Using the Lovász Local Lemma in the configuration model

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When none of the events happen

Assume that $A1, A2, \ldots, An$ are events in a probability space Ω. How can we infer ?

If Ai's are mutually independent, $P(Ai)<1$, then

If , then

A way to combine arguments:

Assume that $A1, A2, \ldots, An$ are events in a probability space Ω.
Graph G is a dependency graph of the events $A1, A2, \ldots, An$, if $V(G)=\{1,2, \ldots, n\}$ and each Ai is independent of the elements of the event algebra generated by

Lovász Local Lemma (Erdős-Lovász 1975)

Assume G is a dependency graph for $A1,A2,\ldots,An$, and d=max degree in G
If for $i=1,2,\ldots,n$, $P(Ai)<p$, and $e(d+1)p<1$, then

Lovász Local Lemma (Spencer)

Assume G is a dependency graph for $A1, A2, \ldots, An$
If there exist $x1, x2, \ldots, xn$ in $[0,1)$ such that

then

Negative dependency graphs

Assume that $A1, A2, \ldots, An$ are events in a probability space Ω.
Graph G with $V(G)=\{1,2,\ldots,n\}$ is a negative dependency graph for events $A1, A2, \ldots, An$, if

implies

LLL: Erdős-Spencer 1991, Albert-Freeze-Reed 1995, Ku

Assume G is a negative dependency graph for $A1,A2,\ldots,An$, exist $x1,x2,\ldots,xn$ in $[0,1)$ such that, , then

Setting $xi=1/(d+1)$ implies the uniform version both for dependency and negative dependency

Needle in the haystack

LLL has been in use for existence proofs to exhibit the existence of events of tiny probability. Is it good for other purposes?
Where to find negative dependency graphs that are not dependency graphs?

Poisson paradigm

Assume that $A1, A2, \ldots, An$ are events in a probability space Ω, $p(Ai)=pi$. Let X denote the sum of indicator variables of the events. If dependencies are rare, X can be approximated with Poisson distribution of mean $\Sigma\, pi$.

X~Poisson means using $k=0$,

Two negative dependency graphs

H is a complete graph K_N or a complete bipartite graph $K_{N,L}$; Ω is the uniform probability space of maximal matchings in H. For a partial matching M, the canonical event

Canonical events A_M and A_{M^*} are in conflict: M and M^* have no common extension into maximal matching, i.e.

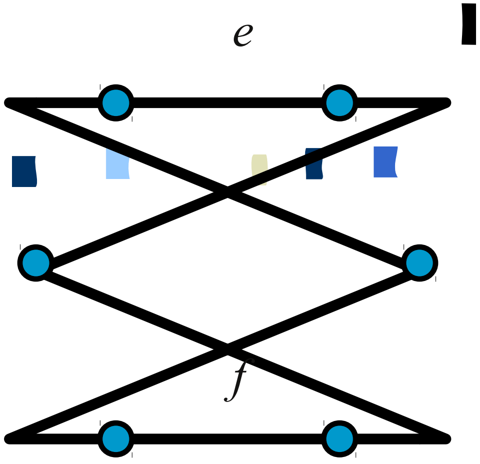
Main theorem

For a graph $H=K_N$ or $K_{N,L}$, and a family of canonical events, if the edges of the graph G are defined by conflicts, then G is a negative dependency graph.

This theorem fails to extend for the hexagon $H=C_6$

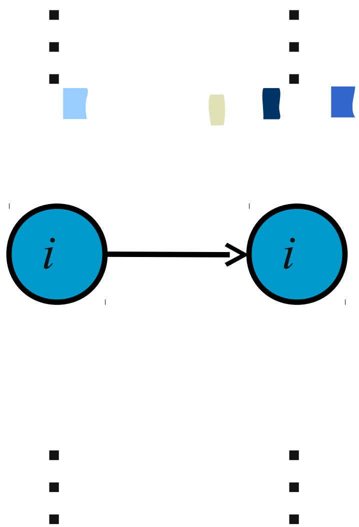
Hexagon example

Two perfect matchings

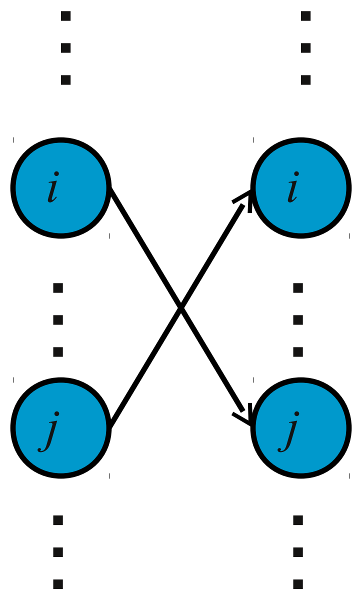


Relevance for permutation enumeration problems

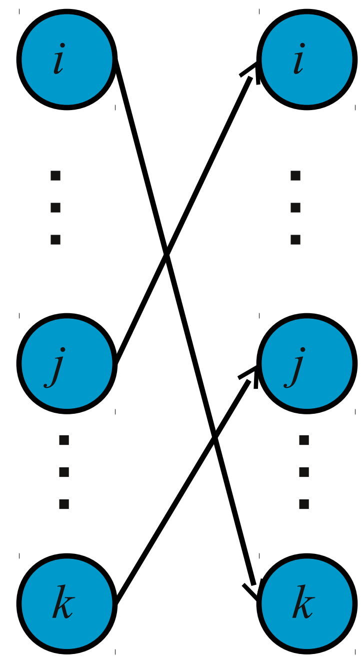
Derangements avoid:



2-cycle free avoids:



3-cycle free avoids:



ε-near-positive dependency graphs

Assume that $A1,A2,\ldots,An$ are events in a probability space Ω.
Graph G with $V(G)=\{1,2,\ldots,n\}$ is an ε–near-positive dependency graph of the events $A1,A2,\ldots,An$,

Main asymptotic theorem (conditions)

M is a set of partial matchings in K_N $(2|N)$ or $K_{N,L}$ $(N \le L)$;
M is antichain for inclusion

r is the size of the largest matching from M

Main asymptotic theorem (conditions continued)

with

for all

Main asymptotic theorem - conclusion

Consequences for permutation enumeration

For k fixed, the proportion of k-cycle free permutations is

(Bender 70's) If $\max K$ grows slowly with n, the proportion of permutations free of cycles of length from set K is

Enumeration of labeled d-regular graphs

Bender-Canfield, independently Wormald 1978: d fixed, nd even

Configuration model (Bollobás 1980)

Put nd (nd even) vertices into n equal clusters
Pick a random matching of Knd
Contract every cluster into a single vertex getting a multigraph or a simple graph
Observe that all simple graphs are equiprobable

Enumeration of labeled regular graphs

Bollobás 1980: *nd* even,

McKay 1985: for

Enumeration of labeled regular graphs

McKay, Wormald 1991: nd even,

Wormald 1981: fix $d \geq 3$, $g \geq 3$ girth

Theorem (from main)

In the configuration model, if $d \geq 3$ and $g6\ (d-1)2g-3=o(n)$, then the probability that the resulting random d-regular multigraph after the contraction has girth at least g, is

hence the number of d-regular graphs with girth at least g is

McKay, Wormald, Wysocka [2004]

Our condition is slightly stronger than in McKay, Wormald, Wysocka [2004]:

$(d-1)2g-3 \; = o(n)$

Configuration model for degree sequences (Bollobás 1980)

Put $N=d1+d2+\ldots+dn$ (even sum) vertices into n clusters, $d1\leq d2\leq\ldots\leq dn$

Pick a random matching of K_N

Contract every cluster into a single vertex getting a multigraph or a simple graph

Observe that all simple graphs are equiprobable

New Theorem (hypotheses)

For a sequence $x1,\ldots,xn$, set

Assume $d1\geq 1$, , set $Dj=dj(dj-1)$ and

New Theorem (conclusion) (McKay and Wormald 1991 without girth condition)

Then the number of graphs with degree sequence $d_1 \le d_2 \le \ldots \le d_n$ and girth at least $g \ge 3$ is

More general results hold:

For excluded sets of cycles (instead of excluding all short cycles)
Also for bipartite degree sequences

Classic Erdős result with the probabilistic method:

There are graphs with girth $\geq g$ and chromatic number at least k, for any given g and k.

Turning the Erdős result universal from existential:

In the configuration model, assume $d_1 \geq 1$,
k fixed,

and

Then almost all graphs with degree sequence $d_1 \leq d_2 \leq \ldots \leq d_n$ and girth at least $g \geq 3$ are not k-colorable.

Recall:

For a graph $H=K_N$ or $K_{N,L}$, and a family of canonical events, if the edges of the graph G are defined by conflicts, then G is a negative dependency graph.

This theorem fails to extend for the hexagon $H=C_6$

A slightly stronger result (Austin Mohr)

Assume r divides N. For a hypergraph $H=KN(r)$, and a family of canonical events, if the edges of the graph G are defined by conflicts, then G is a negative dependency graph.

A conjecture (Austin Mohr)

Ω is the uniform probability space of partitions of a set H. For a set of disjoint subsets M of H, the canonical event is

Canonical events A_M and A_{M^*} are in conflict: M and M^* have no common extension into a partition

CONJECTURE: For a family of canonical events, if the edges of the graph G are defined by conflicts, then G is a negative dependency graph.

A theorem for spanning trees

Ω is the uniform probability space of spanning trees in *KN*. For a circuit-free set of edges *M*, the canonical event

Canonical events *AM* and *AM** are in conflict:

Spanning tree theorem (with Austin Mohr)

For a family of canonical events, if the edges of the graph G are defined by conflicts, then G is a negative dependency graph.

van der Waerden conjecture – Egorychev-Falikman theorem 1981

For a non-negative doubly stochastic nxn matrix A, permanent(A) is minimized, if $a_{ij}=1/n$.

If all $a_{ij}=1/n$ then

Is there a negative dependency graph for doubly stoch. matrices?

Using the doubly stoch. matrix $A=(aij)$, define a random function π on $[n]$ by selecting $\pi(i)$ independently for each i, with

Define event Bi by

is the event that π is a permutation.

Note that

Do the events $B1,\ldots,Bn$ define an edgeless negative dependency graph?

Is there a negative dependency graph for doubly stoch. matrices?

If the events $B1,\ldots,Bn$ define an edgeless negative dependency graph, then

If $aij=1/n$, the events $B1,\ldots,Bn$ define an edgeless negative dependency graph and $(e\text{-}o(1))-n < Permanent(A)$

Leonid Gurvits: not always negative dependency graph