# Control of chaos by time delayed feedback

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# ... in collaboration with

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## 1 Time–delayed feedback method

Goal

stabilisation of
unstable periodic orbits
$\xi(t) = \xi(t+T)$

Limitations

- no model
- no data processing

e.g.. fast time scales

Idea (→ K. Pyragas, PLA **170**, 421, '92)

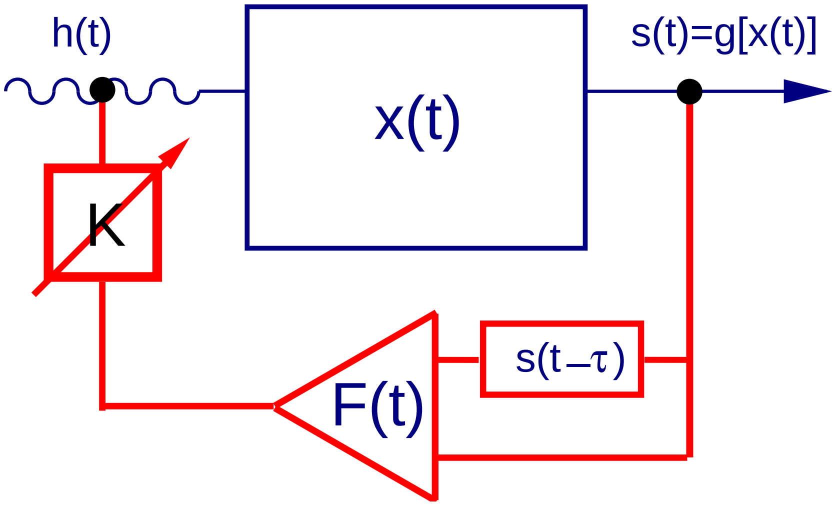
control force

$$F(t) \sim x(t) - \xi(t) \rightarrow x(t) - x(t-\tau)$$

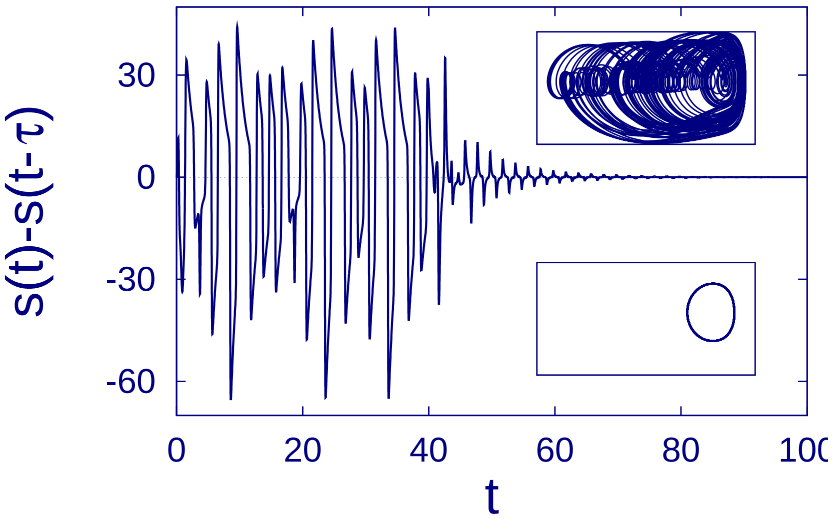
delay $\tau \equiv$ period $T$

non invasive method (cf. → spectroscopy)

## control scheme




## example



# Experimental realisations

- $CO_2$ laser (→ S. Bielawski et.al., PRE **49**, R971, '94)
- discharge gas tube (→ T. Pierre et.al., PRL **76**, 2290, '96)
- photorefractive systems (→ E. Benkler et.al., PRL **84**, 879, '00)
- Taylor–Couette flow (→ O. Lüthje et.al., PRL **86**, 1745, '01)
- FMR on YIG (→ H. Benner et.al., JKPS **40**, 1046, '02)
- electrochemical cell (→ P. Parmananda et.al., PRE **59**, 5266, '99)
- cardiac arrhythmia (→ K. Hall et.al., PRL **78**, 4518, '97)
- electronic circuits (→ K. Pyragas et.al., PLA **180**, 99, '93)
- mechanical pendulum (→ T. Hikihara et.al. PLA **211**, 29, '96)

## 2 Stability analysis

### General equation of motion

$$\dot{x}(t) = f\left(x(t), F(t)\right)$$

### Control schemes

- Pyragas control (→ K. Pyragas, PLA **170**, 421, '92)

$$F_{Pyr}(t) = K\left\{g[x(t)] - g[x(t-\tau)]\right\}$$

- extended time–delayed feedback (→ J. E. S. Socolar et.al., PRE **50**, 3245, '94)

$$F_{ext}(t) = K \sum_{\nu \geq 0} R^{\nu} \left\{g[x(t-\nu\tau)] - g[x(t-\nu\tau-\tau)]\right\}$$

- rhythmic control (→ S. Bielawski et.el., PRA **47**, 2492, '93)

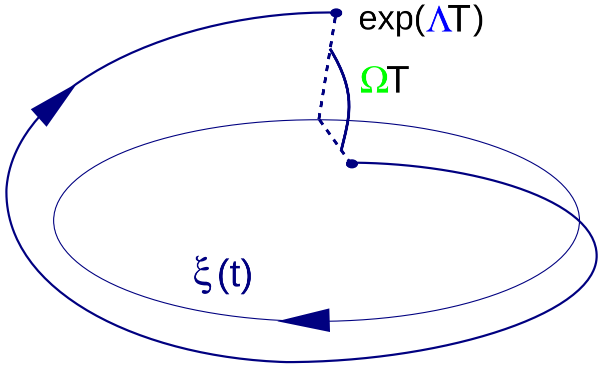
$$F_{rhy}(t) = K(t)\left\{g[x(t)] - g[x(t-\tau)]\right\}$$

# Linear stability analysis

$$x(t) = \xi(t) + \delta x(t)$$

## Floquet decomposition

$$\delta x(t) = e^{(\Lambda + i\Omega)t} Q(t)$$
$$Q(t) = Q(t+T)$$




Λ Expansion, Ω Torsion

$$\delta \dot{x}(t) = D_1 f(\xi(t), 0)\delta x(t) + \underbrace{d_2 f(\xi(t), 0) \otimes Dg[\xi(t)]}_{\text{control matrix}} K \{\delta x(t) - \delta x(t-\tau)\}$$

$$\delta x(t) - \delta x(t-\tau) \to \left\{ 1 - e^{-(\Lambda + i\Omega)\tau} \right\} Q(t)$$

## Characteristic equation

$$\Lambda + i\Omega = \Gamma\left[K\{1 - e^{-(\Lambda+i\Omega)\tau}\}\right]$$

## "Mean–field expansion"

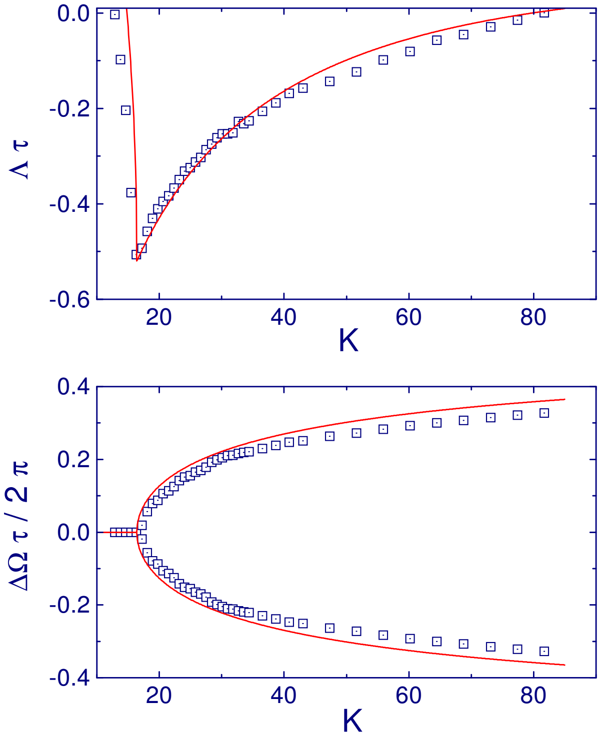
$$\boxed{\Lambda\tau + i\Omega\tau = (\lambda + i\omega)\tau - (-\tau\chi)K\left\{1 - e^{-\Lambda\tau - i\Omega\tau}\right\}}$$

- analytic expression (→ W.J. et.al., PRE **61**, 3675, '00)
- exact for diagonal control
- correct asymptotics for $K \to 0$ and $|K| \to \infty$

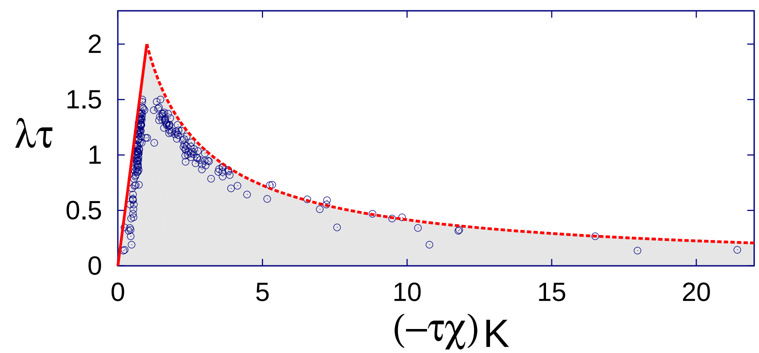
## 3 Control properties

- control domains and instabilities (→ W.J. et.al., PLA **254**, 158, '99)

Floquet exponents



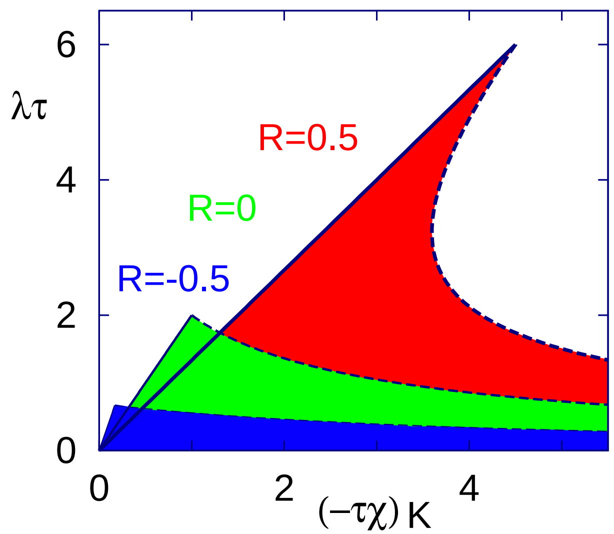

dependence on the Lyapunov exponent




$\lambda\tau \lesssim 2$

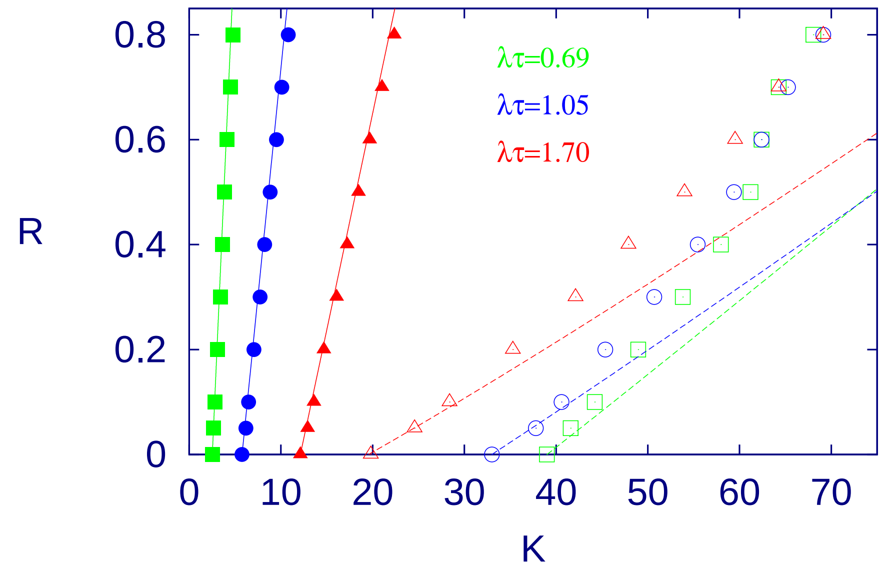
## – extended time–delayed feedback

### analytic expression




$$\lambda\tau \lesssim 2 \times \frac{1+R}{1-R}$$

### circuit experiment



- adjustment of delay (→ A. Kittel et.al., PLA **198**, 433, '95)
  - induced periodic orbits (→ W.J. et.al., PRL **81**, 562, '98)

- topological constraints (torsion) (→ W.J. et.al., PRL **78**, 203, '97)
  - rhythmic control, $K \longrightarrow K(t)$
  - unstable controller (→ K.Pyragas, PRL **86**, 2265, '01)

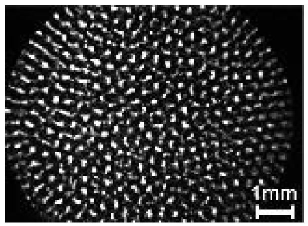
- control loop latency, $F(t) \longrightarrow F(t-\delta)$ (→ W.J. et.al., PRE **59**, 2826, '99)

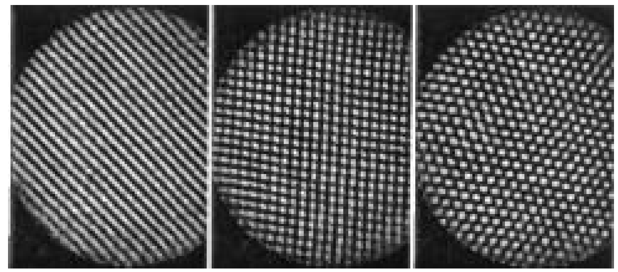
Limitations of the mean–field expansion (→ W.J. et.al., PRE **61**, 5045, '00)

## 4 Oscillating feedback and eigenmode control

### Laser experiment: Fourier filter

(→ E. Benkler et.al., PRL **84**, 879, '00)



→ Control through eigenmodes

## Simulation: Floquet mode control

(→ N. Baba et.al., PRL **89**, 074101, '02)

reaction–diffusion model

$$\partial_t a(x,t) = \frac{u-a}{1+(u-a)^2} - Ta + \partial_x^2 a - f_a(x,t)$$

$$\partial_t u(t) = \alpha \left[ j_0 - \left( u - \int_0^L a dx/L \right) \right] - f_u(t)$$

control force through spatio–temporal filters ...

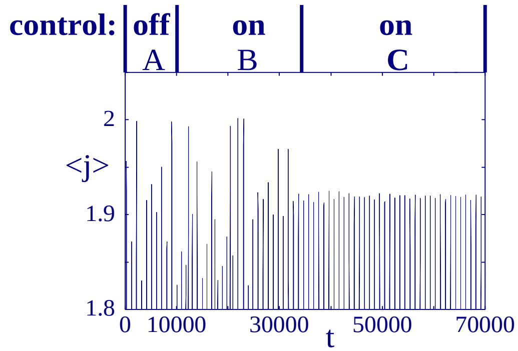
$$f_a(x,t) = K\Psi_a(x,t)[s(t) - s(t-\tau)]$$

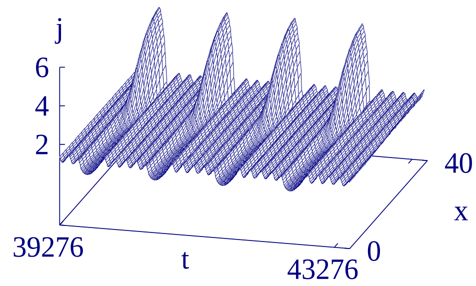
$$f_u(t) = K\Psi_u(t)[s(t) - s(t-\tau)]$$

$$s(t) = \int_0^L \Phi_a^*(x,t) a(x,t)\, dx + \Phi_u^*(t) u(t)$$

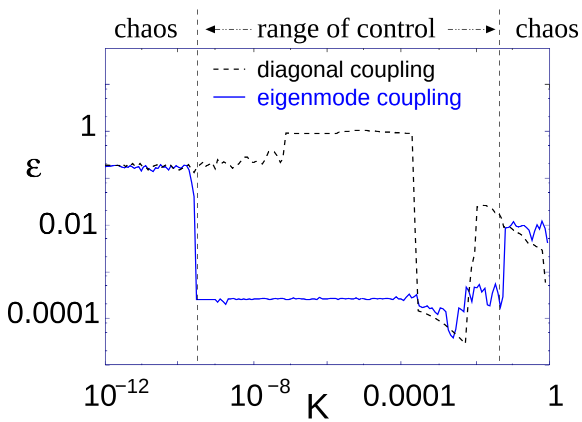
... derived from eigenmodes $(\Psi_a, \Psi_u)$, $(\Phi_a, \Phi_u)$ of the unstable orbit (→ analytic treatment).

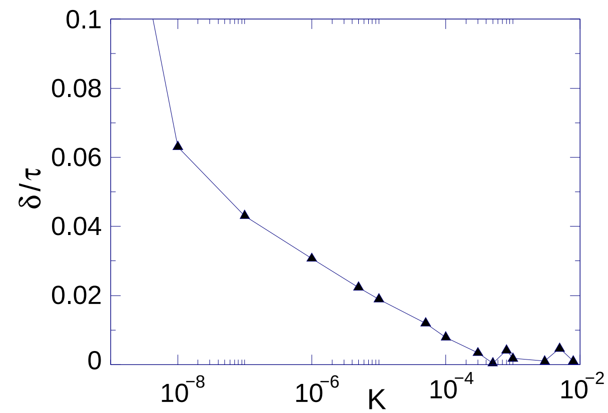
stabilisation of filaments




increase of control domain ... through phase shift $\delta$.



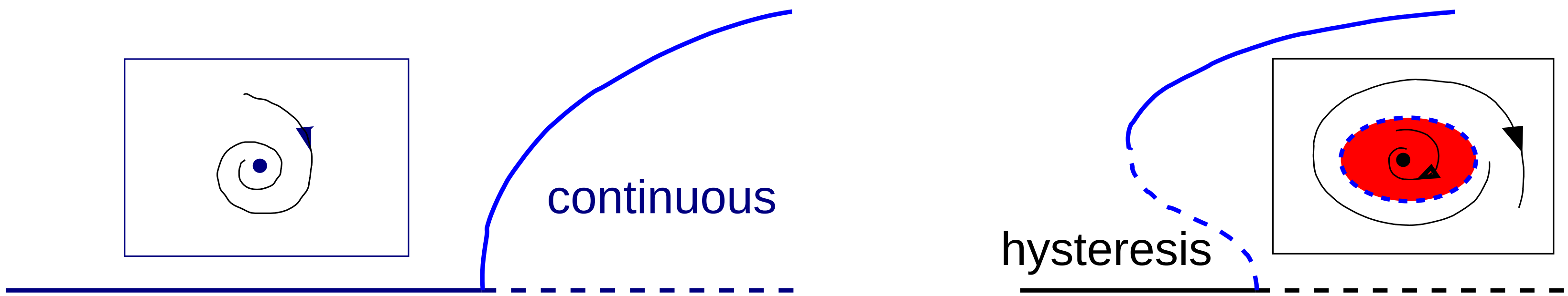

analytic prediction of the control interval (→ W. J. et.al., PRE **67**,026222, ’03)

$$K_{max}/K_{min} \simeq \exp(A\delta^2)$$

## 5 From local to global analysis ?

- local analysis $\longrightarrow$ linear stability
- global features $\longrightarrow$ domain of attraction ?
- delay systems $\longrightarrow$ infinite–dimensional phase spaces

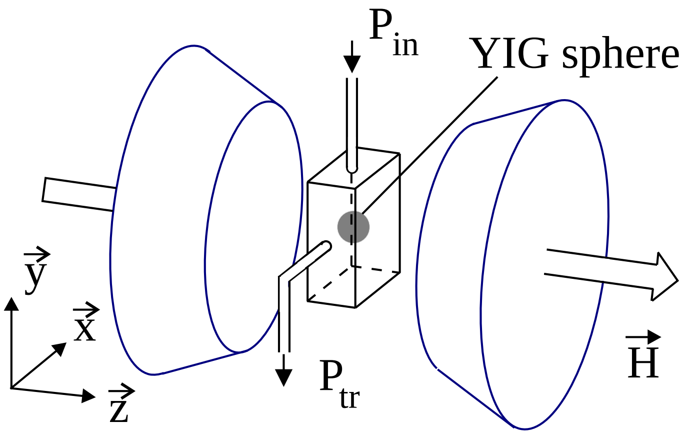
super/sub–critical instabilities

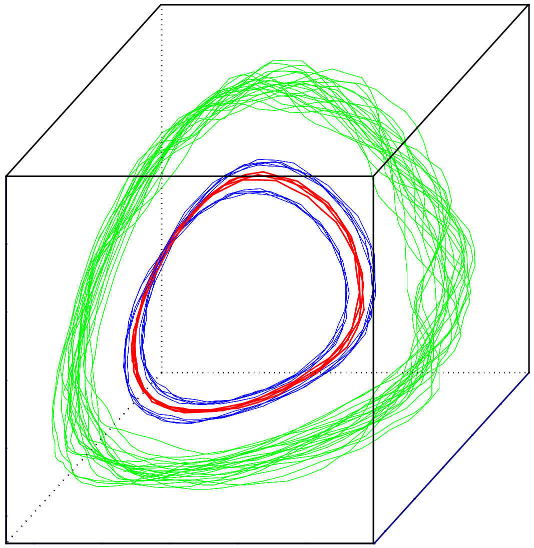



$\longrightarrow$ normal form analysis, amplitude equations, etc.

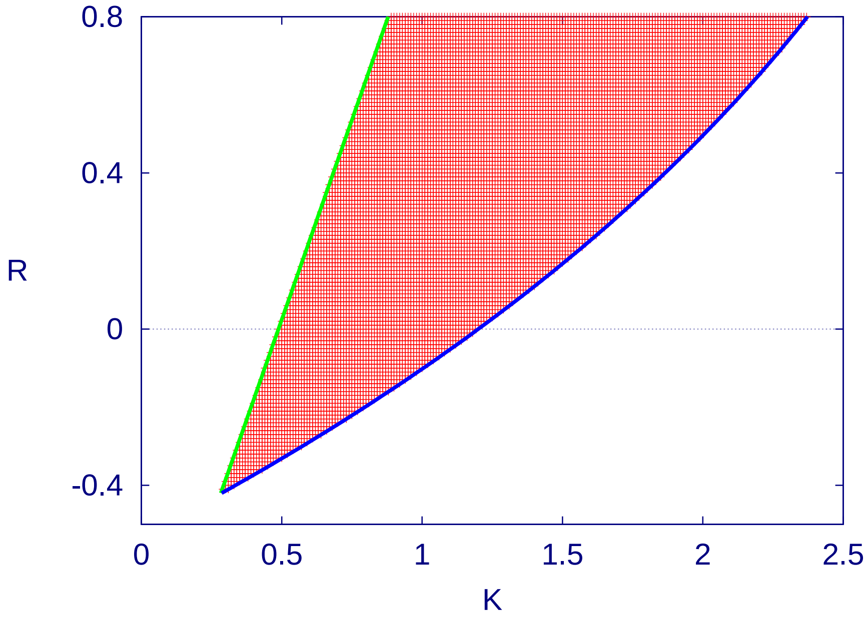
$$\dot{z} = \mu z - r|z|^2 z, \quad \text{sign}(\text{Re}(r)) = ?$$

# Experimental hint (FMR in YIG)



# Theoretical analysis (for extended control scheme)

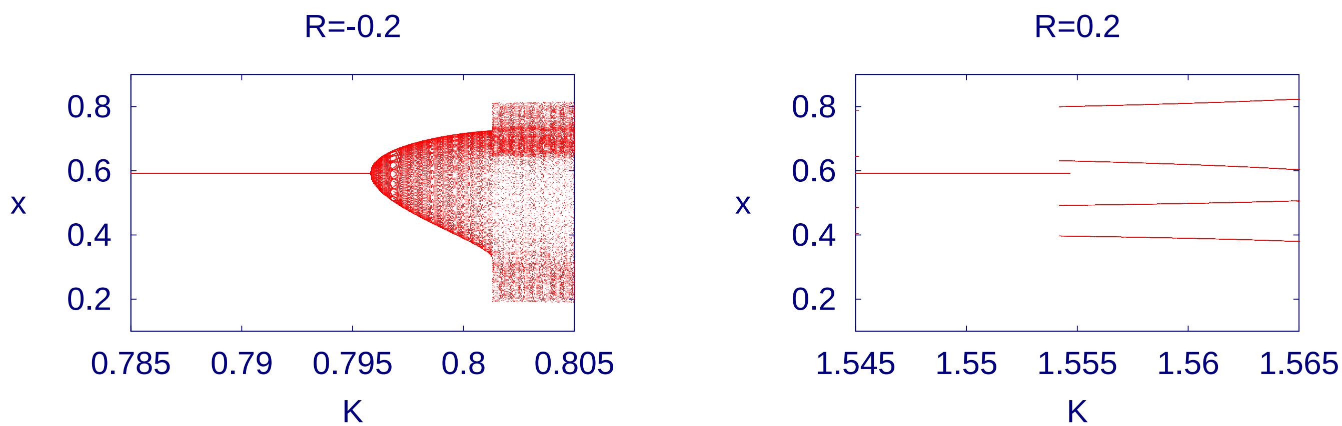



Flip: $K_{fl}/(1+R) =$ const., $r =$ const.

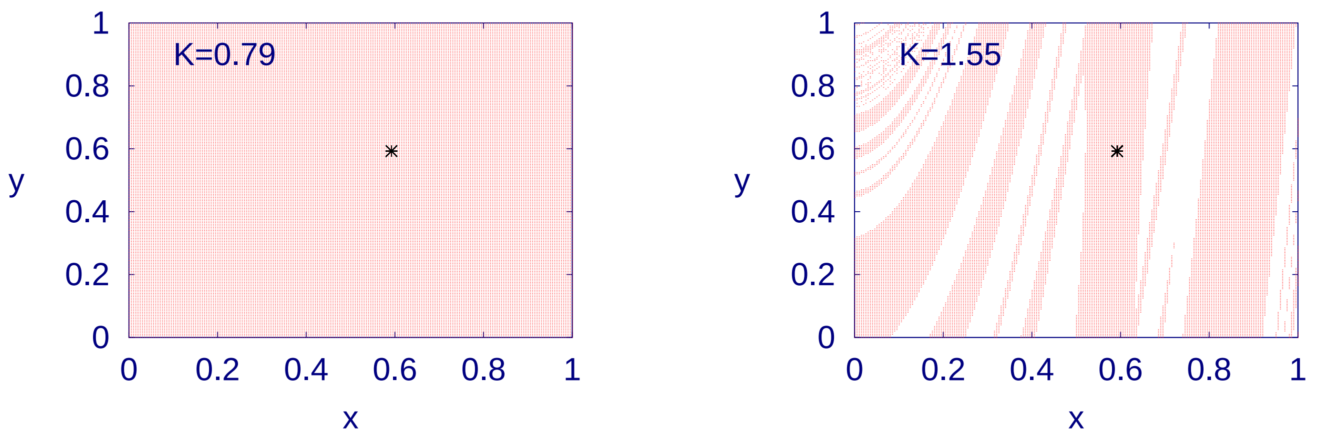
Hopf: super/sub–critical transition possible

# Simulation (Henon map with extended control)

upper control threshold $K_{ho}$




basin of attraction ($F_0 = 0$)



## 6 Outlook

– spatially extended systems
spatio–temporal delay, pattern selection, transport, . . .

– global properties of delay systems
beyond (linear) stability analysis, manifolds, dimensions, visualisation, . . .

– noise & delay
tunnelling with delay (→ Tsimring et.al., PRL **87**, 250602, '01)
time scales, synchronisation, . . .