Complex Semiclassical Approach to Chaotic Tunneling

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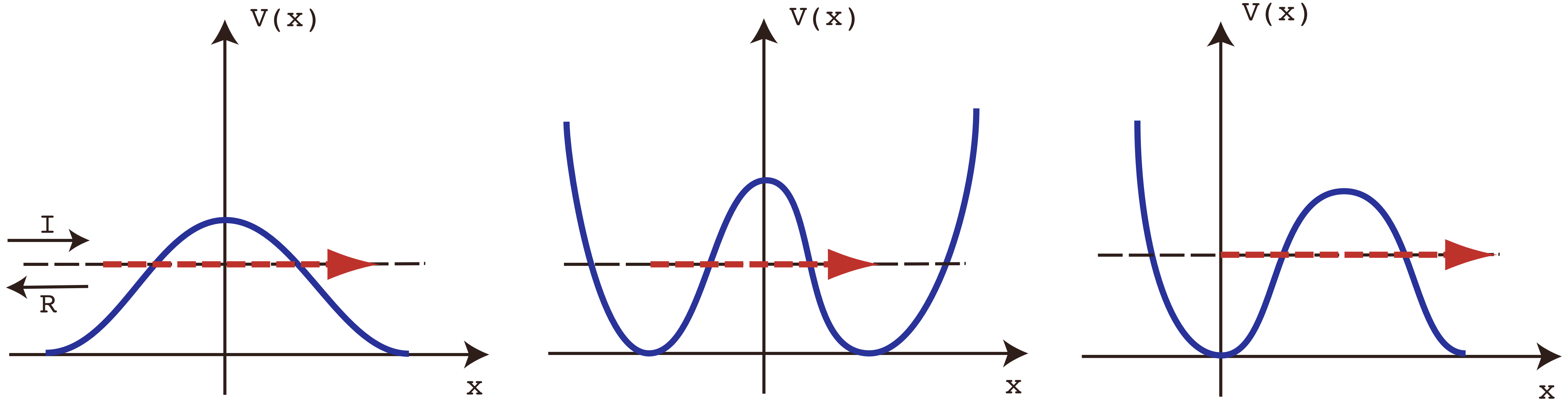
Confinement of classical motions

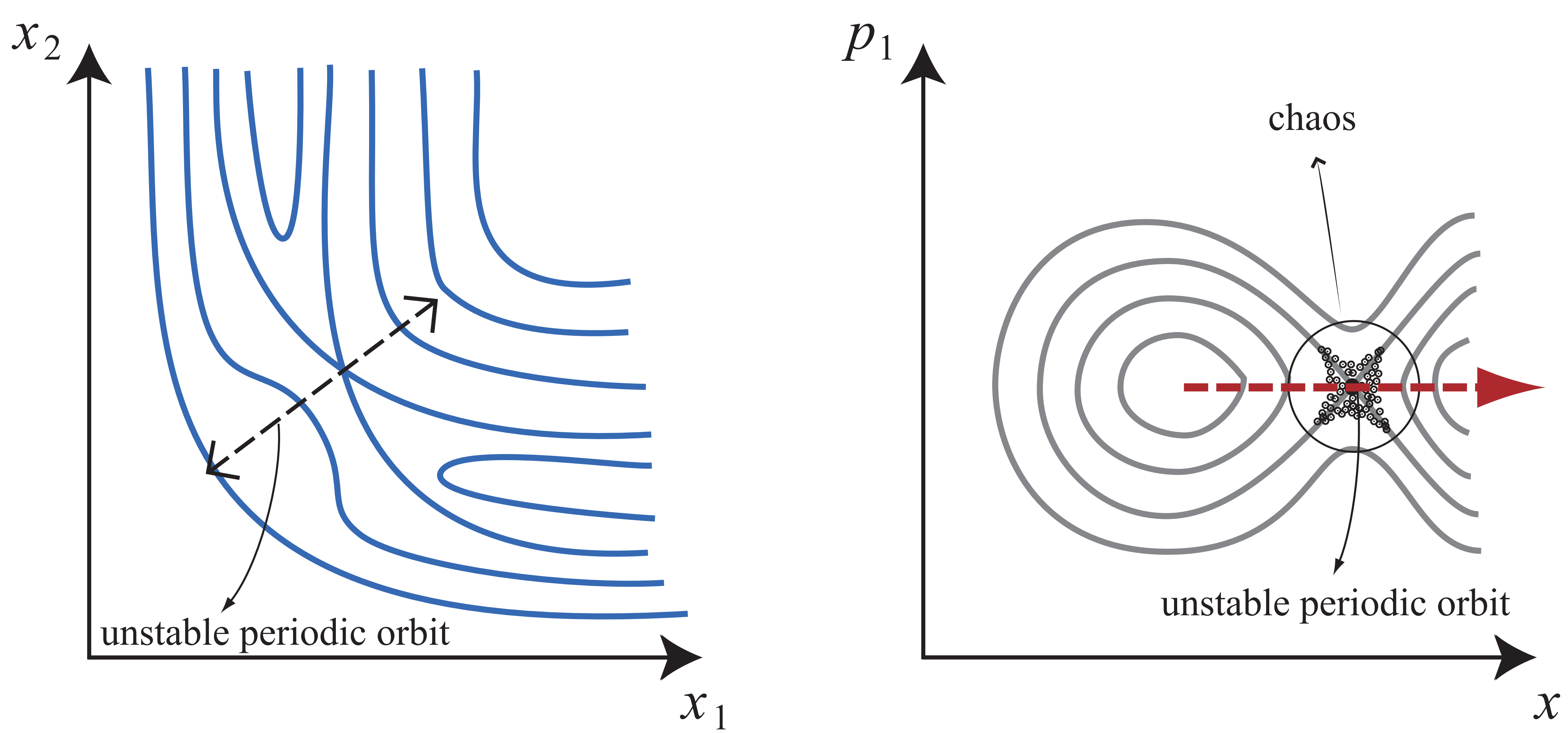
— 1-dimension —



Confinement is due to energy barriers

Confinement of classical motions

— 2-dimensions —



Confinement is not always due to energy barriers *dynamical barriers* also restrict the classical motions

Questions we here ask are

* **essential differences between one- and multi-dimensions ?**
* **dynamically disconnected regions are connected, why and how ?**
* **evaluate or even define the tunneling probability in multi-dimensional systems, is it possible ?**

Plan of Lectures

1. **Time domain semiclassical approach to dynamical tunneling**

2. **Complex dynamics in one variable**

3. **Complex dynamics in two variables**

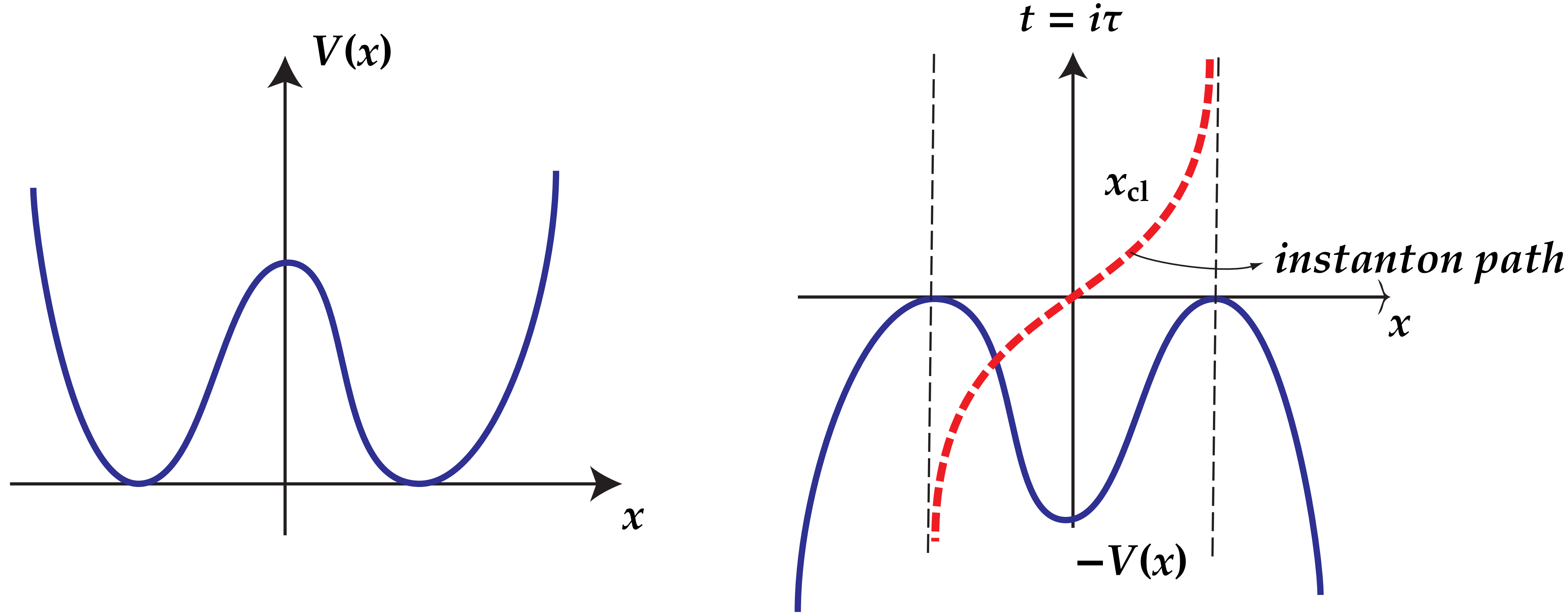
4. **How to apply general theory of complex dynamics to tunneling problems**

1. Time domain semiclassical approach to dynamical tunneling

We here take *complex semiclassical approach*

* **a natural extension of semiclassical analyses in the real domain**
* **it enables us to clarify what dynamical mechanism works**
* **to develop the theory in the complex domain follows original spirits of semiclassical methods**

Classical paths running in the complex space connect classically inaccessible regions.



Discretized dynamics — Classical and Quantum —

Classical dynamics :

$$F : \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} p - V'(q) \\ q + T'(p') \end{pmatrix}$$

where $T(p)$, $V(q)$ are kinetic and potential functions.

Quantum dynamics :

$$K_n(a, b) = \langle b|\hat{U}^n|a\rangle = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_j dq_j \prod_j dp_j \exp\left[\frac{i}{\hbar} S(\{q_j\}, \{p_j\})\right]$$

where the action $S(\{q_j\}, \{p_j\})$ is determined such that the variational condition

$$\delta S(\{q_j\}, \{p_j\}) = 0$$

yields the classical map F.

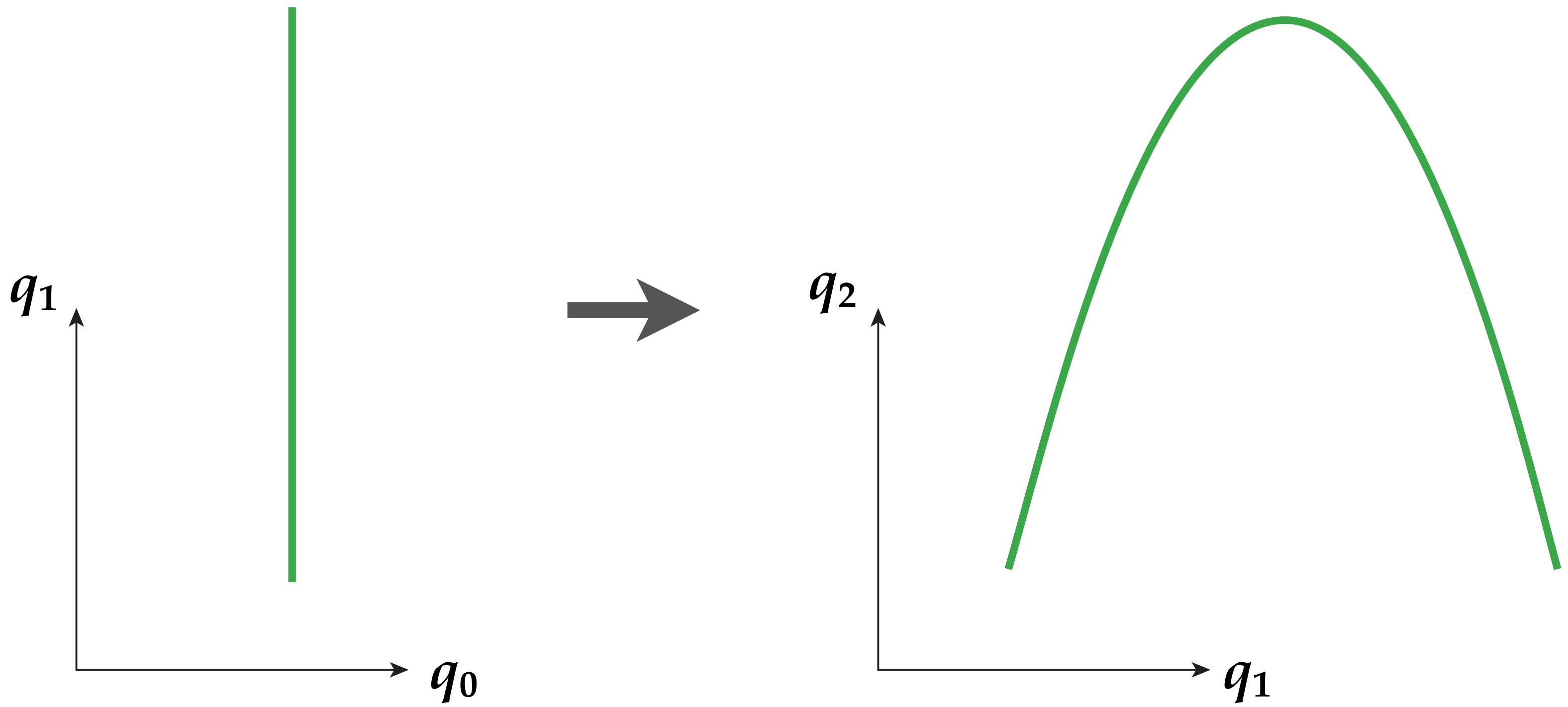
What is "dynamical tunneling" ?

Example 1   Quadratic map

$$F : \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} p_{n-1} - V'(q_{n-1}) \\ q_{n-1} + p_n \end{pmatrix}$$

where $\quad V(q) = \dfrac{1}{3}q^3 + cq$

For simplicity, instead of (q_n, p_n), we use (q_n, q_{n+1}) as phase space variables.



Quantum propagator for the 1-step quadratic map :

$$K(q_2, q_0) = \langle q_2 | U | q_0 \rangle = \int_{-\infty}^{\infty} dq_1 \exp\Big[\frac{i}{\hbar} S(q_0, q_1, q_2)\Big]$$

where

$$S(q_0, q_1, q_2) = \sum_{j=1}^{2} \frac{1}{2}(q_j - q_{j-1})^2 - V(q_1)$$

We here evaluate the propagator $K(q_2, q_0)$ using the *saddle point approximation.* The saddle point condition is given as

$$\frac{\partial S(q_0, q_1, q_2)}{\partial q_1} = 0$$

This yields the equation of motion in the Lagrangian form

$$(q_2 - q_1) - (q_1 - q_0) = q_1^2 + c$$

Of course, this is equivalent to the map F.

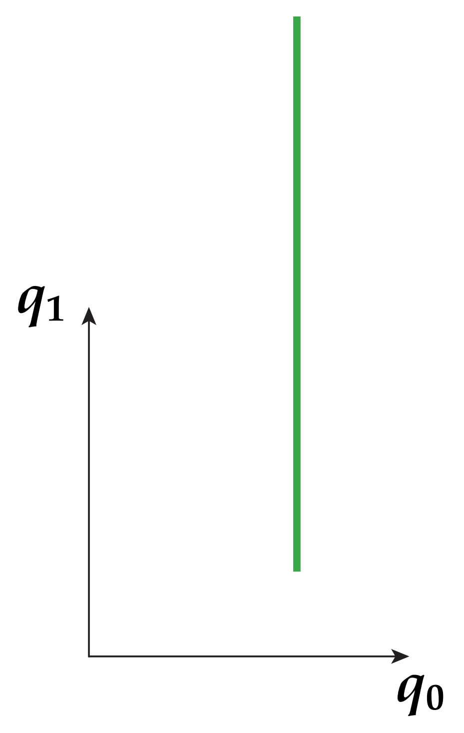
For given $q_0,\ q_2 \in \mathbb{R}$, the saddle point equation

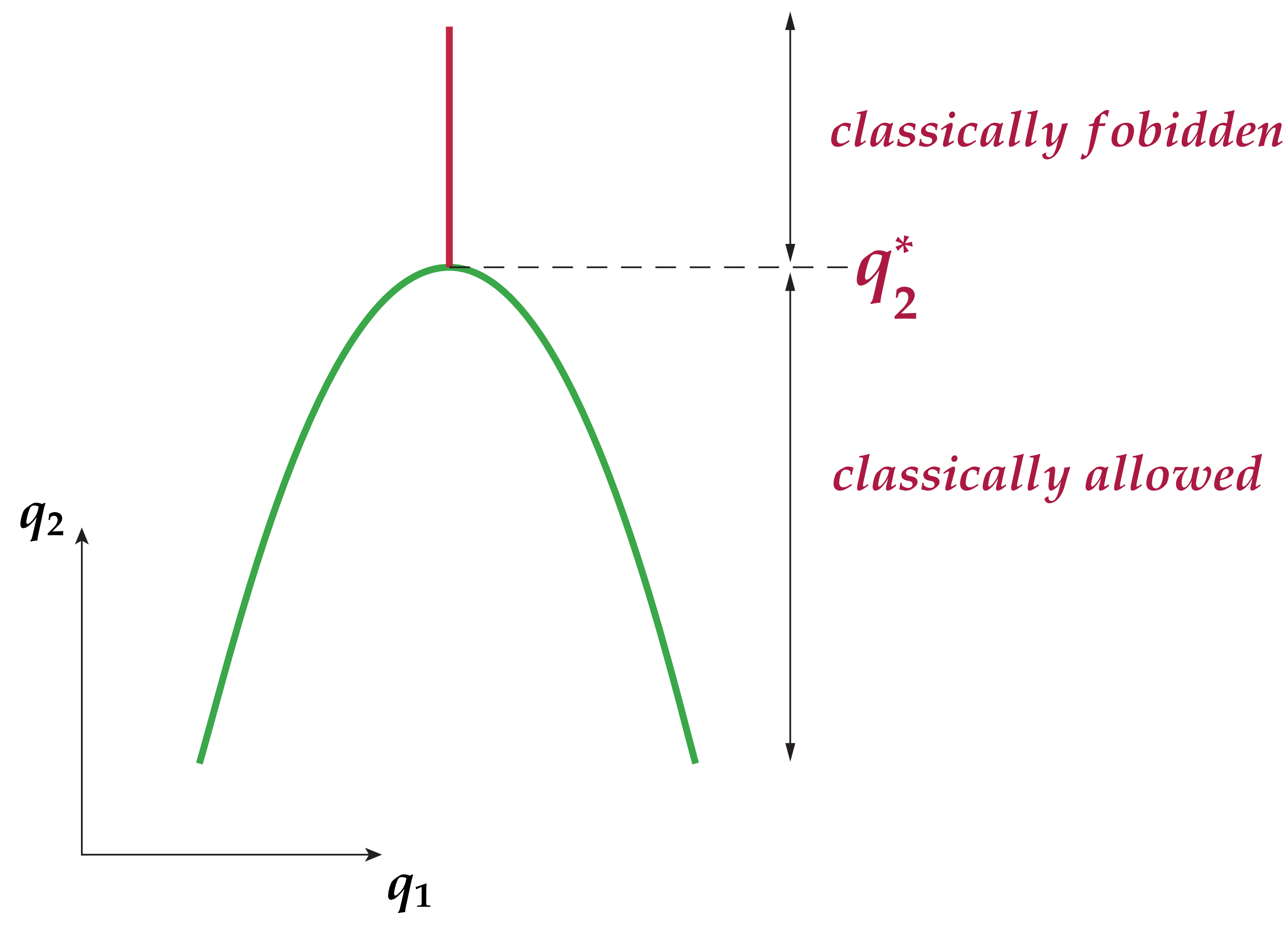
$$(q_2 - q_1) - (q_1 - q_0) = q_1^2 + c$$

has

$$\begin{cases} \text{2 real solutions} & \text{if } q_2 < q_2^* \quad \textit{classically allowed} \\ \text{2 complex (conjugate) solutions} & \text{if } q_2 > q_2^* \quad \textit{classically fobidden} \end{cases}$$

For $q_2 > q_2^*$ the initial and final coordinates $q_0, q_2 \in \mathbb{R}$, but the intermediate $q_1 \in \mathbb{C}$.





Quantum propagator for the 2-step quadratic map :

$$K(q_3, q_0) = \langle q_3|U|q_0\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dq_1 dq_2 \exp\Big[\frac{i}{\hbar} S(q_0, q_1, q_2, q_3)\Big]$$

where

$$S(q_0, q_1, q_2, q_3) = \sum_{j=1}^{3} \frac{1}{2}(q_j - q_{j-1})^2 - \sum_{j=1}^{2} V(q_j)$$

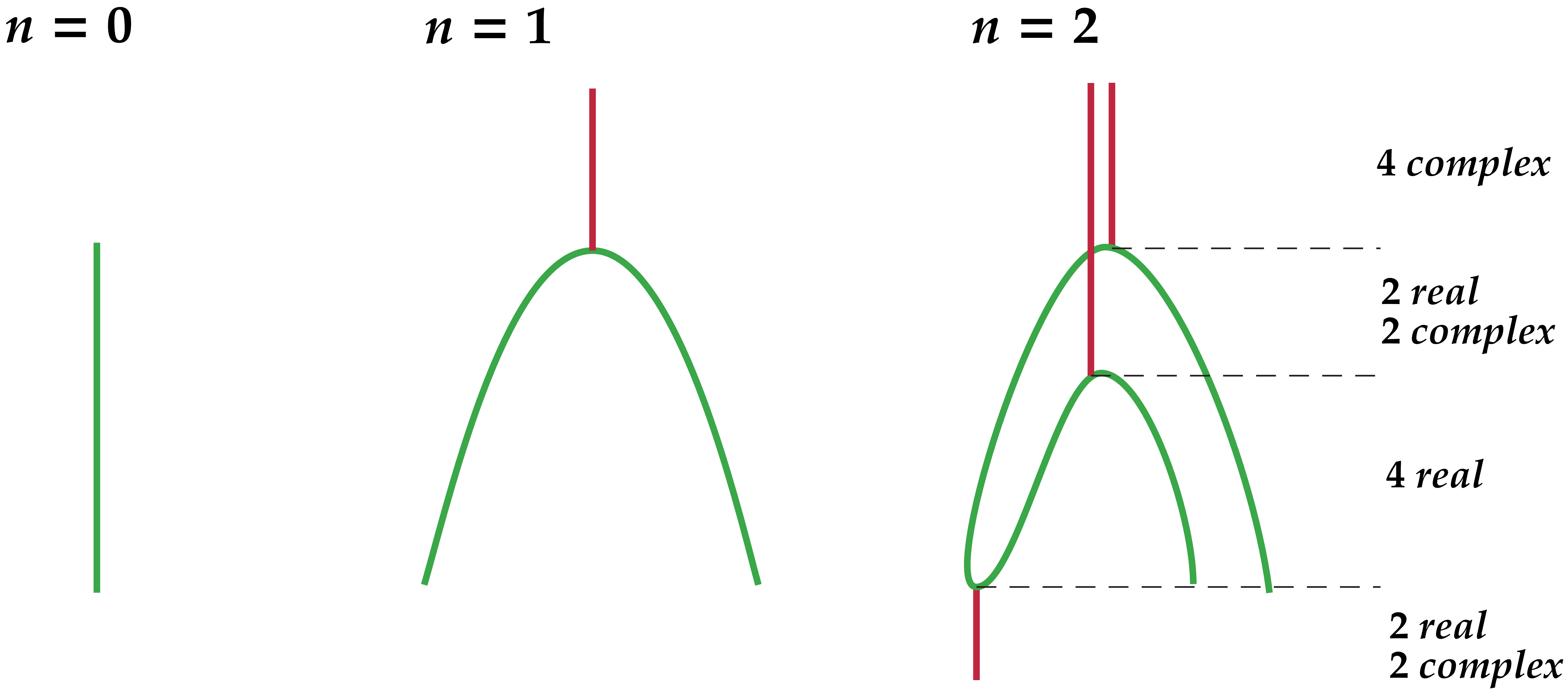
The saddle point condition is given as

$$\frac{\partial S(q_0, q_1, q_2, q_3)}{\partial q_1} = 0\,, \qquad \frac{\partial S(q_0, q_1, q_2, q_3)}{\partial q_2} = 0$$

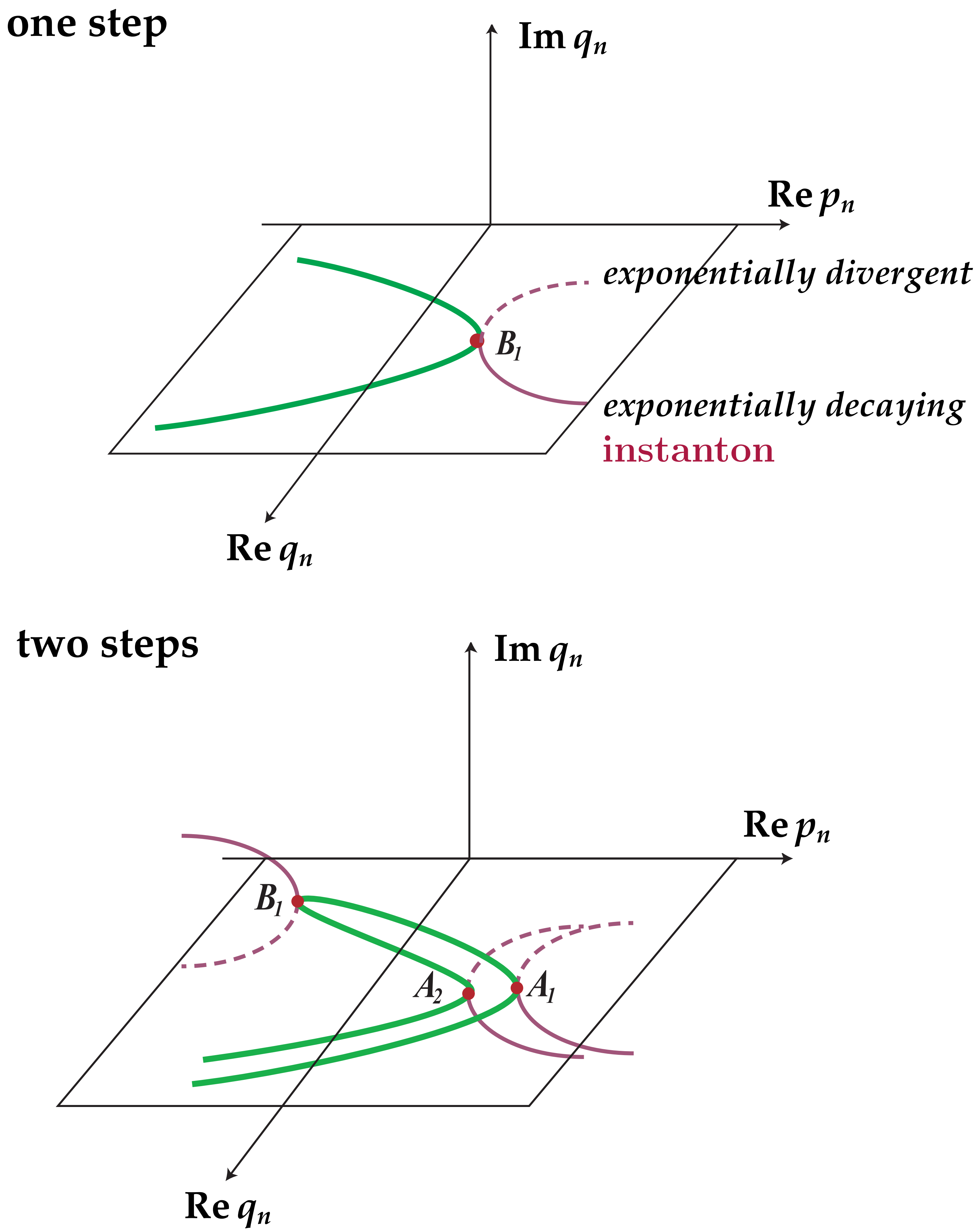
This yields the equations of motion

$$(q_2 - q_1) - (q_1 - q_0) = q_1^2 + c, \qquad (q_3 - q_2) - (q_2 - q_1) = q_2^2 + c$$

For given $q_0,\ q_3 \in \mathbb{R}$, the saddle point equation has 4 solutions.



$n = 0$
$n = 1$
$n = 2$
4 complex
2 real
2 complex
4 real
2 real
2 complex

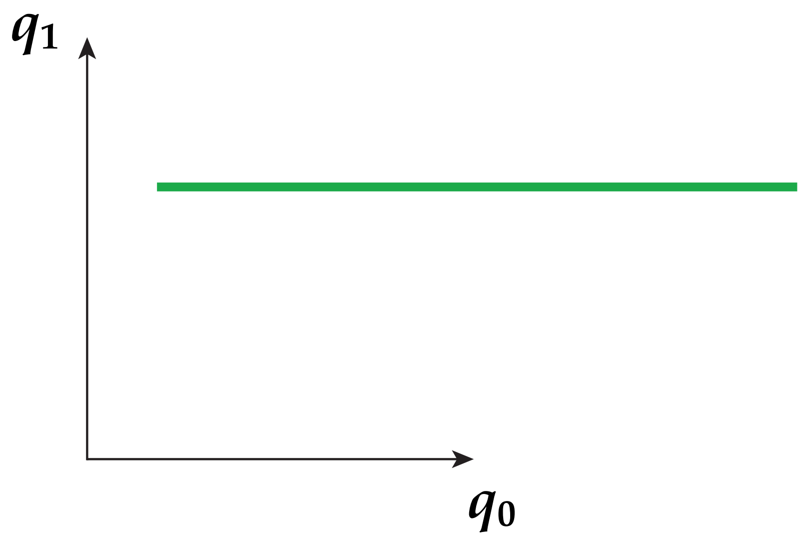


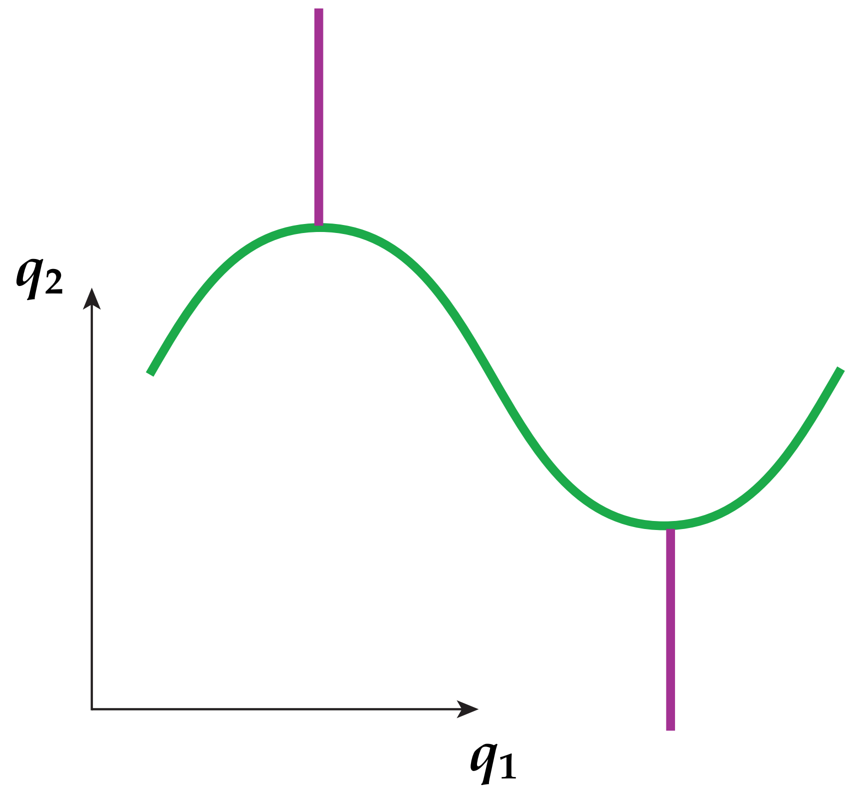
one step
Im q_n
Re p_n
exponentially divergent
B_1
exponentially decaying
instanton
Re q_n
two steps
Im q_n
Re p_n
B_1
A_2
A_1
Re q_n

Example 2 Standard map

$$F : \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} p_{n-1} - V'(q_{n-1}) \\ q_{n-1} + p_n \end{pmatrix}$$

where $V(q) = K \cos q$





Quantum propagator for the 1-step standard map :

$$K(q_2, q_0) = \langle q_2 | U | q_0 \rangle = \int_{-\infty}^{\infty} dq_1 \exp\Big[\frac{i}{\hbar} S(q_0, q_1, q_2)\Big]$$

where

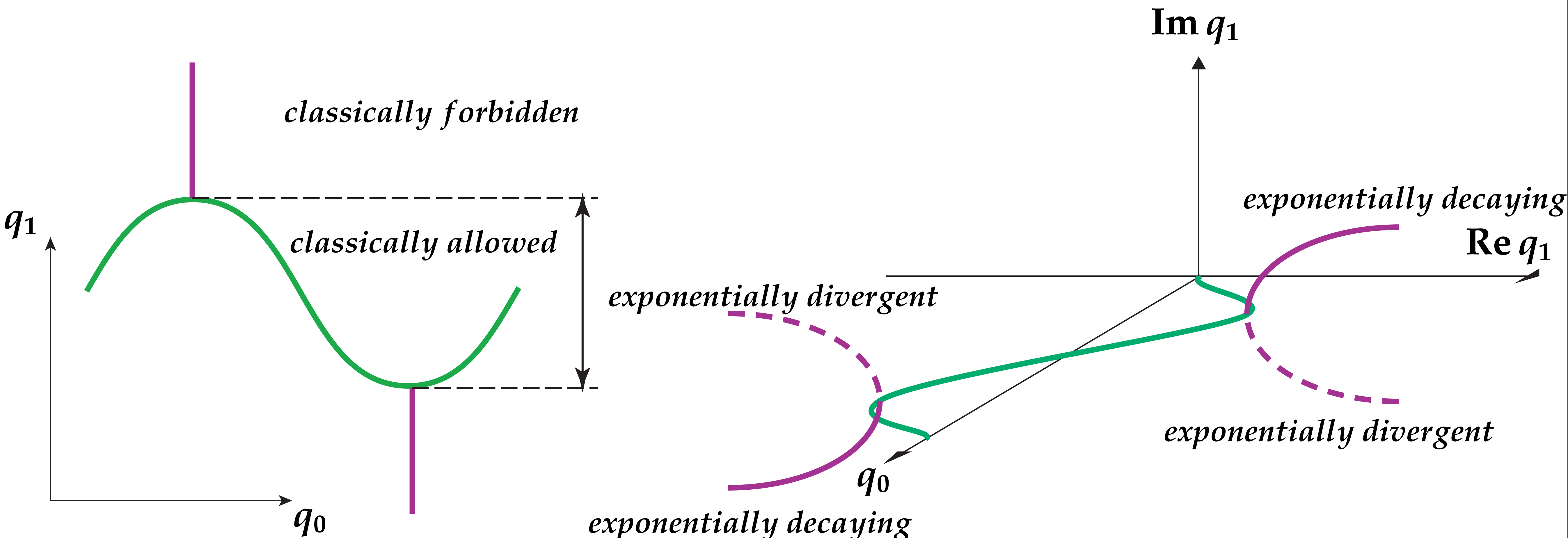
$$S(q_0, q_1, q_2) = \sum_{j=1}^{2} \frac{1}{2}(q_j - q_{j-1})^2 - K \cos q_1$$

The saddle point condition yields,

$$(q_2 - q_1) - (q_1 - q_0) = K \sin q_1$$

For given $q_0,\ q_2 \in \mathbb{R}$, the saddle point equation has the solutions

$$\begin{cases} \text{2 real solutions} & \text{if } |q_2| < q_2^* \\ \text{2 complex (conjugate) solutions} & \text{if } |q_2| > q_2^* \end{cases}$$



classically forbidden
q_1
classically allowed
q_0
Im q_1
exponentially decaying
Re q_1
exponentially divergent
exponentially divergent
q_0
exponentially decaying

Quantum propagator for the 2-step standard map :

$$K(q_3, q_0) = \langle q_3|U|q_0\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dq_1 dq_2 \exp\Big[\frac{i}{\hbar}S(q_0, q_1, q_2, q_3)\Big]$$

where

$$S(q_0, q_1, q_2, q_3) = \sum_{j=1}^{3} \frac{1}{2}(q_j - q_{j-1})^2 - \sum_{j=1}^{2} V(q_j)$$

The saddle point condition yields,

$$(q_2 - q_1) - (q_1 - q_0) = K \sin q_1, \qquad (q_3 - q_2) - (q_2 - q_1) = K \sin q_2$$

This gives

$$q_3 - 2q_0 = q_1 + 2(2q_1 + K \sin q_1) + K \sin(q_0 + 2q_1 + K \sin q_1)$$

For given $q_0,\ q_3 \in \mathbb{R}$, this has infinitely many solutions.

Note : In the case of real semiclassics, the number of stationary phase solutions is always finite within a finite time step n.

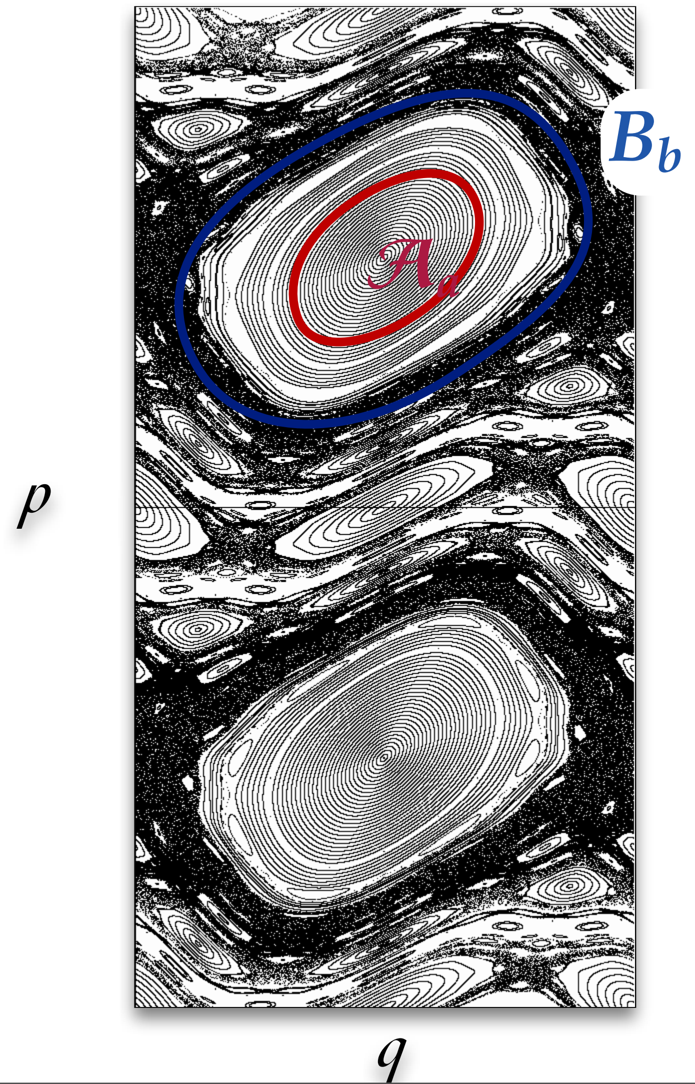
Dynamical tunneling in mixed phase space

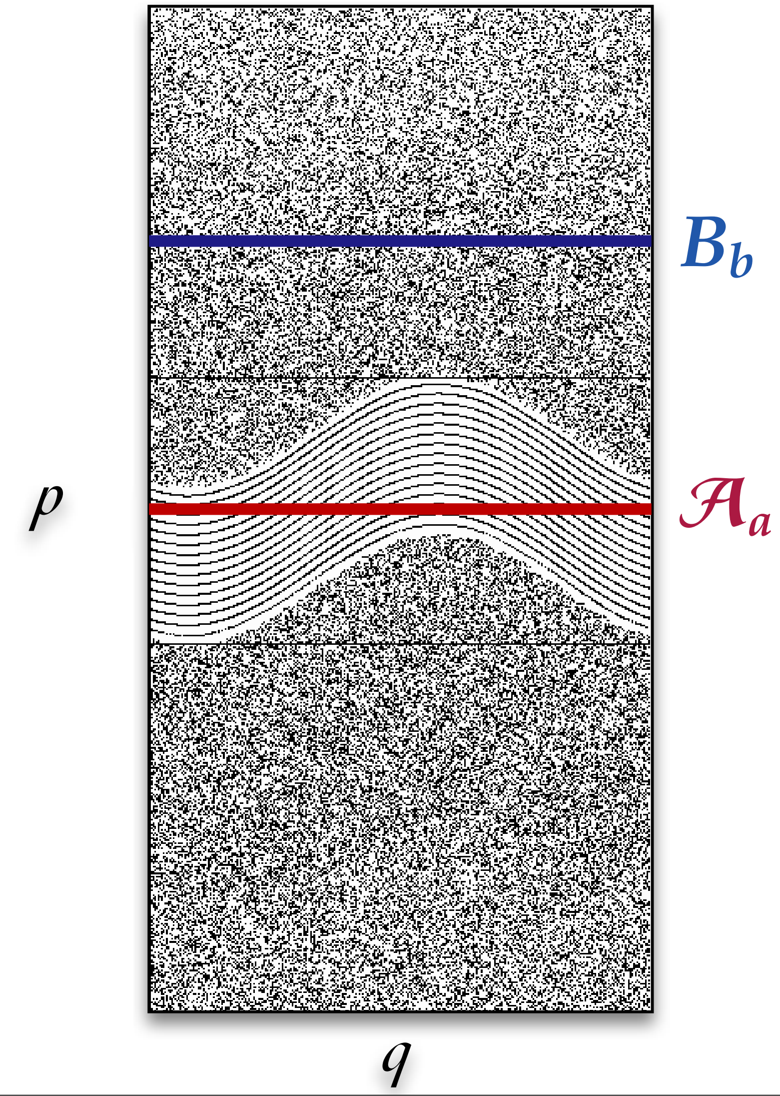
Classical dynamics

$$F : \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} p - V'(q) \\ q + p' \end{pmatrix}$$

Forbidden process in classical dynamics

$\mathcal{A}_a \cap F^{-n}(B_b) = \emptyset$ for $\forall n$, if $\mathcal{A}_a, B_b (\in \mathbb{R})$ are dynamically separated.





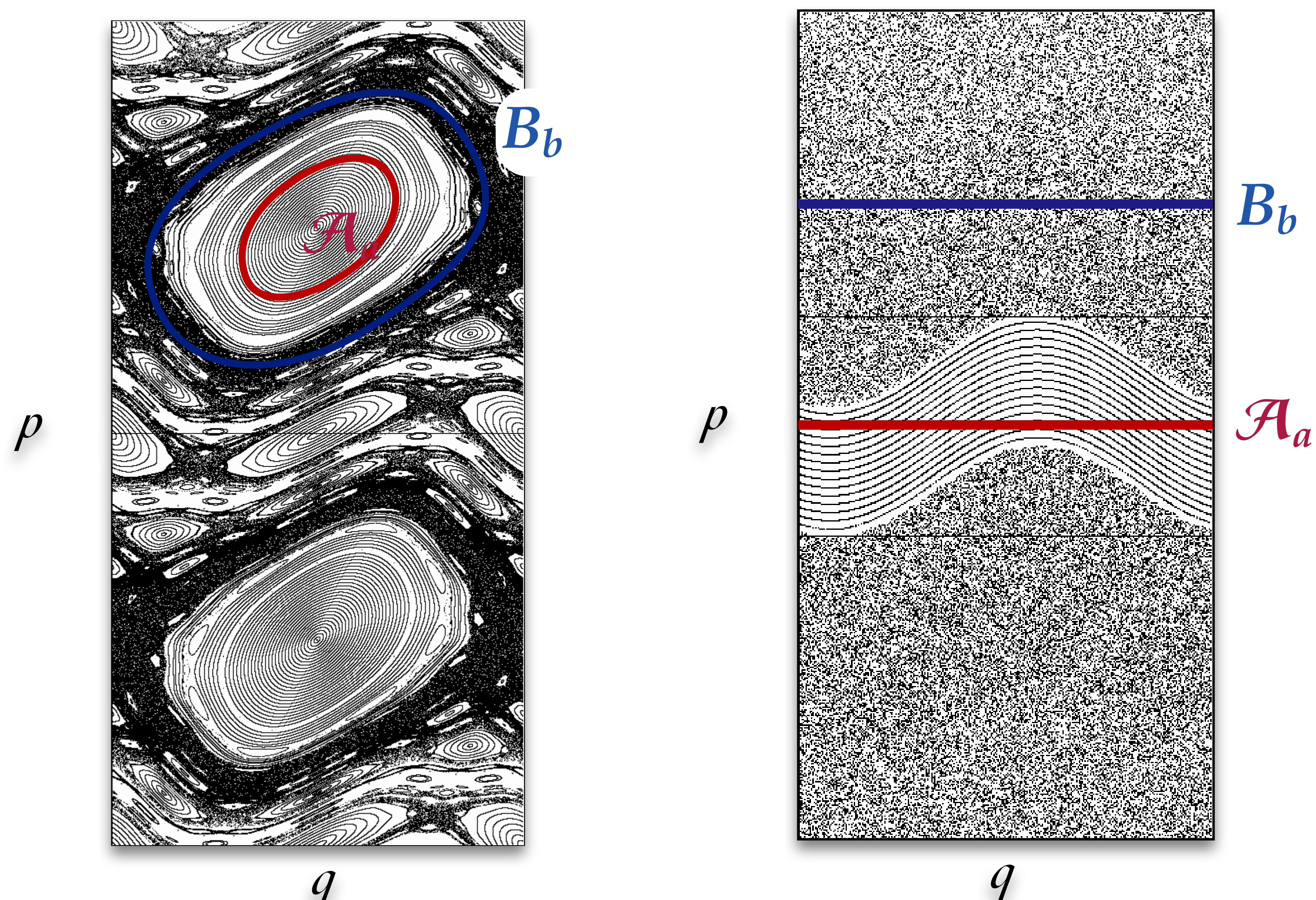
Dynamical tunneling in mixed phase space

Quantum dynamics

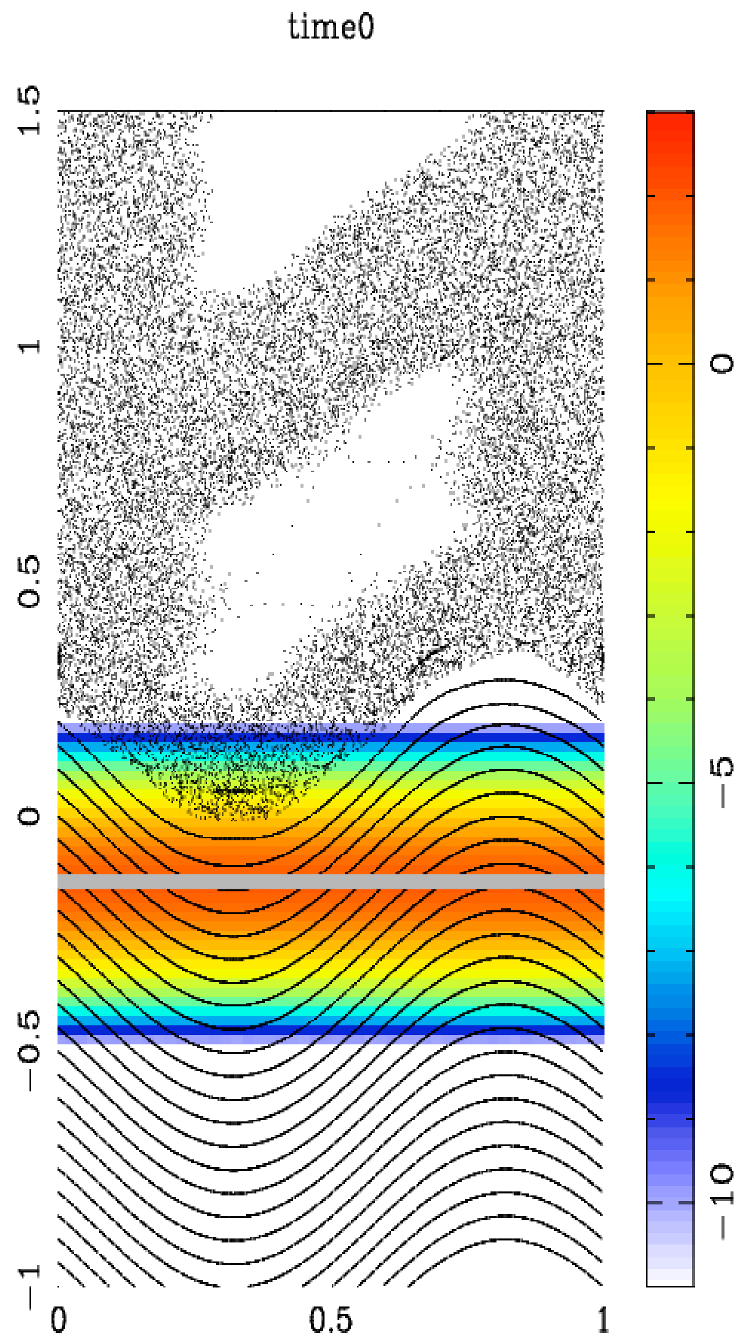
$$K(a,b) = \langle b|\hat{U}^n|a\rangle = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_j dq_j \prod_j dp_j \exp\left[\frac{i}{\hbar}S(\{q_j\},\{p_j\})\right]$$

Tunneling process in quantum dynamics

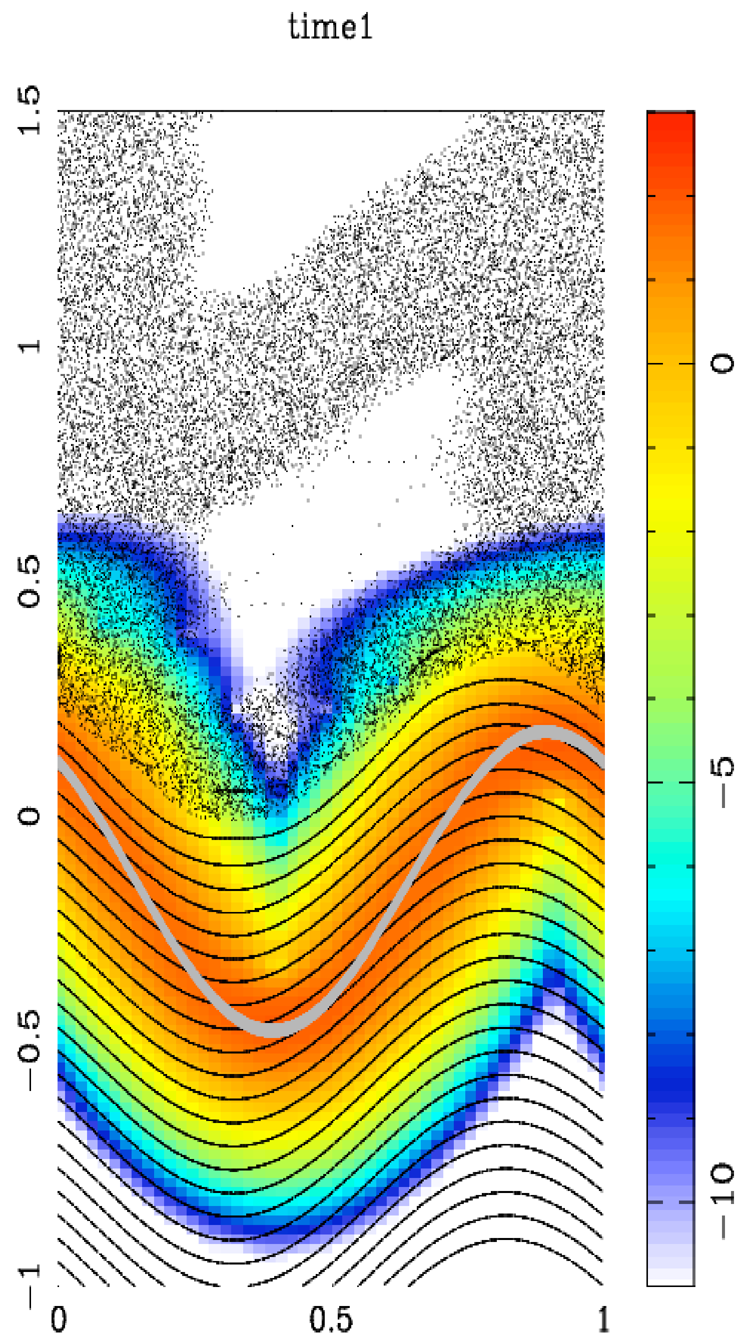
$K(a,b) \neq 0$ even if $\mathcal{A}_a, B_b (\in \mathbb{R})$ are dynamically separated.



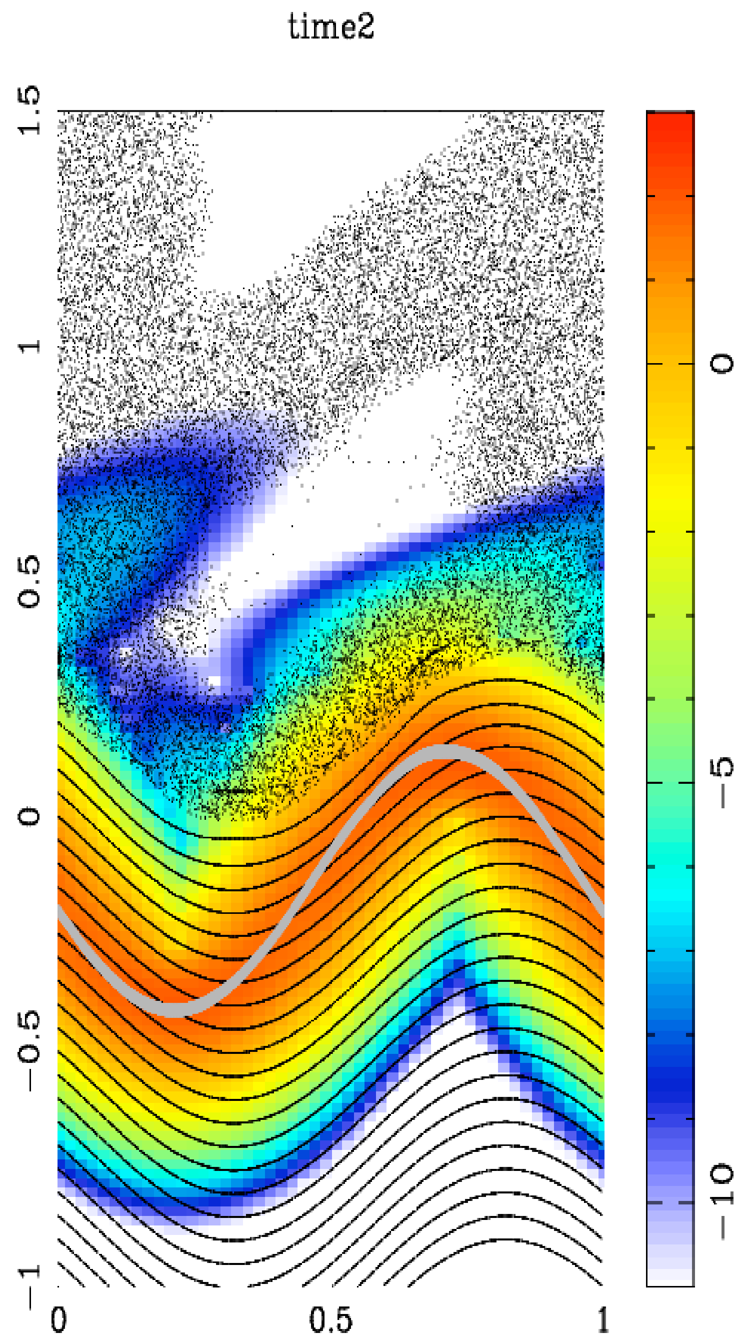
Dynamical tunneling in mixed phase space



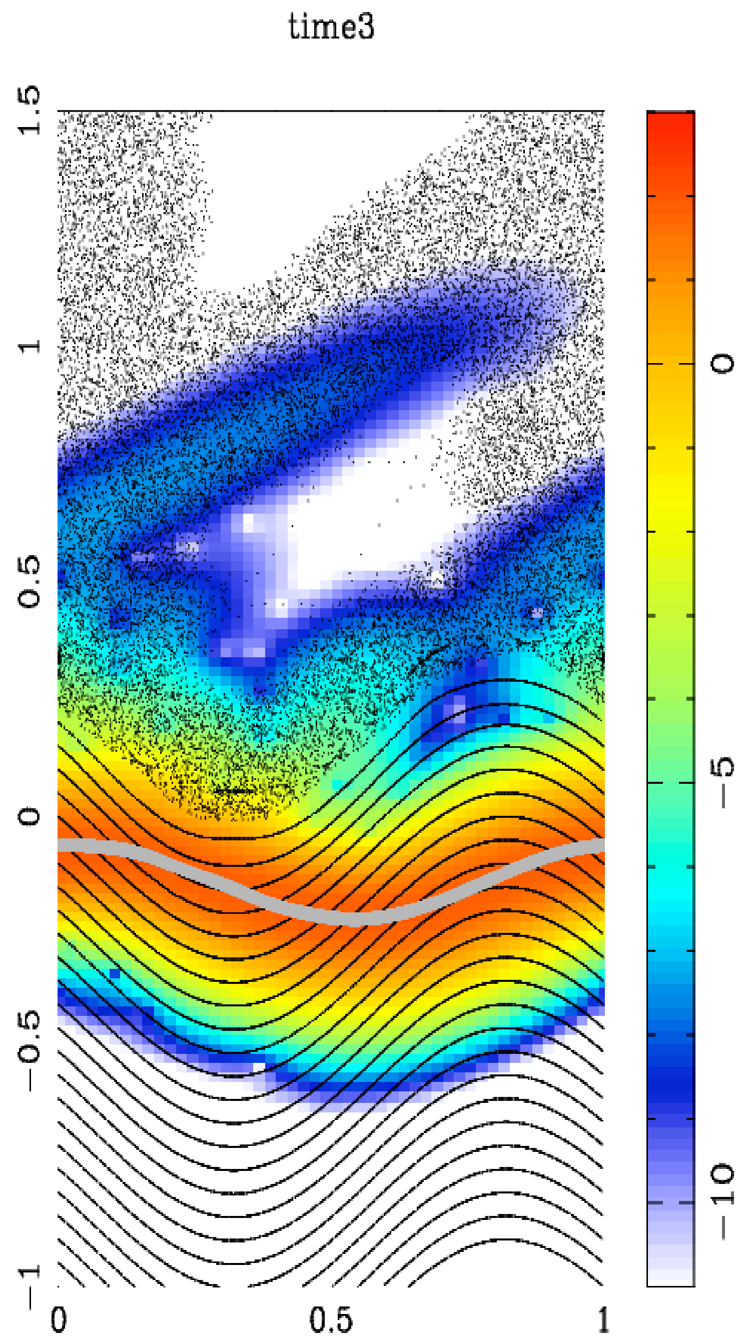
Dynamical tunneling in mixed phase space



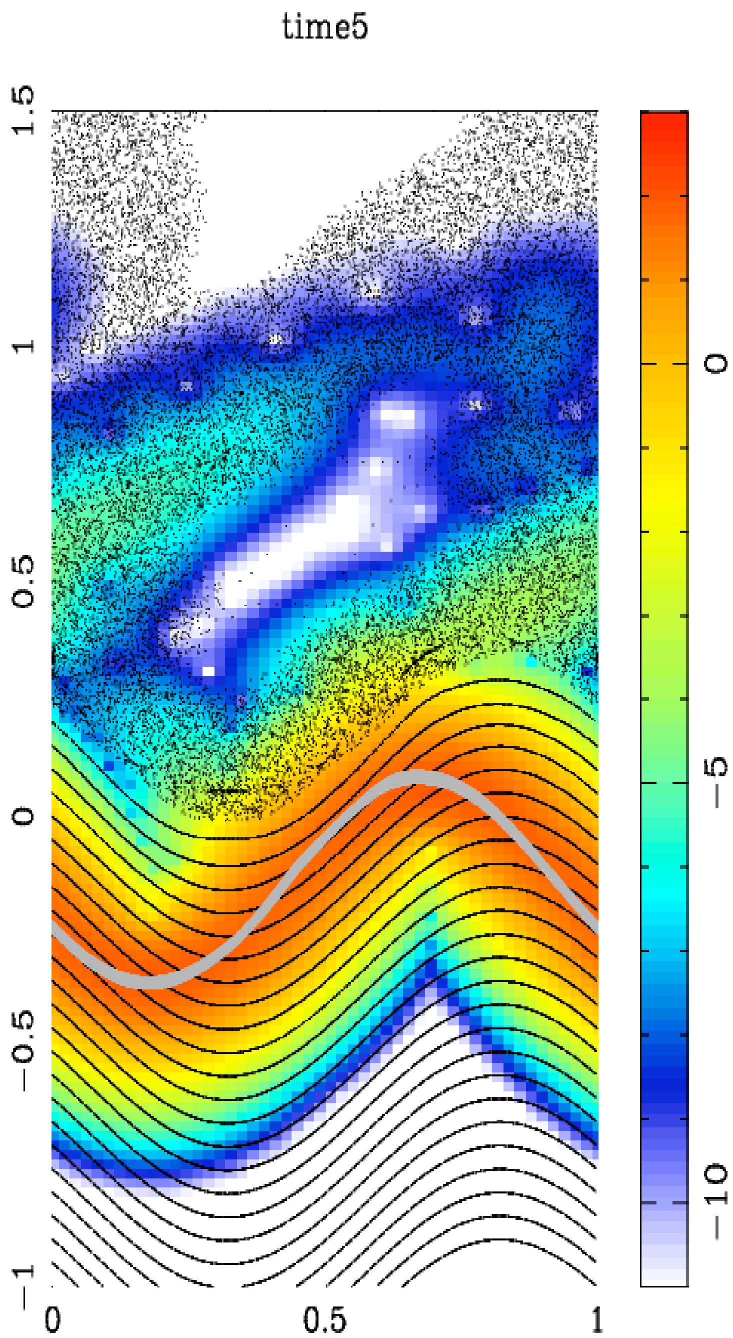
Dynamical tunneling in mixed phase space



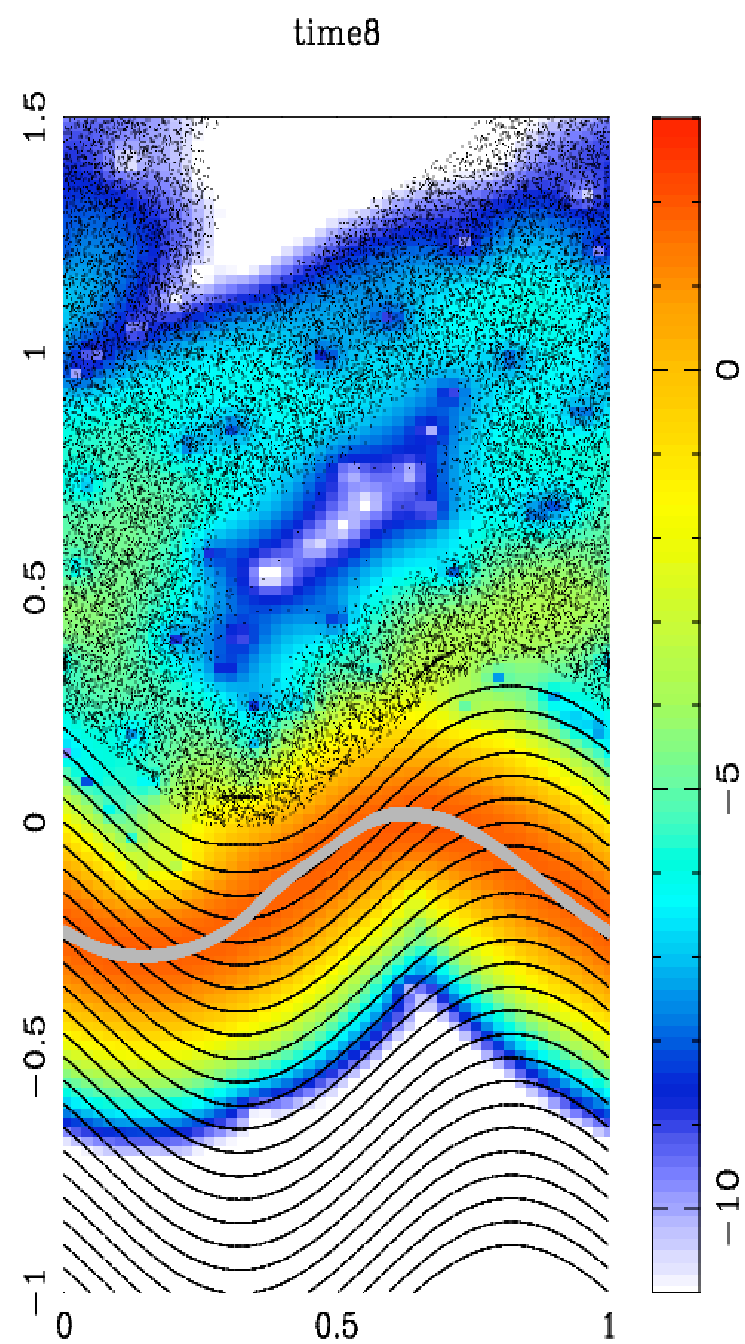
Dynamical tunneling in mixed phase space



Dynamical tunneling in mixed phase space



Dynamical tunneling in mixed phase space



Complex Semiclassical approach to dynamical tunneling

Quantum propagator

$$K(a,b) = \langle b|\hat{U}^n|a\rangle = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_j dq_j \prod_j dp_j \exp\left[\frac{i}{\hbar}S(\{q_j\},\{p_j\})\right]$$

$|a\rangle$: initial state $\quad |b\rangle$: final state

Semiclassical propagator (⇐ saddle point evaluation of $K(a,b)$)

$$K^{sc}(a,b) = \sum_\gamma A_n^{(\gamma)}(a,b)\exp\left\{\frac{\mathrm{i}}{\hbar}S_n^{(\gamma)}(a,b)\right\}$$

$\mathcal{A}_a = \{\,(q,p)\in\mathbb{C}^2 \mid A(q,p) = a\,\}$: initial manifold

$\mathcal{B}_b = \{\,(q,p)\in\mathbb{C}^2 \mid B(q,p) = b\,\}$: final manifold

Example　The quantum propagator in *p*-representation

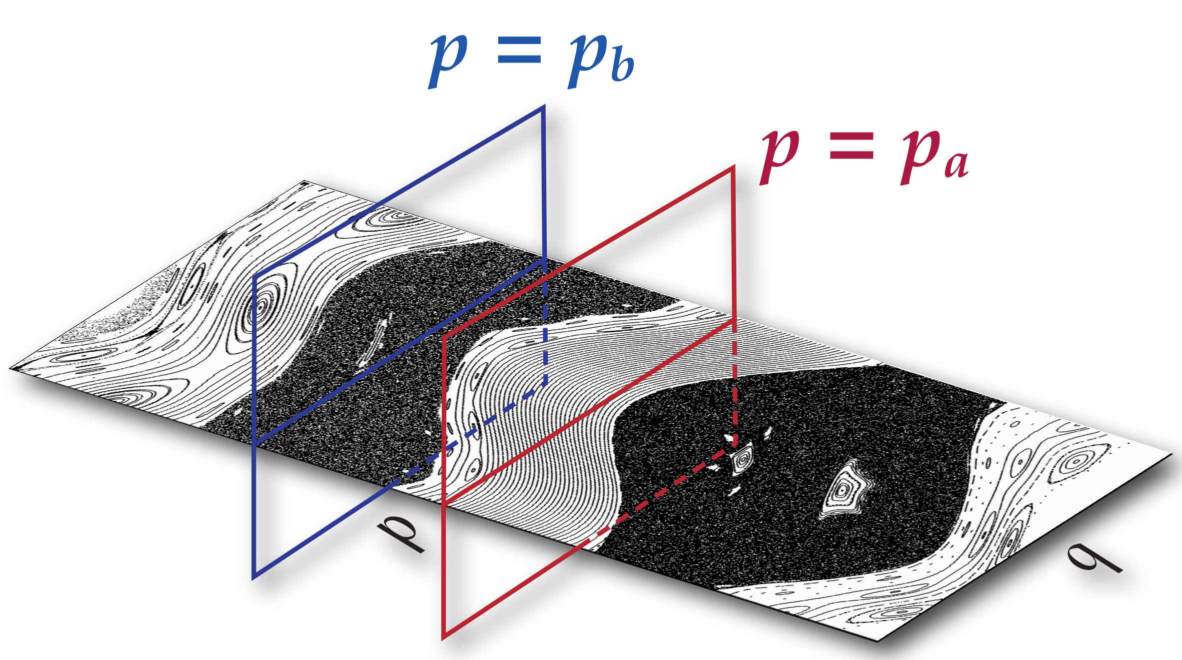
$$K(p_a, p_b) = \langle p_b|\hat{U}^n|p_a\rangle = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_j dq_j \prod_j dp_j \exp\left[\frac{i}{\hbar}S(\{q_j\},\{p_j\})\right]$$

$\mathcal{A}_a = \{ (q,p) \in \mathbb{C}^2 \mid p = p_a \in \mathbb{R}\}$: initial manifold

$\mathcal{B}_b = \{ (q,p) \in \mathbb{C}^2 \mid p = p_b \in \mathbb{R}\}$: final manifold

$\Rightarrow$ $\mathcal{A}_a$ and $\mathcal{B}_b$ are both 1-dimesional complex lines in $\mathbb{C}^2$.

Note : This holds in arbitrary representations, for example in the *coherent state representation* .



Problems When $\mathcal{A}_a$ and $\mathcal{B}_b$ are dynamically separated ,

1. how are dynamically disconnected regions $\mathcal{A}_a$ and $\mathcal{B}_b$ in $\mathbb{R}^2$ are connected under the dynamics in $\mathbb{C}^2$?

2. is it possible to relate the dynamics from $\mathcal{A}_a$ to $\mathcal{B}_b$ to some invariant structures in $\mathbb{C}^2$?

3. how to evaluate the tunneling probability from $\mathcal{A}_a$ to $\mathcal{B}_b$?

4. does some specific relevant orbit(s) (like the instanton) exclusively control the transition from $\mathcal{A}_a$ to $\mathcal{B}_b$, or are there any other principles ?

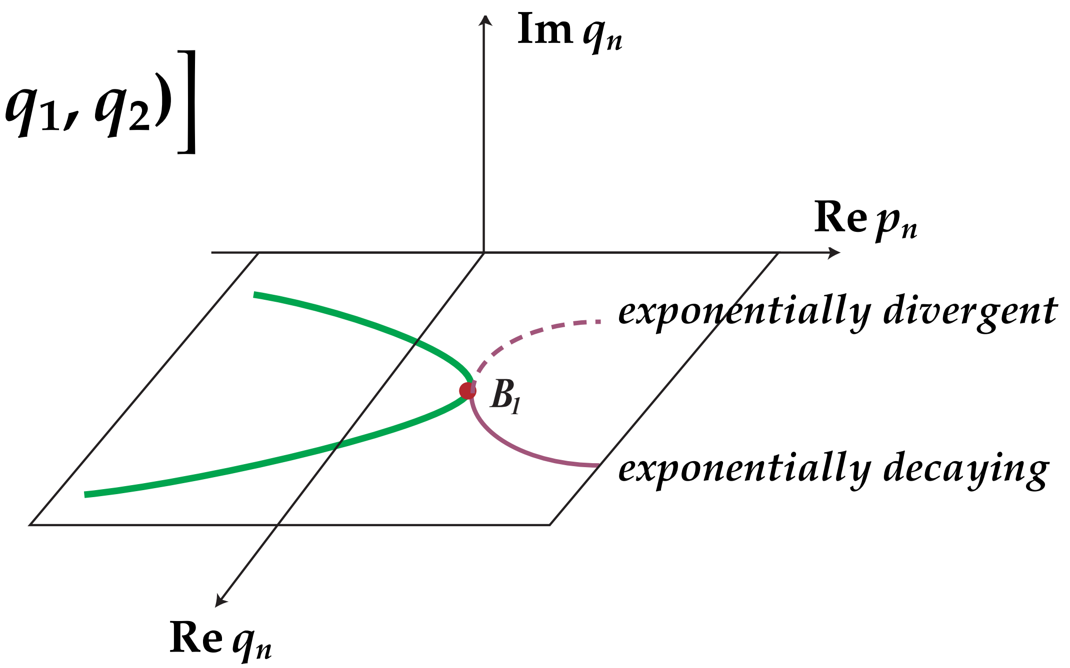
Not all the complex orbits contribute in the saddle point evaluation

Quantum propagator for the 1-step quadratic map :

$$K(q_2, q_0) = \langle q_2|U|q_0\rangle = \int_{-\infty}^{\infty} dq_1 \exp\Big[\frac{i}{\hbar}S(q_0, q_1, q_2)\Big]$$

where

$$S(q_0, q_1, q_2) = \sum_{j=1}^{2} \frac{1}{2}(q_j - q_{j-1})^2 + \frac{1}{3}q_1^3 + cq_1$$



The saddle point condition gives

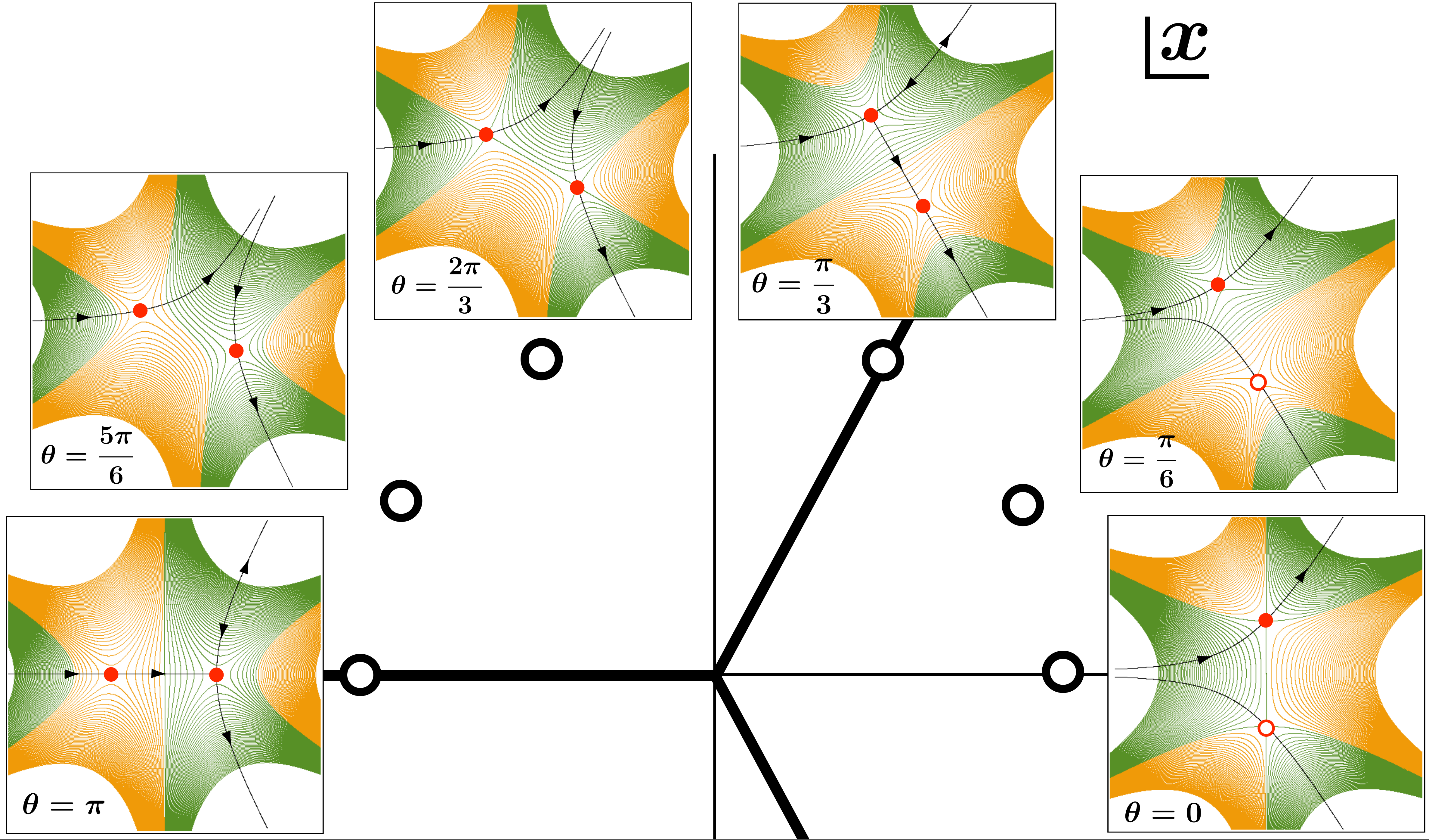
$$\begin{cases} \text{2 real solutions} & \text{if } q_2 < q_2^* \\ \text{2 complex (conjugate) solutions} & \text{if } q_2 > q_2^* \end{cases}$$

For the classically forbidden side, one solution gives exponentially decay, but the other exponentially divergent.

$K(q_2, q_0)$ can be transformed into a canonical form of the Airy integral

$$\mathrm{Ai}(x) = \int_C d\xi \exp i\left[\frac{1}{3}\xi^3 + x\xi\right] = \int_{C_1} + \int_{C_2}$$

where C_1 and C_2 denote the steepest descent contours passing respectively through the saddle points $\xi_1 = i|x|^{\frac{1}{2}}\mathrm{e}^{\frac{1}{2}i\theta}$ and $\xi_2 = -i|x|^{\frac{1}{2}}\mathrm{e}^{\frac{1}{2}i\theta}$



Saddle point method in multiple integrals

— Stokes phenomenon in multidimensions —

Quantum propagator for the 2-step quadratic map :

$$K(q_3, q_0) = \langle q_3|U|q_0\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dq_1 dq_2 \exp\Big[\frac{i}{\hbar}S(q_0, q_1, q_2, q_3)\Big]$$

where

$$S(q_0, q_1, q_2, q_3) = \sum_{j=1}^{3}\frac{1}{2}(q_j - q_{j-1})^2 - \sum_{j=1}^{2} V(q_j)$$

- **Steepest descent surfaces ?**
- **Uniqueness ?**
- **Practical recipe ?**

We do not discuss the issue of Stokes phenomena in our lectures

2. Complex dynamics in one variable

For the moment, we forget about

- **boundary conditions (initial and final)**
- **Stokes phenomenon (non-contributing complex orbits)**

just focus on the dynamics in $\mathbb{C}$ or $\mathbb{C}^2$.

Why the dynamics in $\mathbb{C}$? (our interest is the dynamics in $\mathbb{C}^2$)

- **not so familiar even in $\mathbb{C}$**
- **better understood in $\mathbb{C}$ than $\mathbb{C}^2$**
- **need technically hard tools in $\mathbb{C}^2$**

1-dimensinol polynomial maps and the Julia set

Consider 1-dimensinol polynomial maps with degree d

$$P : z \mapsto P(z)$$

where

$$P(z) = z^d + a_1 z^{d-1} + \cdots + a_d \qquad (d \geq 2)$$

Classify the orbits according to the behavior of $n \to \infty$

$$F_P = \{ z \in \mathbb{C} \mid \lim_{n\to\infty} P^n(z) = \infty \} \quad : \quad \text{Fatou set}$$

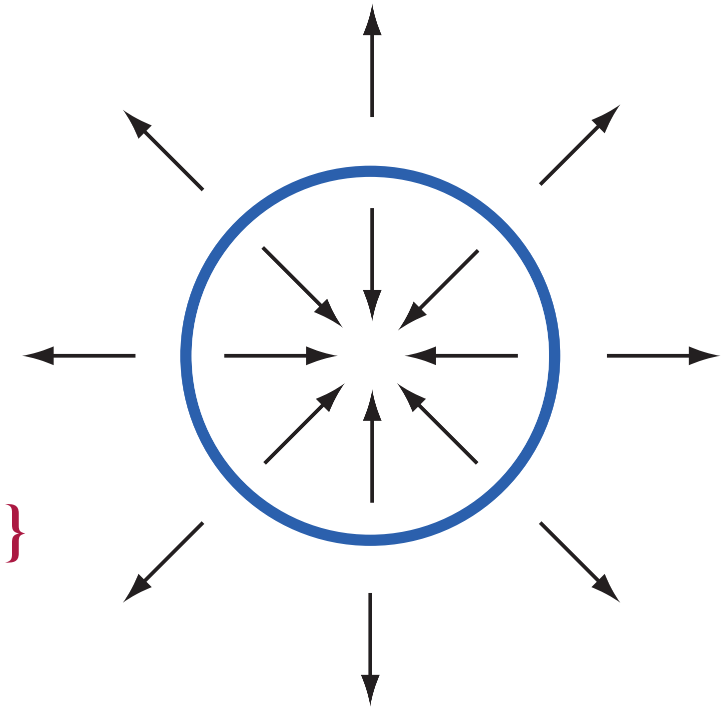
$$K_P = \{ z \in \mathbb{C} \mid \lim_{n\to\infty} P^n(z) \text{ is bounded} \} \quad : \quad \text{Filled Julia set}$$

$$K_P = \mathbb{C} - F_P$$

In particular

$$J_P = \partial K_P \quad : \quad \text{Julia set}$$

Example 1 $\quad P(z) = z^2$



It is easy to show that

$$F_P = \{\, |z| > 1 \,\}, \quad K_P = \{\, |z| \le 1 \,\}, \quad J_P = \{\, |z| = 1 \,\}$$

- $z = \infty$ is an attracting fixed point of P.

  The points $z \in F_P$ tend to ∞ montonically.

- $z = 0$ is also is an attracting fixed point of P.

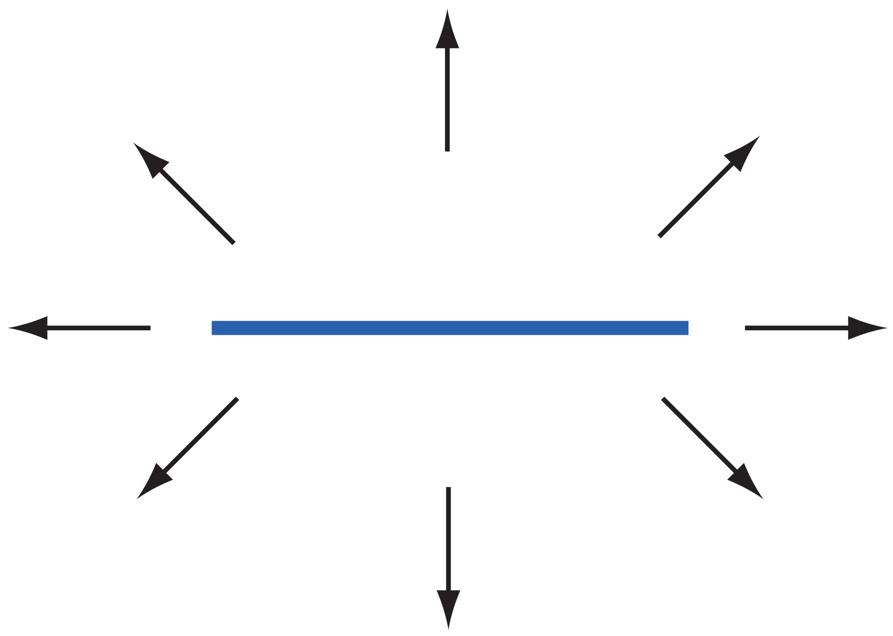
  The points $z \in K_P - J_P$ converge to $z = 0$ monotonically.

- The orbits $z \in J$ are chaotic.

  Putting $z = e^{2\pi i\theta}$, then the map on J_P can be reduced to $\theta \mapsto 2\theta \pmod 1$.

Note: K_P has interior points and Area(K_P)> 0.

Example 2 $\quad P(z) = 2z^2 - 1$



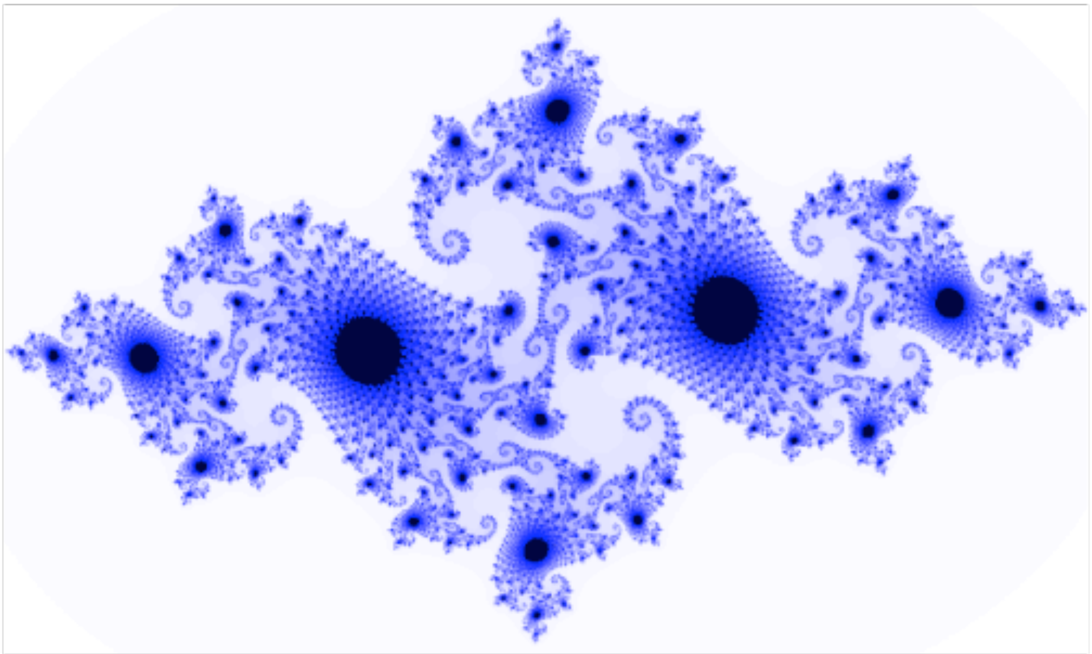
It is also easy to show that

$$F_P = \mathbb{C} - [-1,1], \quad K_P = J_P = [-1,1]$$

- Since $P(\cos\theta) = \cos(2\theta)$, we generally have $P^n(\cos\theta) = \cos(2^n\theta)$.
- Then the iteration on $z \in [-1,1]$ is described by $\theta \mapsto 2\theta \pmod 1$.
- One can show that if $z \in \mathbb{C} - [-1,1]$, $P^n(z) \to \infty$ as $n \to \infty$.

Note: K_P has no interior points and Area(K_P)= 0.

Properties of the Julia set



"P is chaotic on J_P"

1. Sensitive dependence on initial conditions

   there exists $\delta > 0$ such that, for any $z \in J_P$ and any nbd U of z,

   there exists $\zeta \in U$ and $n \geq 0$ such that $|P^n(z) - P^n(\zeta)| > \delta$

2. Density of repelling periodic orbits

   $$J_P = \partial K_P = \overline{\{\text{ repelling fixed points }\}}$$

3. Topological transitivity

   For any open sets $U, V \subset J_P$, there exists $k > 0$ such that $P^k(U) \cap V \neq \emptyset$

Why polynomial maps ?

- **Polynomial maps have "filtration property".**

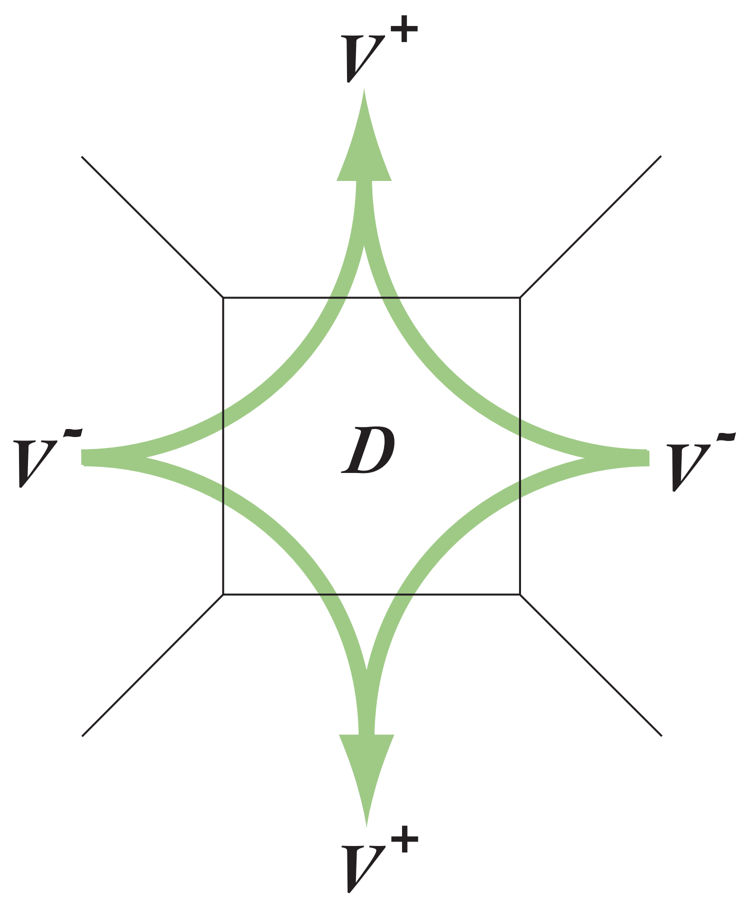
  For sufficiently large R, one can show that $P(V) \subset V$ where $V = \{\,|z| > R\,\}$.

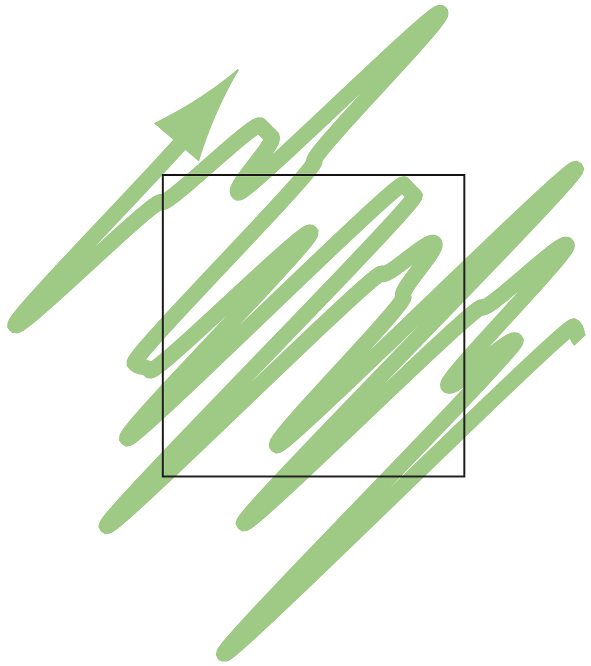
  Hence,

$$F_P = \bigcup_{n=1}^{\infty} P^{-n}(V)$$

- **Transcendental maps do not have.**

  ex) $P(z) = z + \sin(2\pi z)$, $P(z) = z + e^z + 1, \cdots$





Why polynomial maps ?

- **Polynomial maps do not have "wandering domain".**

  Theorem (Sullivan) for any component of Ω in the Fatou set F_P, $\Omega, P(\Omega), P^2(\Omega), \cdots$ is eventually periodic.

- **Transcendental maps can have.**

  ex) $P(z) = z + \sin(2\pi z)$

Definition A component Ω in the Fatou set F_P is:

(a) *periodic* if $\exists\, n > 0$ such that $P^n(\Omega) = \Omega$,

(b) *eventually periodic* if $\exists\, m > 0$ such that $P^m(\Omega)$ is periodic,

(c) *wandering* if the sets $P^n(\Omega)$ for $n \geq 0$ are pairwise disjoint.

The dynamics around $z = 0$ or $z = \infty$

Suppose

$$P : z \mapsto a_1 z + a_2 z^2 + \cdots a_d z^d \qquad (a_1 \neq 0)$$

Then, $z = 0$ and $z = \infty$ are attracting fixed points.

The dynamics around $z = 0$ and $z = \infty$ are rather simple.

$P(z) \sim z$ **around** $z = 0$

$P(z) \sim z^d$ **around** $z = \infty$