MPIPKS Summer School Lecture

Boundary Conformal Field Theory Approach to QIP's

- There is considerable interest in models involving a single localized quantum impurity interacting with a gapless ~~continuum~~ delocalized excitations
- eg Kondo model, multi-channel, multi-impurity [charge & spin varieties]
- magnetic impurity in spin system
- construction in quantum ~~wire~~ wire
- multi-terminal junction
- X-ray edge problems.
- magnetic monopoles (dyons) interacting with charged particles

:

- these models often exhibit interesting cross-over in behavior from high E to low E that can be described by RG
- getting an accurate description of full cross-over is difficult - sometimes done with Bethe Ansatz & or NRG or ~~Appell~~ with MFT's but all these methods have limitations
- high & low energy behavior is often relatively simple & can be accurately calculated
- very generally these high and low energy RG fixed points correspond to CIBC's in CFT's
- often they have equivalent but

more familiar descriptions [eg. free fermions]

- but in some cases they correspond to NFL QCP's - eg. 2 channel (or multi-channel) Kondo, 2 impurity Kondo

- in these lectures I will review this general formalism and discuss the 2-channel Kondo example in some detail, including experiments by Goldhaber-Gordon groups.

- the first step in mapping to a BCFT is to map onto 1D

- this is possible, and convenient, due to fact that all interactions take place at $\vec{r}=0$ - short range

eg - Kondo model

- for spherically symmetric case we can expand in spherical harmonics

- we ignore

$$H = \int d^3\vec{k}\, \psi^\dagger_{\vec{k}\alpha} \psi_{\vec{k}\alpha}\, \epsilon(k) + J \psi^\dagger \frac{\vec{\sigma}}{2} \psi(r=0)\cdot\vec{S}$$

$$\epsilon = \frac{k^2}{2m} - \mu$$

- we can expand in spherical harmonics

$$\psi_{\vec{k}} = \frac{1}{\sqrt{\text{const}\cdot k}}\, \psi_0(k) + \text{higher harmonics}$$

ψ_0 - s-wave

- only s-wave interacts with impurity

- NB - we ignore bulk interactions - justified?

- to study low energy physics (at $J\nu \ll 1$ where ν = density of states $= mk_F/\pi^2$)

we linearize $\epsilon(k)$ near k_F: $\epsilon = v_F(k - k_F)$

and define incoming and outgoing waves [left, right movers]

$$\psi_{L/R}(r) = \int_{-\Lambda}^{\Lambda} dk\, e^{\pm ikr}\, \psi_0(k_F + q)$$

$$H = \frac{v_F}{2\pi} i \int_0^\infty dr \left[\psi_L^\dagger \partial_r \psi_L - \psi_R^\dagger \partial_r \psi_R \right]$$
$$+ v_F \lambda \, \psi_L^\dagger(0) \frac{\vec{\sigma}}{2} \psi_L(0) \cdot \vec{S} + \cdots$$

where $\lambda = J\nu$, NB $\psi_L(0) = \psi_R(0)$

- a Lorentz-invariant ($v_F \leftrightarrow c$)

Dirac fermion field theory in $(1+1)$ D

with impurity interactions at $r=0$

- defined on half-line

at $\lambda = 0$, this is an example of a CFT

with a boundary [+ decoupled spin]

- fermions are massless $\Rightarrow$ no scale

imaginary time Lagrangian is

$$2\pi \mathcal{L} = \psi_L^\dagger \left(\partial_\tau + i v_F \partial_r \right) \psi_L + \psi_R^\dagger \left(\partial_\tau - i v_F \partial_r \right) \psi_R$$

let $z = \tau + i r$ [and set $v_F = 1$]

- then, without boundary, $S = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} dr \, \mathcal{L}$

is invariant under conformal transformation

$\psi_L \to \left(\frac{dW}{dz}\right)^{+1/2} \psi_L$, $\psi_R \to \left(\frac{dW}{d\bar{z}}\right)^{+1/2} \psi_R$.

$z \to w(z)$, $\psi_L \to W^{1/2} \psi_L$, $\psi_R \to W^{1/2} \psi_R$

where $\bar{z} = \tau - i\sigma$

Dirac $S = \int d\tau d\bar{z} \left[\psi_L^+ \frac{\partial}{\partial z} \psi_R + \psi_R^+ \frac{\partial}{\partial \bar{z}} \psi_R \right]$

- ∞-dimensional set of transformations

$w(z) = \sum_{n=0}^{\infty} a_n z^n$, $a_n \in C$

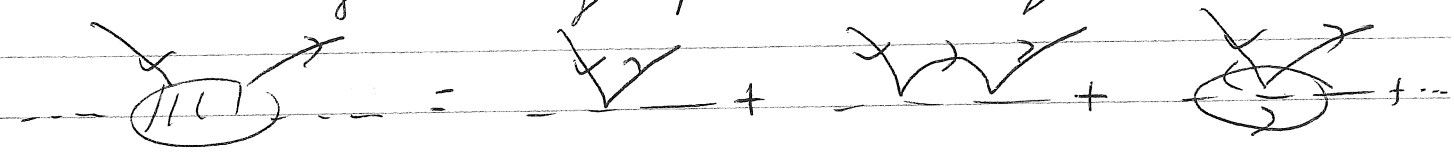
$a_1 \in R$ - scale transformation etc.

∞ symmetry has powerful consequences.

- if system is defined on $1/2$-line $\sigma > 0$, $\mathrm{Im}\, z > 0$, full conformal symmetry is broken but CI BC's preserve ∞ subset that maps boundary, $\mathrm{Im}\, z = 0$ onto itself $a_n \in R \Rightarrow W^*(\tau) = w(\tau)$

- in above example $\psi_L(\sigma = 0, \tau) = \psi_R(\sigma = 0, \tau)$ is a CI BC.

of course Kondo interaction breaks BCI
- in fact it introduces a characteristic energy scale – Kondo temperature
- renormalization group (RG) eqns:



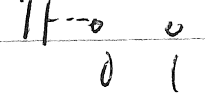
$$\frac{d\lambda}{d\ln D} = -\lambda^2 + \frac{\lambda^3}{2} + \cdots$$

$\Rightarrow \lambda(D)$ is $O(1)$ at $T_K \pm D_0 e^{-\frac{1}{\lambda_0}} \sqrt{\lambda_0}$

where D_0 a band-width – energy scale at which bare coupling is evaluated

- but low energy behavior is again described by a CIBC – at $E \ll T_K$
- this is most easily seen by considering a tight-binding model with a large J

$$H = -t \sum_{j=0}^{\infty} (f_j^\dagger f_{j+1} + h.c.) + J f_0^\dagger \frac{\vec{\sigma}}{2} f_0 \cdot \vec{S}$$ (spin indices suppressed)



if $J \gg t$, 1 electron is trapped at site 0, forming a singlet with impurity spin

- electrons on other sites can do anything but go to site 0
- at ½-filling p-h symmetry, this corresponds to a $\pi/2$ phase shift

bc modified from $\psi_L(0) = \psi_R(0)$ to $\psi_L(0) = -\psi_R(0)$

- this follows from writing $\psi_j = e^{i\frac{\pi}{2}j}\psi_R(j) + e^{-i\frac{\pi}{2}j}\psi_L(j)$

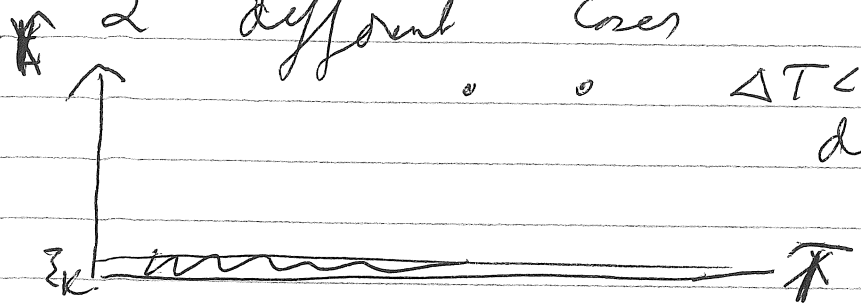
open bc: $\psi_{-1} = 0 \Rightarrow \psi_L = \psi_R$
$J \to \infty$: $\psi_0 = 0 \Rightarrow \psi_L = -\psi_R$

- another CIBC occurs [e impurity spin disappears] - only change from strong to weak coupling is in BC.
[+ elimination of impurity]
- starting with a small bare J, this low energy description only becomes

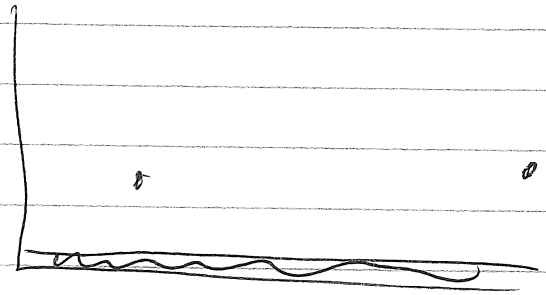
valid at $E \ll T_K$

- ~~any~~ how "nearby" electron which forms singlet is at distance of $O\left(\frac{v_F}{T_K} \equiv \xi_K\right)$
- this is a simple example of a very general phenomenon:
- low energy effective Hamiltonian is a CFT with a CIBC for a wide range of problems. - tested above
- why? - no rigorous proof but -
- far from boundary compared to all microscopic length scales (such as ξ_K) and at long times ($\gg 1/T_K$) we might expect universal behavior characteristic of a critical point

2 different cases



$\Delta T \ll r$ – boundary doesn't matter



$\Delta T \gg r$ – boundary influences Green's function

but only in a universal way consistent with system being critical – i.e only ratios of lengths enter $\Delta T / r_i$ etc.

+ powers of length

eg $G(r_1, T_1; r_2, T_2) = \frac{1}{|T_1 - T_2|^n} f\left(\frac{r_1}{T_1 - T_2}, \frac{r_2}{T_1 - T_2}\right)$

$f \to$ const at $\frac{r_1'}{|T_1 - T_2|} \gg 1$

– this type of scale-invariant long-distance behavior is described by CFT with a CIBC

NB - bulk critical theory doesn't change due to short-range interaction only at boundary

- all that happens is an RG flow from 1 CIBC to another

- CIBC is a "termination" of bulk CI behavior as boundary is taken into account (but not approached too closely) that preserves a larger subset of conformal symmetry

- some low energy, long distance properties ~~are~~ afternoon are determined ~~by the CIBC~~ completely by the conformally invariant Hamiltonian + CIBC - these include all long time & distance (& low T) Green functions, the T=0 impurity entropy, & g, the T=0 ~~resistivity~~ resistivity (or conductance) for

dilute random array of Kondo impurities
(or single quantum dot acting as a
Kondo impurity) . . .

[start here]

- other low energy long-distance properties require consideration of leading irrelevant operator (i.e. interactions) at fixed point
i.e. CFT of CIBC is exact description of zero energy limit only
- it is a fixed point Hamiltonian ~~that~~ that system flows to
- at any finite energy scale H_{eff} contains additional interacting terms which break the conformal symmetry
- these:

1) don't involve the impurity degrees of freedom

itself [eg impurity spin] because it is screened at much higher energies [$\sim T_K$]

2) are ~~not~~ added to the model with specified CIBC - the set of possible interactions is therefore determined by CIBC [+ other symmetries like SU(2)]

- of this ∞ set of interactions, all of which are irrelevant, the "least irrelevant" lowest dimension, LI! controls asymptotic behavior of various quantities: Kondo

- impurity susceptibility
- T-dependence of impurity entropy (⇒ specific heat)
- T-dependence of resistivity / susceptibility
- leading corrections to all Green's functions
- to calculate these effects we need

to know coupling constant(s) of LIO in H_{eff}

$$H = H^{CFT}_{CIBC} + g\, O_{LI}$$

- this cannot usually be determined exactly [unless we can use Bethe ansatz or a numerical method]
- but we can estimate it in order of magnitude up to a factor of $O(1)$
- it drops out in various ratios of physical quantities [eg Wilson ratio of impurity susceptibility & specific heat or "Kondo ratio" of resistivity to square of susceptibility] (to here)
- how are CIBC specified?
- surprisingly, in NFL cases we don't usually explicitly ~~say~~ specify a BC

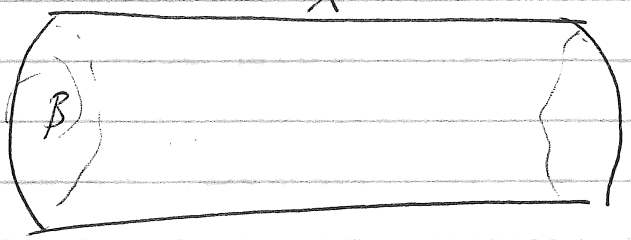
- instead a "boundary state" $|A\rangle$ is specified
- this allows all of above quantities to be calculated (e.g. finite size spectrum with arbitrary pair of CIBCs at 2 ends of finite system) & it is equivalent to specifying BC explicitly
- FSS is very useful for comparing CFT predictions to NRG calculations
- consider CI partition function, at inverse temperature β, for finite system of length l, with CIBCs A, B at 2 ends:
  H^l_{AB} is corresponding Hamiltonian
- all eigenvalues are of form $\frac{\pi}{l} X$ for dimensionless numbers X:

$$Z_{AB} = \mathrm{tr}\, e^{-\beta H^l_{AB}}$$

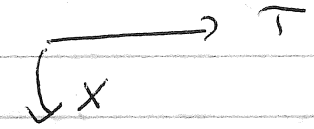
- now we use a basic property of (1+1) dimensional Lorentz invariant QFT's:

- we may interchange space and imaginary time [modular transformation]

- ~~how~~ Z_{AB}



originally  Cylinder - periodic in imaginary time

now consider 

- now periodic in space (no boundary) with system length β

- not periodic in τ [not a trace]

- instead system propagates for time l between initial & final states $|A\rangle$, $|B\rangle$

$$Z_{AB} = \langle A | e^{-l H_p^\beta} | B \rangle$$

H_p^β is Hamiltonian with PBC, length β

- the boundary states $|A\rangle$, $|B\rangle$ determine Z_{AB} & in fact all other needed properties
- these states must obey certain properties to respect boundary conformal invariance - determined by J Cardy
- in some cases boundary conformal invariance + other symmetries allow only a finite set of possible boundary states
- then it may be relatively straightforward to determine boundary state corresponding to RG fixed point of a given condensed matter Q I P - eg multi-channel Kondo

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~~Simple Example = Quantum Ising Chain~~

Quantities Determined completely by fixed point Hamiltonian (CIBC)

① Impurity Entropy

- how is it defined & measured experimentally

- for dilute Kondo impurities in a metal, specific (heat) capacity is $C(T) = C_0(T) + n_{imp} C_{imp}(T) + \dots$

$C_{imp}(T)$ is extra entropy due to a single impurity - it is $O(1)$ in ∞ volume limit

- we can calculate it from single impurity model [eg 1D version] as

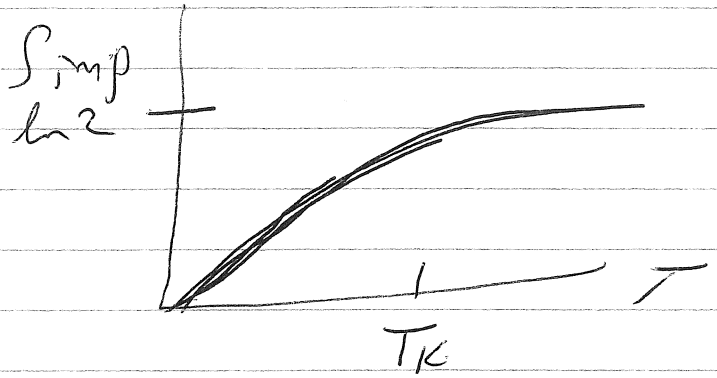
$$C_{imp}(T) = \lim_{L\to\infty} \left[C_L(T, J) - C_L^{(0)}(T) \right]_{J=0}$$

where $C_L^{(0)}$ is ~~spec~~ heat capacity without impurity

~~NB $C_L(T, J=0) = C_L^{(0)}(T) + \ln(2S+1)$~~

impurity entropy is $S_{imp}(T) = \int_0^T \frac{dT}{T} C_{imp}(T)$

- for usual $S = \frac{1}{2}$, single channel Kondo model



in scaling limit where $\log T \to 0$

$S_{imp} = \ln 2$ at $T \gg T_K$ since

the $J_{eff}(T) \to J = 0$ & we get entropy

of free spin $= \ln 2$ [2 states]

at $T \to 0$, impurity is screened

& doesn't contribute $\Rightarrow S_{imp}(0) = 0$

NB - can also define $S_{imp}(T) = \lim_{t \to \infty} [S(T) - S^{(0)}(T)]$

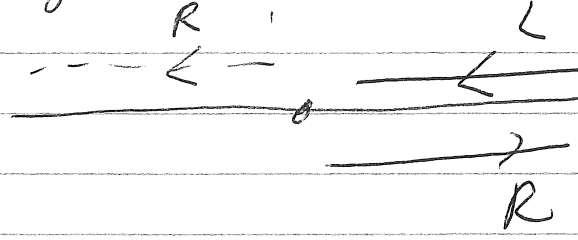
- a general formula can be found for S_{imp} at $T \to \infty$, $T \to 0$ - property of

RG fixed points in ultra-violet
e confirmed for any CIBC.
- it is a pure number [setting $k_B = 1$]
in dimensional grounds [for $T \to 0$ or $\omega \to \infty$
~~there is no~~
- it can be expressed in terms of a simple
matrix element of B. state $|A\rangle$

(2) S-matrix in single particle sector
[resistivity or conductance in Kondo model]

- in 1D Dirac fermion theory we
calculate $\langle \psi_L^+(\tau + i\delta)\, \psi_R(\tau' - i\delta')\rangle$
or more simply $\langle \psi_L^+(\tau)\, \psi_R(\tau)\rangle$
(equal time)
- for $\lambda = 0$, bc $\psi_R(0) = \psi_L(0)$
implies that we may consider

$\psi_R(r) = \psi_L(-r)$ – analytic continuation to negative r-axis



$$\Rightarrow \langle \psi_L^+(r)\, \psi_R(r') \rangle = \langle \psi_L^+(r)\, \psi_L(-r') \rangle = \frac{1}{2\pi i r}$$

– in single-channel Kondo model with simple bc $\psi_R(0) = -\psi_L(0)$

$$\psi_R(r) = -\psi_L(-r)$$

$$\Rightarrow \langle \psi_L^+(r)\, \psi_R(r') \rangle = -\frac{1}{2\pi i r}$$

– more generally, when we break p–h symmetry, we could have an arbitrary phase shift, δ, at the Fermi surface

$$\langle \psi_L^+(r)\, \psi_R(r') \rangle = \frac{e^{2i\delta}}{2\pi i r}$$

– above example was $\delta = \pi/2$

- from ϕ or $\langle \psi_k^+ | \psi_k \rangle = \frac{S_{(1)}}{2\pi i}$

where $S_{(1)}$ is S-matrix element for single particle to single particle scattering

- in general [eg multi-channel Kondo] we find $|S_{(1)}|$ can be < 1

- we can interpret this as a finite probability for 1 particle $\to$ 1 particle + h particle-hole pair scattering at $\epsilon \to 0$ - this violates usual Fermi liquid behavior pointed out by Landau.

T-matrix is $T = -\frac{i}{2\pi\nu}[1 - S_{(1)}]$

- resistivity at $T=0$ for dilute Kondo scatterers [& conductance thru quantum dot in Kondo regime]

is proportional to $T_2 [1 - f(1)]$

- so maximum resistance [$\frac{2e^2}{h}$ conductance] occurs for single channel case

- it is reduced at NFL critical point

Explicit Realization of 2 Channel Kondo QCP

$$H = \frac{VF}{2\pi} i \int_0^\infty dr \left[\psi_L^{\dagger 1\alpha} \frac{d}{dr} \psi_{L 1\alpha} - \psi_R^{\dagger 1\alpha} \frac{d}{dr} \psi_{R 1\alpha} \right]$$
$$+ VF \lambda \, \psi_L^{\dagger 1\alpha}(0) \frac{\vec{\sigma}_{\alpha\beta}}{2} \psi_{L\beta a}(0) \cdot \vec{S}, \quad a = 1, 2$$

NB- equal Kondo couplings to both channels.

- why does this have a NFL unstable fixed point?

- to consider stability of FL fixed point we can again consider lattice model with a strong local coupling, $J \gg t$

- but now 2 electrons get trapped at

site $j=0$ (1 from each channel) going with a $S=1$ state

- large AF coupling to impurity spin (of size $\frac{1}{2}$) leads to an $S=\frac{1}{2}$ ground state $\uparrow\downarrow - \downarrow$

- there is now a weak (Kondo) coupling $\sim t^2/J$ of this $S=\frac{1}{2}$ spin to site $j=1$

- also turns out to be AF

- so, it is not consistent to assume $\frac{J}{t}$ renormalizes to ∞

- instead, we get an intermediate coupling fixed point

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- we found CIBC - checked against Bethe ansatz & NRG

strip
[$g = \frac{1}{2}$ - fractional "ground state degeneracy"]

$S(\omega) = 0$ - normal inelastic scattering in this sense

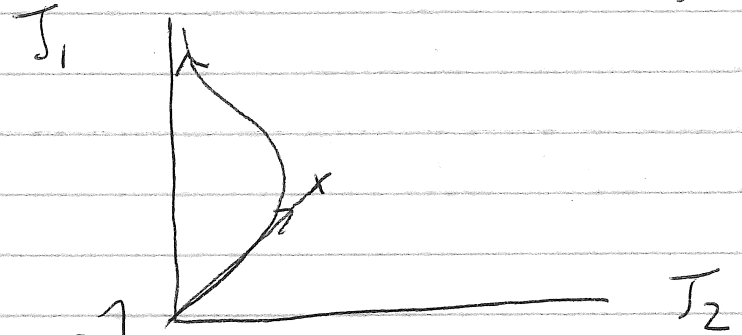
$\Rightarrow T = -\frac{i}{2\pi\nu}[1 - S(\omega)]$ has $\frac{1}{2}$ value for single channel case

$G = \frac{e^2}{h}$ not $\frac{2e^2}{h}$ at $T = V = 0$

strip
- Channel asymmetry is a relevant perturbation of dimension 1/2
- this was established (I A & A Ludwig) by finding most relevant operator allowed by symmetry when $J_1 \neq J_2$
- stable fixed point corresponds to most strongly coupled channel screening impurity spin & other one decoupled at low E - i.e. $\delta_1 = \pi/2$, $\delta_2 = 0$

- relevant channel symmetry breaking perturbation at the QCP has dimension 1/2

$\Rightarrow \left(J_1 - J_2\right)_{eff}^{(T)} = \left(J_1 - J_2\right)_0 \left(\frac{D}{T}\right)^{1/2}$ for an appropriate UV cut-off scale, D



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- right at the QCP, $L \cdot I \cdot O$ has dimension 3/2

$\Rightarrow$ Coupling constant $\frac{1}{\sqrt{T_K}}$]

- various experimental realizations of 2CKE were proposed over the years:

low-lying

- atoms with a quadrupolar state or other orbital degrees of freedom playing the role of "spin" with actual electron spin playing the role of "channel"

= atomic impurities in tunnel barriers

- hi-Tc superconductors,

-

- all were questioned

- D Goldhaber-Gordon et al. finally produced an indisputable expt. realization

- a double quantum dot device

- you might think that a single q.d. device corresponded to a 2CK system with left + right leads being 2 channels

- actually, this doesn't work

$$H_A \sim H_L^0 + H_R^0 + \left(d^\dagger \left[t_L \psi_L(0) + t_R \psi_R(0)\right] + h.c.\right) + U(n_d - 1)^2$$

$$H_K \sim \frac{1}{D}\left(t_L^* \psi_L + t_R^* \psi_R\right) + \vec{S} \cdot \left(t_L^* \psi_L + t_R^* \psi_R\right) \cdot \vec{J}$$

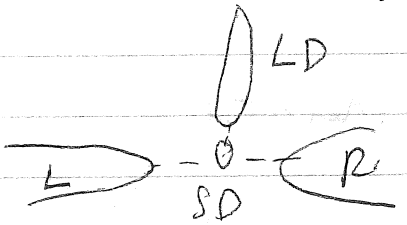
$$J \sim t^2/U$$

- a particular broad combination acts as a

single channel in Kondo model

[Nozières - Ruckh], other channel decouples

- in GG exp [Y Oreg e GG] a large 2^{N-1} dd acts as a 2^{N-1} channel



- again a linear combination of 3 "channels"

$$U \sum_{i=1}^{3} \epsilon_i \psi_i \quad + \quad \vec{\sigma} \sum_{i=1}^{3} \epsilon_i \psi_i \cdot \vec{s}$$

- however, due to its finite size charging energy of LD is important [$\epsilon \sim \psi_3$]

$$\delta H = \frac{U_c}{2} (\hat{n}_3 - n_0)^2$$

- at energy scales below U_c tunneling on or off LD is suppressed, leaving 2 CKM

: ψ_3, $\frac{\epsilon_1 \psi_1 + \psi_2 \psi_2}{\sqrt{\epsilon_1^2 + \epsilon_2^2}}$

- size of LD must be small enough that $T_K \ll U$ but large enough that finite-size single particle level spacing $\ll T_K$
- then must tune gates so $J_1 \simeq J_2$
- CFT methods predict dependence of conductance on T & source-drain voltage of form:

$$G \equiv \frac{dI}{dV_{sd}} = \frac{e^2}{h}\left[1 - \left(\frac{T}{T_K}\right)^{1/k} F\left(\frac{eV_{sd}}{T}\right)\right]$$

(where F is not known, in general)

$\frac{e^2}{h}$ factor [$\frac{1}{2}$ of $\frac{2e^2}{h}$] & exponent $\frac{1}{2}$ follow simply from CFT results

- cross-over energy scale is determined from combining weak & strong coupling RG:

$$T_C \propto T_K \frac{(\lambda_1 - \lambda_2)^2}{(\lambda_1 + \lambda_2)^4}$$ (Pustilnik, Borda, Glazman, von Delft)

- Conclusion: BCFT is a general approach to studying low energy RG fixed points of QIP's
- impurity degrees of freedom are integrated out, and we are left with a CIBC
- this method can reproduce in a systematic way FL f.p's found ~~so~~ earlier as well as NFL QCP's that occur in some models