mathcal{D}^{p,q}(\Omega)$) are called (p, q)-*currents*. Here $I = (i_1, \cdots, i_p)$, $J = (j_1, \cdots, j_q)$.

Note: (2, 2)-currents whose coefficients have compact supports can be identified with the *measures* on Ω.

An alternative definition for the Green function $G(x, y)$

Recall also that in 1-dimensional cases we can define the Green function as

$$G(z) \equiv \log|\varphi(z)|$$

where $\varphi(z)$ denotes the *Böttcher function*.

In the same way, for 2-dimensional cases, we define as

$$G(x, y) \equiv \log|\varphi(x, y)|$$

where $\varphi(x, y)$ is a 2-dimensional analog of *Böttcher function*, which is constructed to satisfy

$$\varphi\Big(P(x, y)\Big) = \Big\{\varphi(x, y)\Big\}^2$$

Note : $G(x, y)$ also satisfies the functional relation: $G(z) = \dfrac{1}{2^n} G\Big(P^n(z)\Big)$

Theorem (Bedford-Smillie)

$$\text{supp}\,\mu^{\pm} = J^{\pm}$$

where $J^{\pm}$ is the forward (resp. backward) Julia set.

Note : $\text{supp}\,\mu(z) = J_P$ for 1-dimensional polynomial maps (Brolin)

(Proof)

($\text{supp}\,\mu^+ \subset J^+$)

Since G^+ is pluriharmonic on F^+ (Fatou set), that is $dd^cG^+ = 0$, which implies $\mu^+ = 0$ on F^+. Recall that $G^+ = 0$ on K^+(by definition of G^+). Thus, $\text{supp}\,\mu^+ \subset J^+$.

($\text{supp}\,\mu^+ \supset J^+$)

Suppose that there exist a point $z \in J^+$ and its neighborhood W such that $\text{supp}\,\mu \cap W = \emptyset$. This implies $dd^cG^+ = 0$ on W (that is, G^+ is pluriharmonic on W). On the other hand, $G^+ \equiv 0$ in $(W \cap K^+)$ (by definition of G^+) and $G^+ \geq 0$ on $\mathbb{C}^2$, thus $G^+ \equiv 0$ on the whole W due to the principle of minimum values (since G^+ is pluriharmonic). This contradicts that $G > 0$ on $W \cap F^+$ (G^+ is positive on K^+).

Complex equilibrium measure

Theorem (Bedford-Smillie)

1. $\mu = \mu^+ \wedge \mu^-$ is an invariant measure of the map P
2. Define

$$J^* \equiv \operatorname{supp} \mu$$

where μ is the *potential theoretic invariant measure* defined in 1.

Then we can prove

$$J^* \subset J = \partial J^+ \cap \partial J^-$$

In particular, if P is *hyperbolic*, then

$$J^* = J$$

Stable and unstable convergent theorem

Theorem (Bedford-Smillie) Let M be an algebraic variety, then there is a constant $c > 0$ such that

$$\lim_{n\to\infty} \frac{1}{2^n}[P^{\mp n}M] = c\mu^{\pm}$$

in the sense of current, where $[M]$ is the current of integration of M.

Note 1:

For $u \in \mathcal{D}^{1,1}(\Omega)$, the current of integration of M is defined by $\int [M] \wedge u = \int_M u$

Note 2 :

An algebraic variety is given as the zero set of polynomials

Ex) Line $(z_1 + z_2 - 1 = 0)$, Sphere $(z_1^2 + z_2^2 - 1 = 0)$, and so on.

Some important properties derived from the convergent theorem

Theorem (Bedford-Smillie)

1. For any unstable periodic orbit p, $\overline{W^s(p)} = J^+$, $\overline{W^u(p)} = J^-$

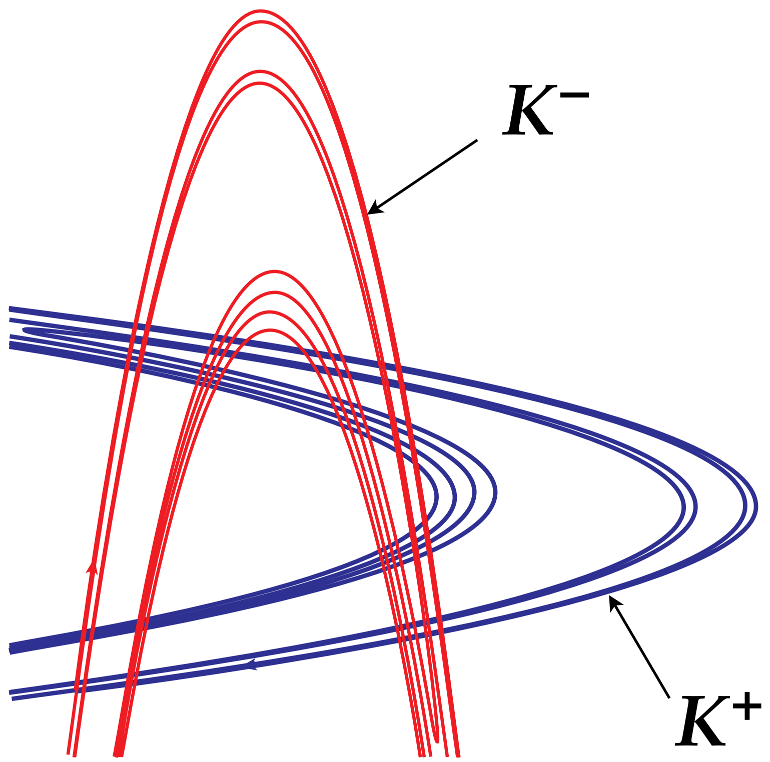
2. μ satisfies the mixing property and is hyperbolic measure, where $\operatorname{supp}\mu = J^*$

3. $\overline{\{\text{Unstable periodic points}\}} = J^*$

Note : The measure μ is said to be hyperbolic measure, if characteristic exponents satisfy $\lambda_1 > 0 > \lambda_2$.

Hénon map $P : \mathbb{R}^2 \mapsto \mathbb{R}^2$

$$P : \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ y^2 - x + a \end{pmatrix}$$



For the horseshoe case, we can prove the followings :

1. For any unstable periodic orbit p, $\overline{W^s(p)} = J^+$, $\overline{W^u(p)} = J^-$

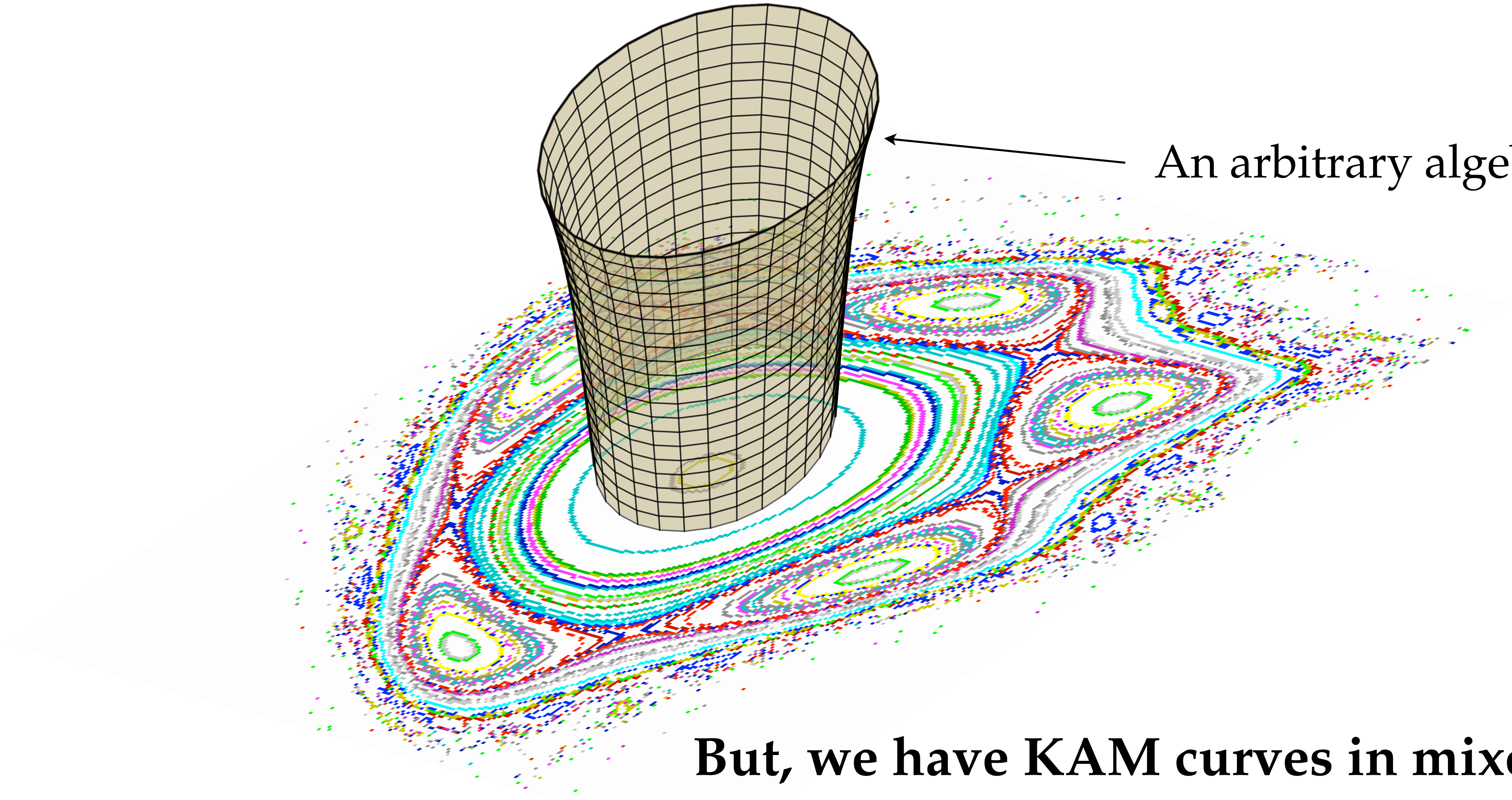
2. The measure on the invariant set satisfies the mixing property and is hyperbolic measure,

3. $\overline{\{\text{Unstable periodic points}\}} = J^*$

What does the convergent theorem tell us ?

Since supp $\mu^{\pm} = J^{\pm}$,

any (algebraic) manifold $M \to J^{-} = \overline{W^{u}(p)}$ $\quad (n \to +\infty)$



But, we have KAM curves in mixed phase space ···

Interior points of $K^{\pm}$ and K
— 2-dimensional area preserving maps —

1. Linearization around a fixed point

Linearization around a fixed pont is not possible in the area preserving map because the non-resonant condition for eigenvalue of the linearized matrix A is be satisfied.

$\Rightarrow$ Siegel disks cannot appear in 2-dimensional area preserving maps

We say that the matrix A satisfies the non-resonant condition if we have

$$\prod_{i=1}^{2} \lambda_i^{k_i} - \lambda_j \neq 0$$

for any $j = 1, 2$ and $(k_1, k_2) \in \mathbb{N}^2$ with $\left|\sum_{i=1}^{2} k_i\right| \geq 2$.

However, for an elliptic fixed point of 2-dimensional area preserving maps, we necessarily have a pair of eigenvalues $\lambda = e^{i\alpha}$ and $\lambda^{-1} = e^{-i\alpha}$ with $\alpha \in \mathbb{R}$ which clearly breaks the non-resonant condition.

KAM curves in $\mathbb{C}^2$

For a given rotation number ω, the motion on the KAM curve C_ω is expressed as a constant rotation in a sutable coordinate θ:

$$\sigma : \theta \mapsto \theta + 2\pi\omega \pmod{2\pi}$$

In order to have such a coordinate θ, the conjugation function φ satisfying

$$\begin{array}{ccc} F : C_\omega & \longmapsto & C_\omega \\ \varphi \downarrow & & \downarrow \varphi \\ \sigma : T^1 & \longmapsto & T^1 \end{array}$$

has to be analytic with respect to θ.

Assume

$$\varphi(\theta, \omega) = \sum_n a_n(\omega) \mathrm{e}^{in\theta}$$

KAM theorem claims $\varphi(\theta, \omega)$ converges on the strip $|\,\mathrm{Im}\,\theta\,| < \rho_c$ for sufficiently irrational $\omega \Rightarrow$ Complexfied KAM curves.

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How fat are KAM curves in $\mathbb{C}^2$?

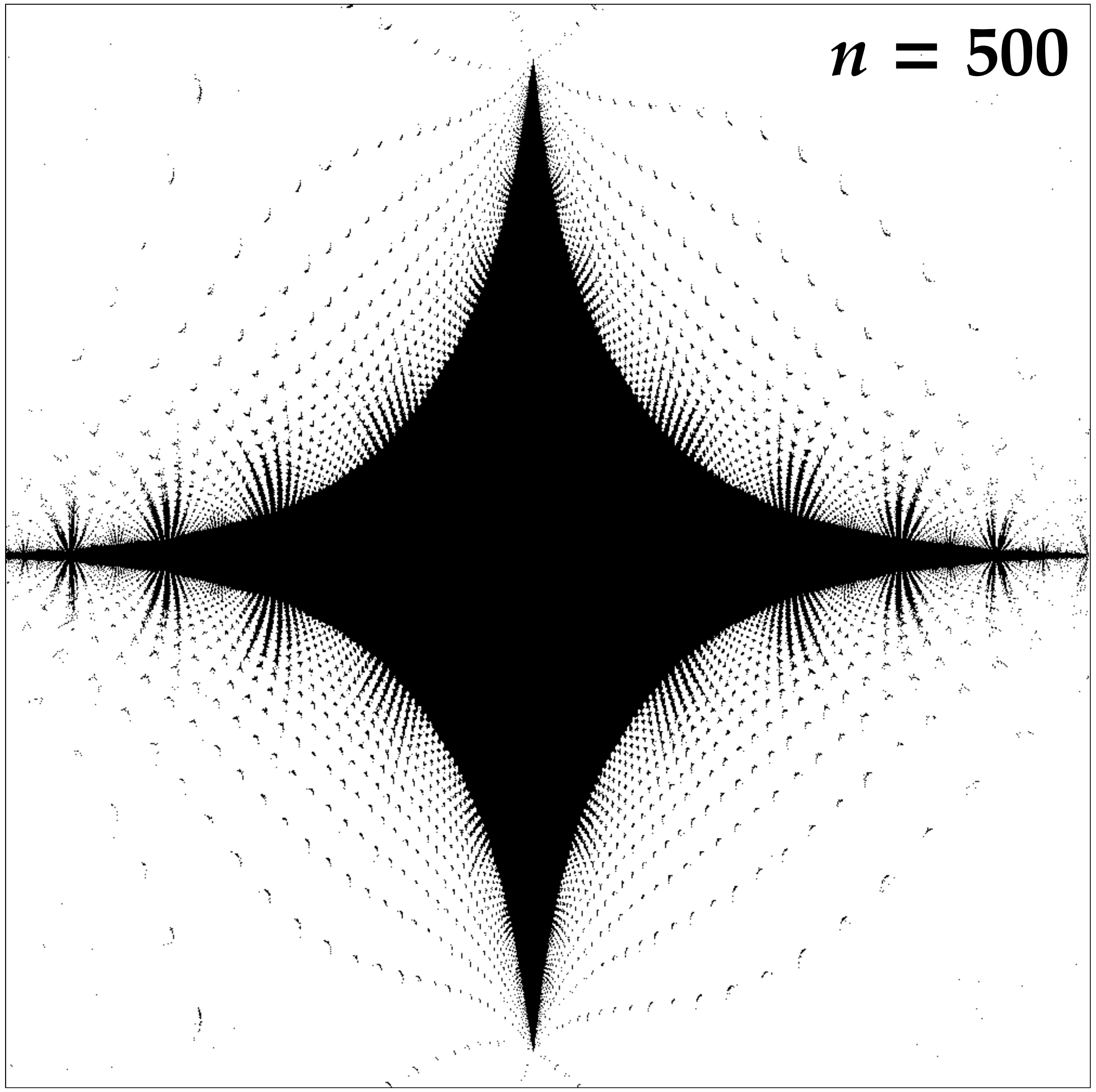
1. For a given $\omega \in \mathbb{R}$, θ can be complexified as $\theta = \theta' + i\theta''$
   $\Longrightarrow$ 2-dim

2. KAM theorem claims that the measure of $\omega \in \mathbb{R}$ with $\rho_c > 0$ is positive, but KAM curves do not exist for rational ω, which also have positive measure:
   $\Longrightarrow$ α-dim $(0 < \alpha < 1)$

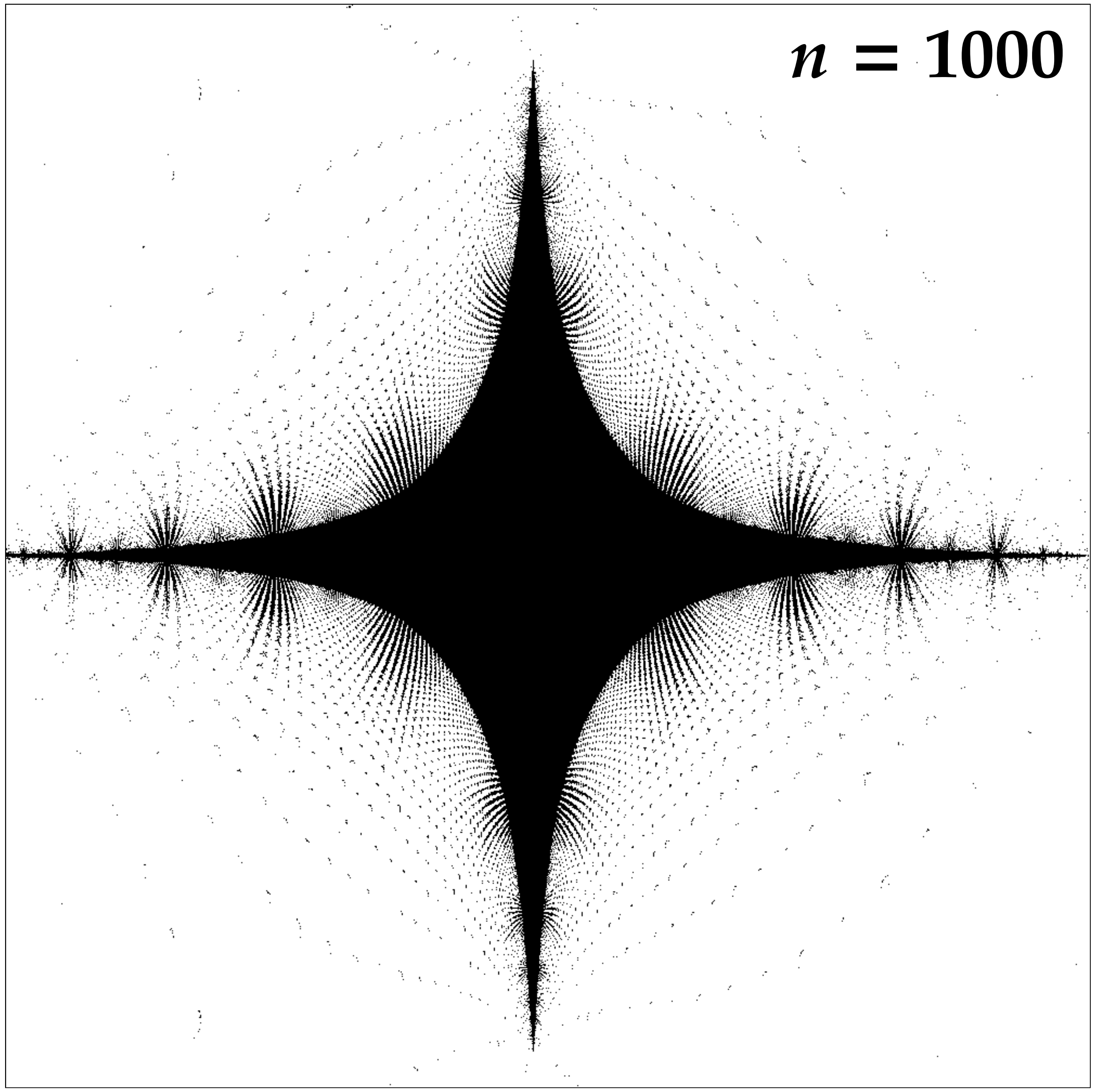
3. If there exist rotation domains with $\omega = \omega' + i\omega''$, which are not necessarily KAM curves
   $\Longrightarrow$ 1-dim

In total, the (Hausdorff) dimension of rotational domains associated with the convergent conjugating function $\varphi(\theta, \omega) = \sum_n a_n(\omega) e^{in\theta}$ is at most $(3 + \alpha)$.

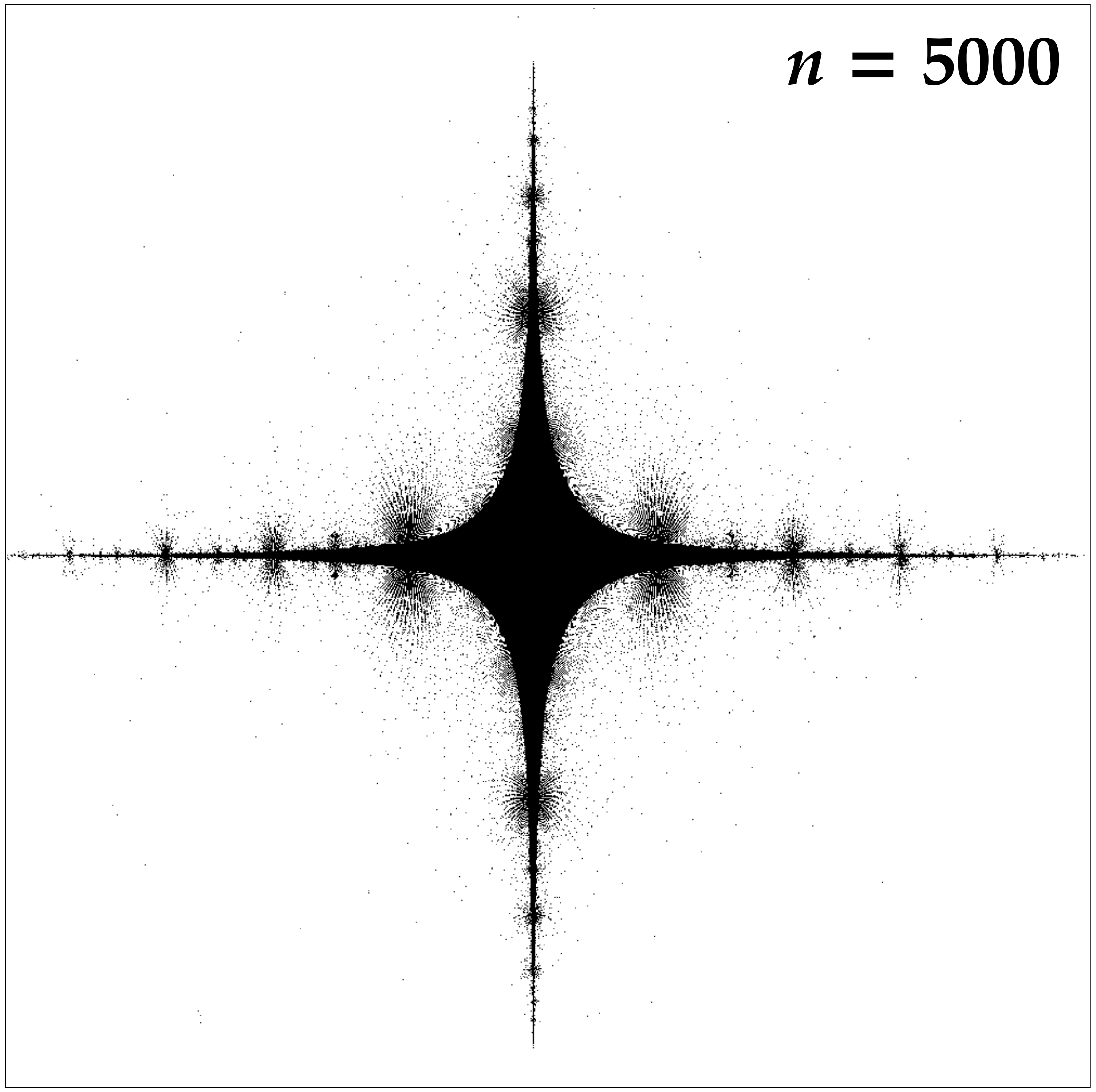
2-dimensional slice of K^+
— Hénon map —



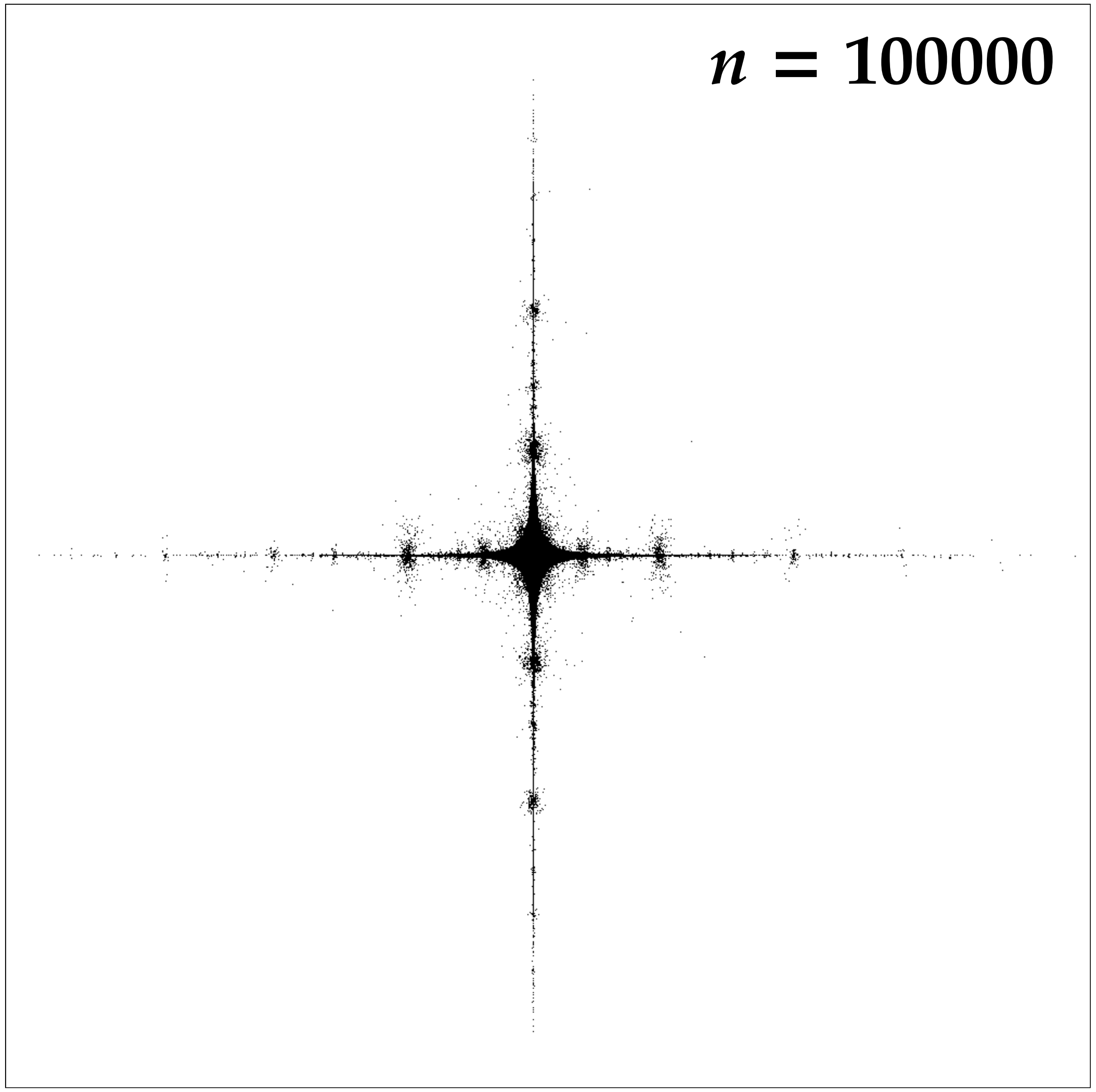
2-dimensional slice of K^+
— Hénon map —



2-dimensional slice of K^+
— Hénon map —



2-dimensional slice of K^+
— Hénon map —



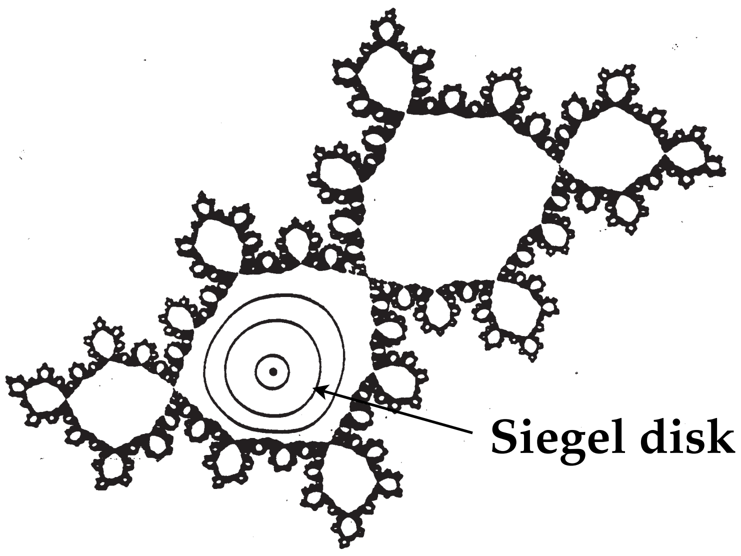
Speculations on

1. Linearization around a fixed point
2. Complexified KAM curves

and

3. Numerical observations

lead us



Vacant interior conjecture

The filled Julia sets of the area-preserving map have no interior points :

$$J^{\pm} = K^{\pm} \quad \text{hence} \quad J = K$$

Fundamental working hypothesis

1. Vacant interior conjecture ($J^{\pm} = K^{\pm}$ and $J = K$)
2. $J^{*} = J$

Note : $J^{*} \subset J$ **for generic cases and** $J^{*} = J$ **for hyperbolic cases.**

“Dynamics” connecting KAM curves

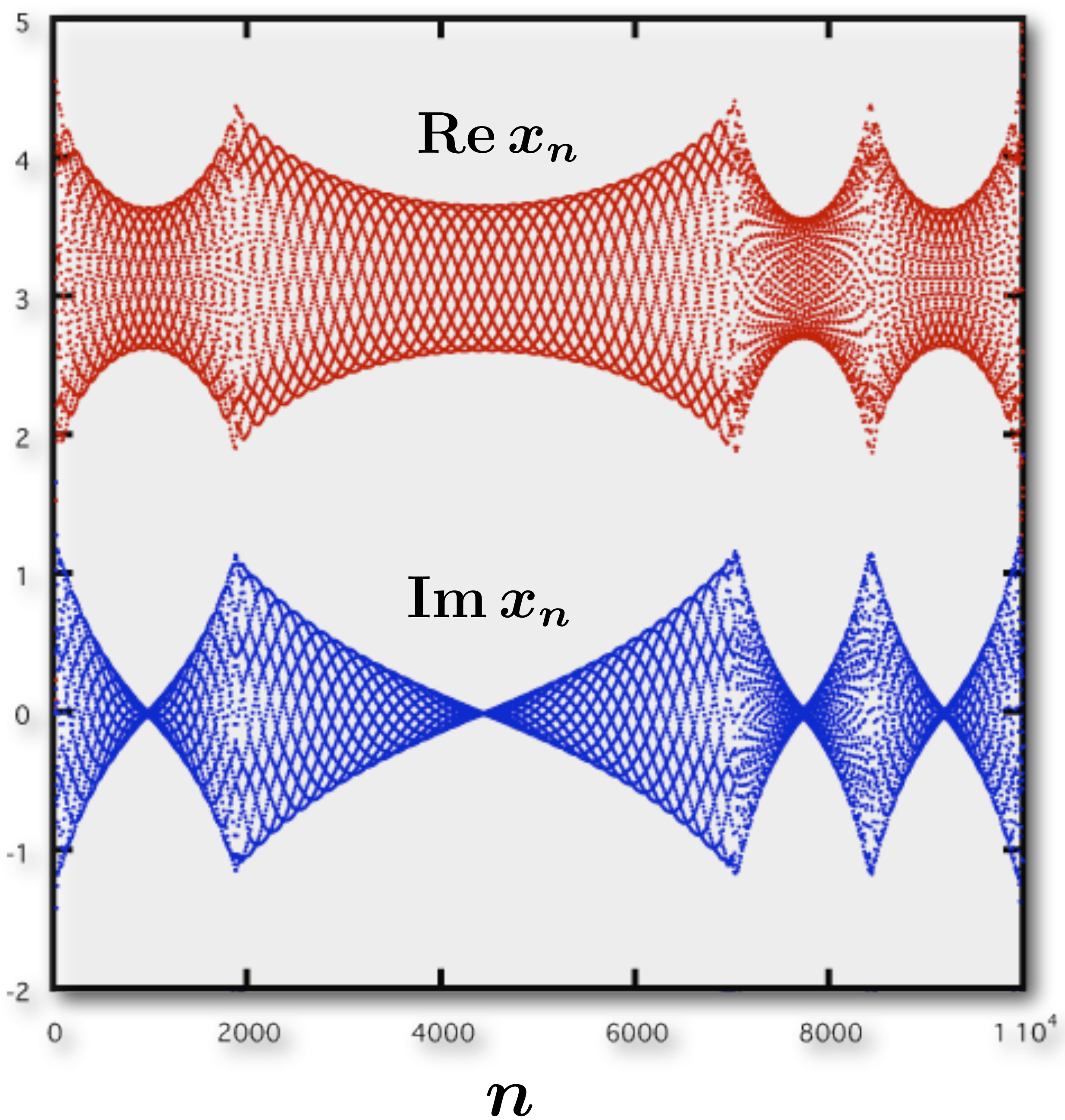
“KAM curves are no more dynamical barriers in $\mathbb{C}^2$”

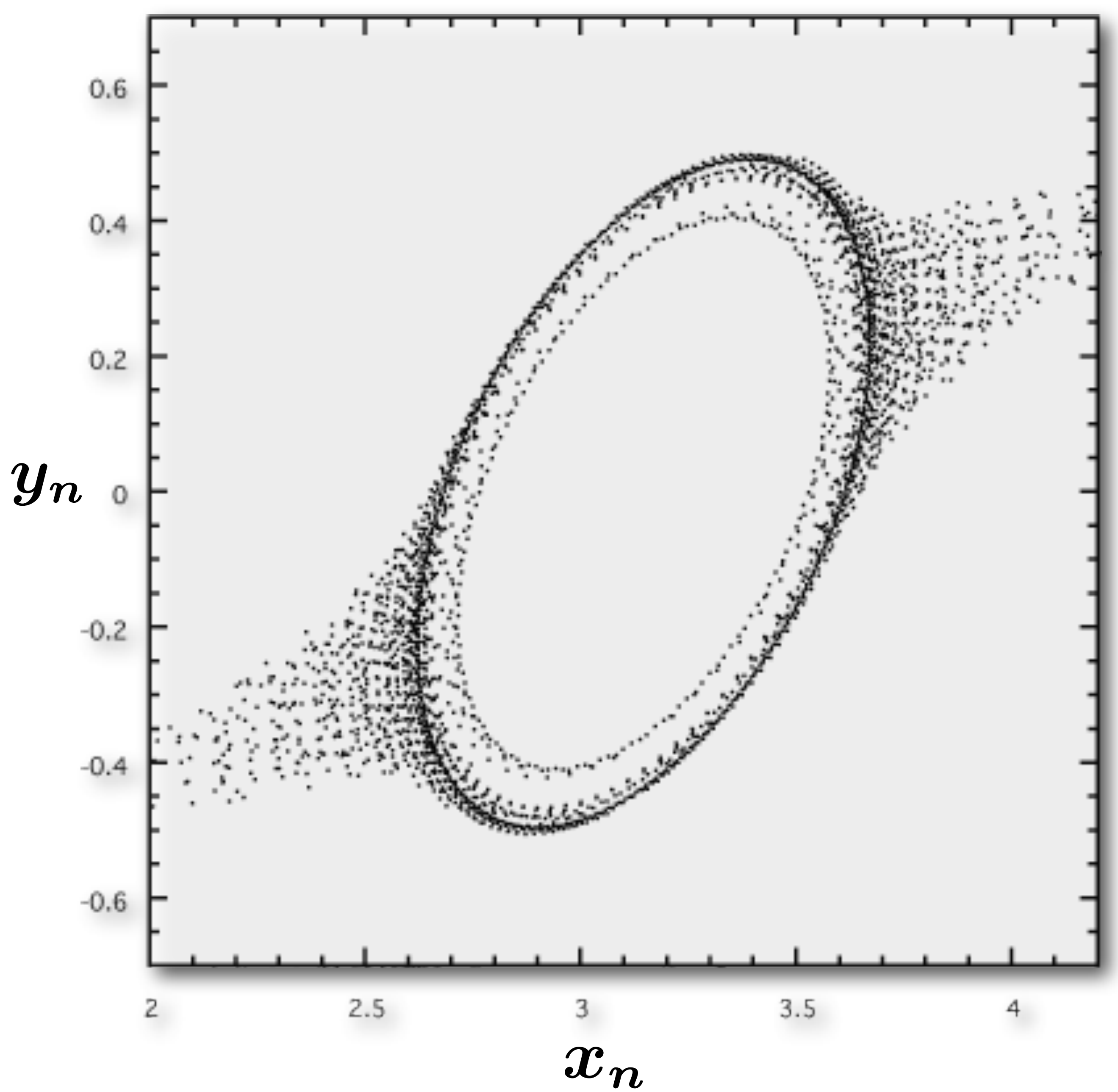
More precisely, for arbitrary neighborhoods $U(z_1)$ and $U(z_2)$ of any two points z_1 and z_2 in {KAM curves (either real or complex)}, there exists n such that $U(z_1) \cap P^n(U(z_2)) \neq \emptyset$.

(Proof)

- $\mathrm{supp}\,\mu = J^*$ ($\Leftarrow$ Bedford-Smille)
- μ is mixing and ergodic ($\Leftarrow$ Bedford-Smille)
- $K = J = J^*$ ($\Leftarrow$ working hypothesis)
- { KAM curves (either real or complex) } $\subset K$

An orbit itinerating among different complex KAM curves





Summary of part II

1. The Hénon map in $\mathbb{R}^2$ has three characteristic parameter regimes: full horseshoe, hyperbolic but not horseshoe, mixed

2. Julia sets and Fatou sets are introduced in $\mathbb{C}^2$ as well as $\mathbb{C}$

3. Techniques using the Green function are explained in analogy with that in $\mathbb{C}$.

4. Convergent theorem (Bedford-Smillie) and some important properties derived from it are shown.

5. On the basis of the vacant interior conjecture together with the assumption $J = J^*$, it is shown that KAM curves do not any more play the role of barriers in the complex plane.