Robust large-scale properties in networks

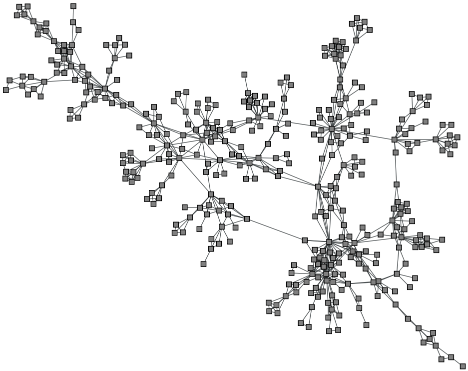
Stochastic blockmodels and their applications

Tiago P. Peixoto

Universität Bremen

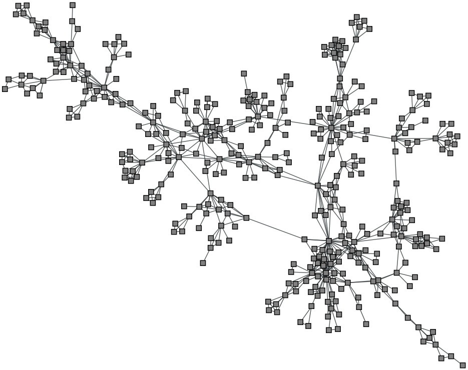
Dresden, May 2012

Levels of network description

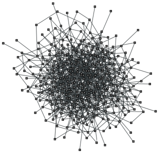


Real Network
(Network scientists)

LEVELS OF NETWORK DESCRIPTION

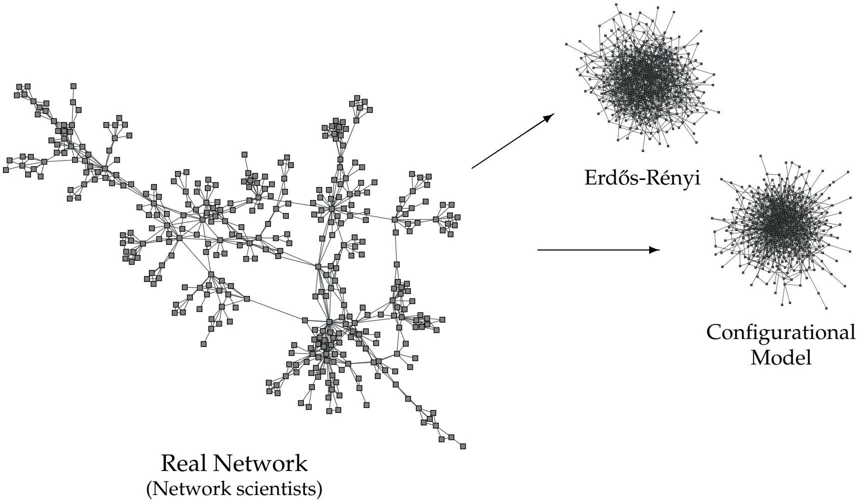


Real Network
(Network scientists)

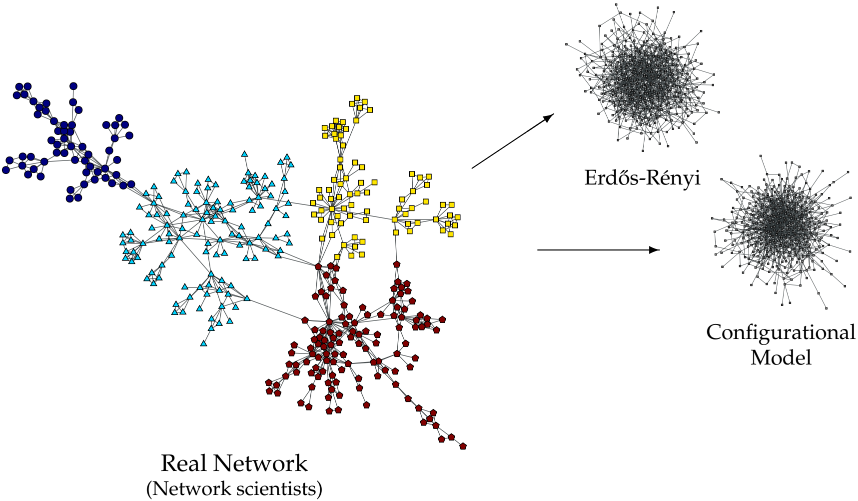


Erdős-Rényi

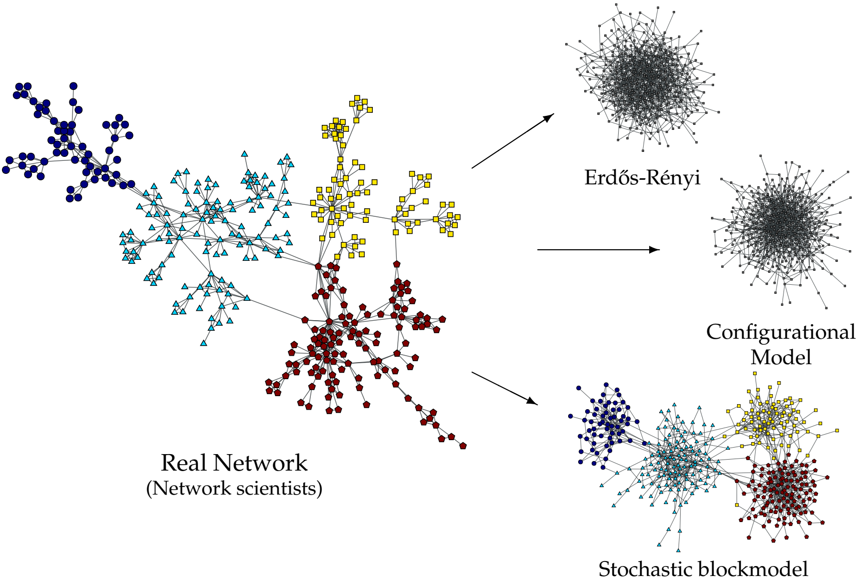
LEVELS OF NETWORK DESCRIPTION



LEVELS OF NETWORK DESCRIPTION



LEVELS OF NETWORK DESCRIPTION



Stochastic blockmodel

B node groups ("blocks")

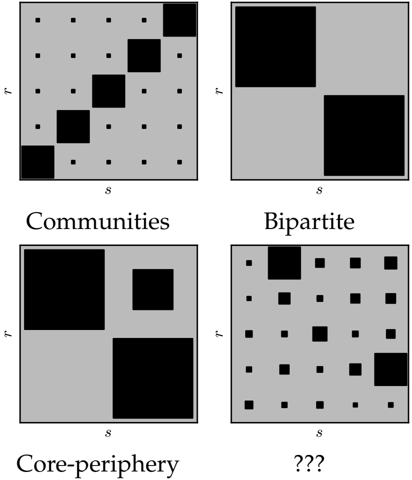
- $n_r \rightarrow$ size of block r
- $e_{rs} \rightarrow$ number of edges between blocks r and s
- $p_k^r \rightarrow$ degree distribution of block r

$$\kappa_r = \sum_s e_{rs}/n_r, \qquad e_r = \sum_s e_{rs}$$

P.W. Holland, K.B. Laskey, and S. Leinhardt, Social Networks 5, 109 (1983)
B. Karrer and M. E. J. Newman, PRE 83, 016107 (2011)

Not community structure!

Or better, not *only* community structure...



Analytical tractability

Suitable for analytical calculations

Heterogeneous mean-field approximations, block averages, etc.

Microcanonical entropy, $\mathcal{S} = \ln \Omega$

$$\mathcal{S} = -E - \sum_k N_k \ln k! - \frac{1}{2} \sum_{rs} e_{rs} \ln \left(\frac{e_{rs}}{e_r e_s} \right)$$

T. P. Peixoto, "Entropy of stochastic blockmodel ensembles" arXiv:1112.6028

- Percolation (size of giant component)
- Dynamical stability of Boolean dynamics

What is the most robust large scale structure?

Network evolution

Exponential random graph approach

Main objective: *null model* for robustness

Ensemble of networks in thermodynamic equilibrium

$i \rightarrow$ network realization , $R_i \rightarrow$ robustness

Obtain a desired robustness, but otherwise maximize entropy.

$$\pi_i \propto e^{\beta N R_i}$$

$\beta \rightarrow$ selective pressure

$\beta = 0$ (fully random) $\beta \rightarrow \infty$ (optimal topology)

Free energy minimization

$$\boxed{\mathcal{F} = -NR - \mathcal{S}/\beta}$$

$$\mathcal{S} = -N\sum_k p_k \ln k! - \frac{1}{2}\sum_{rs} e_{rs} \ln\left(\frac{e_{rs}}{e_r e_s}\right)$$

$R \to$ Heterogeneous mean-field

Constraint: $\langle k \rangle$ kept fixed (edges are expensive)

PERCOLATION

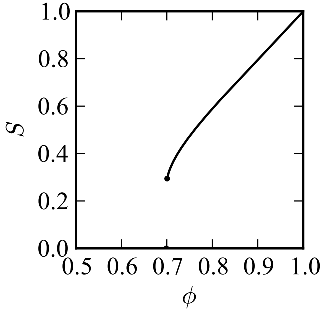
$S \to$ size of macroscopic component

$\phi \to$ dilution (random or targeted)

$$\boxed{R = 2\int_0^1 S(\phi)d\phi}$$

$$R \in [0, 1]$$

Interdependence! 



Buldyrev et al. Nature 2010

T. P. Peixoto, S. Bornholdt, arXiv:1205.2909

PERCOLATION

GENERATING FUNCTION FORMALISM

$$u_r = \sum_s m_{rs}\left[1 - \hat{\phi}_s f_1^s(1) + \hat{\phi}_s f_1^s(u_s)\right]$$

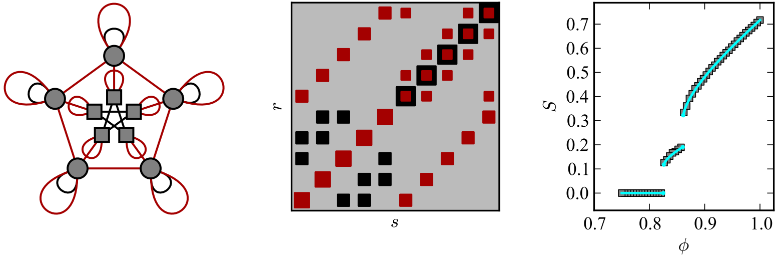
$$\hat{\phi}_r = 1 - \hat{f}_0^r(1 - \textstyle\sum_s \hat{m}_{rs} S_s^0) + \hat{f}_0^r(0)$$

$u_r \rightarrow$ prob. not in GC via neighbor
$\hat{\phi}_r \rightarrow$ prob. has not failed via dependency neighbor
$m_{rs} = e_{rs}/n_r\kappa_r, \quad \hat{m}_{rs} = \hat{e}_{rs}/n_r\hat{\kappa}_r,$

$S_r = \hat{\phi} S_r^0, \quad S_r^0 = f_0^r(1) - f_0^r(u_r)$
$S = \sum_r w_r S_r, \quad w_r = n_r/N, \quad \phi = \sum_{r,k} w_r p_k^r \phi_k^r$

T. P. Peixoto, S. Bornholdt, arXiv:1205.2909

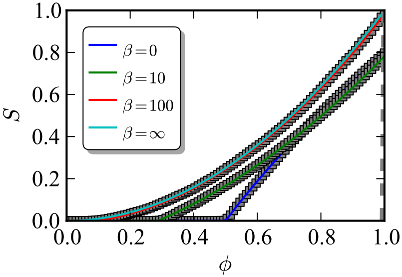
Percolation: Example



Generalization of two-interdependent networks, “network of networks”, etc.

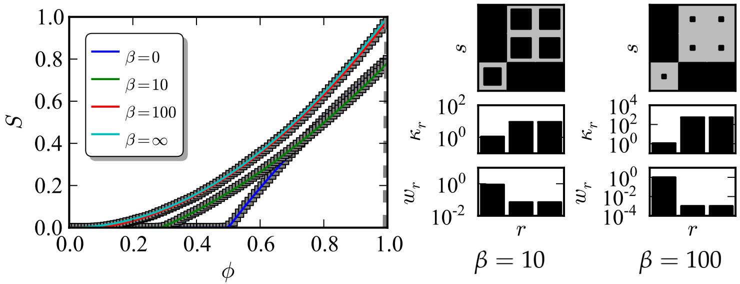
T. P. Peixoto, S. Bornholdt, arXiv:1205.2909

PERCOLATION: OPTIMIZATION



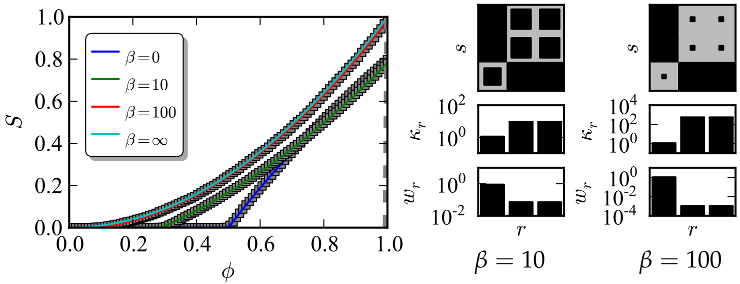
T. P. Peixoto, S. Bornholdt, arXiv:1205.2909

Percolation: Optimization



T. P. Peixoto, S. Bornholdt, arXiv:1205.2909

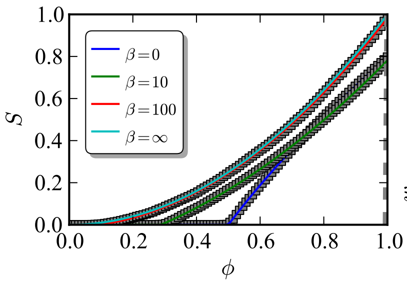
Percolation: Optimization

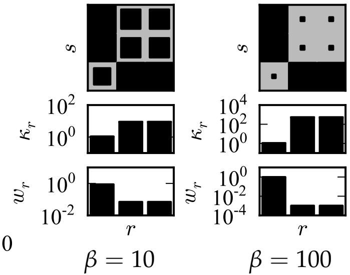


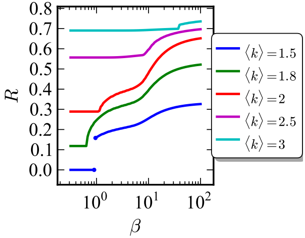
- Core-periphery structure!
- Independent of the number of blocks, B!

T. P. Peixoto, S. Bornholdt, arXiv:1205.2909

Percolation: Optimization



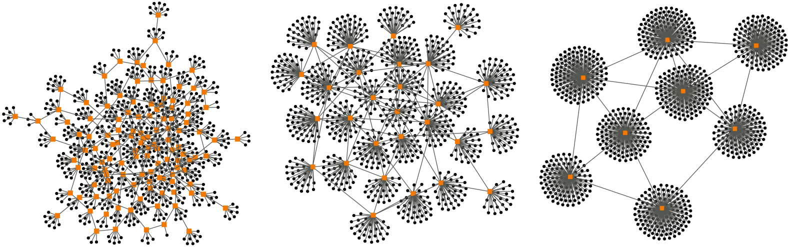




- Core-periphery structure!
- Independent of the number of blocks, B!

T. P. Peixoto, S. Bornholdt, arXiv:1205.2909

Percolation: Optimization



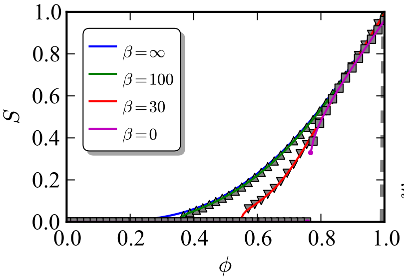
$\beta = 10$ $\beta = 20$ $\beta = 40$

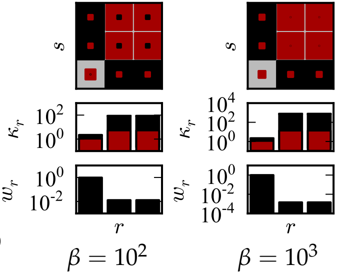
Backbones...

T. P. Peixoto, S. Bornholdt, arXiv:1205.2909

Percolation: Optimization

With interdependence

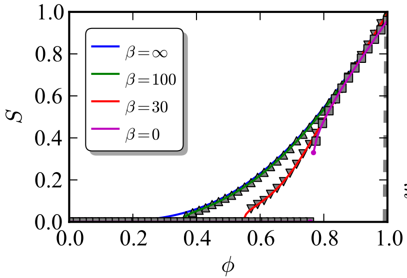


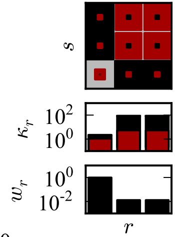


T. P. Peixoto, S. Bornholdt, arXiv:1205.2909

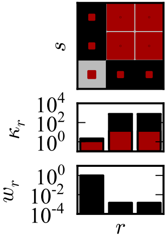
Percolation: Optimization

With interdependence

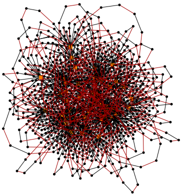




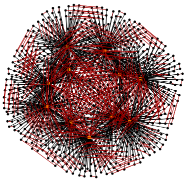
$\beta = 10^2$



$\beta = 10^3$



$\beta = 10$



$\beta = 30$

T. P. Peixoto, S. Bornholdt, arXiv:1205.2909

PERCOLATION: OPTIMIZATION

TARGETED ATTACKS?

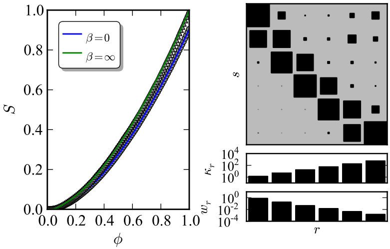
How about targeted attacks?

Fully random topologies are almost always better...

PERCOLATION: OPTIMIZATION

DEGREE CONSTRAINTS

Random failure

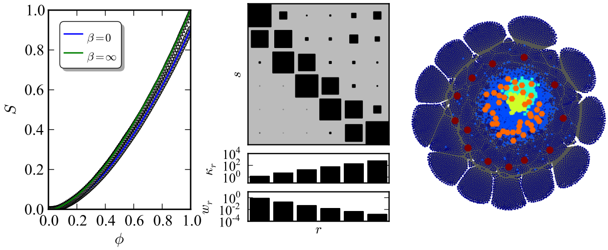


T. P. Peixoto, S. Bornholdt, arXiv:1205.2909

Percolation: Optimization

Degree constraints

Random failure

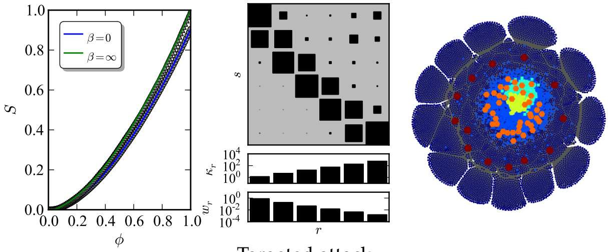


T. P. Peixoto, S. Bornholdt, arXiv:1205.2909

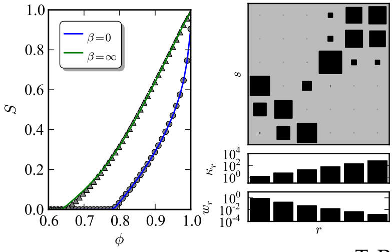
Percolation: Optimization

Degree constraints

Random failure



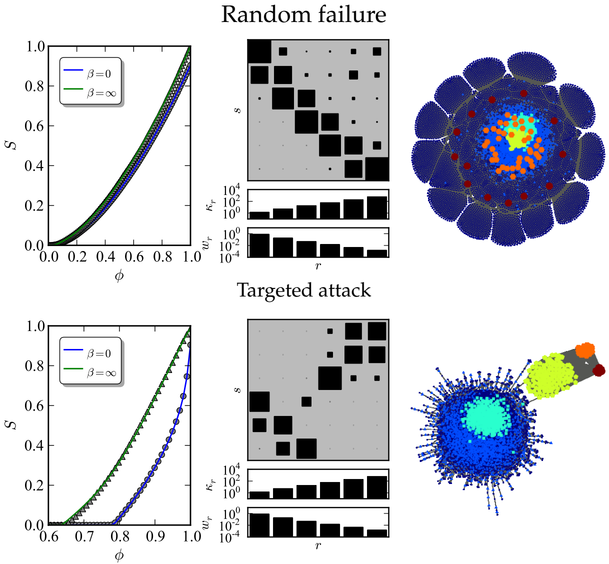
Targeted attack



T. P. Peixoto, S. Bornholdt, arXiv:1205.2909

PERCOLATION: OPTIMIZATION

DEGREE CONSTRAINTS



"Onionlike" topology
Schneider et at.
PNAS 2011

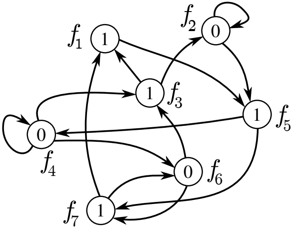
T. P. Peixoto, S. Bornholdt, arXiv:1205.2909

DYNAMICAL ROBUSTNESS AGAINST NOISE

BOOLEAN NETWORKS AND GENE REGULATION

$$\sigma_i(t+1) = f_i\left(\boldsymbol{\sigma}(t)\right)$$

Noise P



σ_4	σ_6	f_3
0	0	1
0	1	0
1	0	0
1	1	1

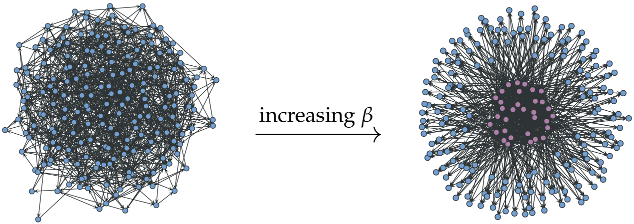
$$b_r(t+1) = \sum_k p_k^r m_k \left((1-2P) \sum_s w_{s\to r} b_s(t) + P \right)$$

$$\boxed{R = -\sum_r w_r b_r(\infty)}$$

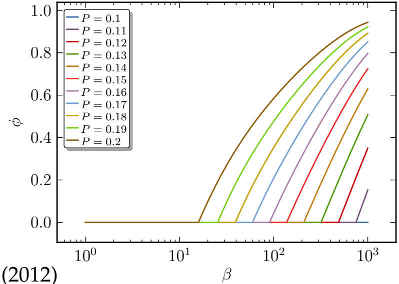
T. P. Peixoto, PRE 85, 041908 (2012)

Dynamical robustness against noise

Boolean networks and gene regulation



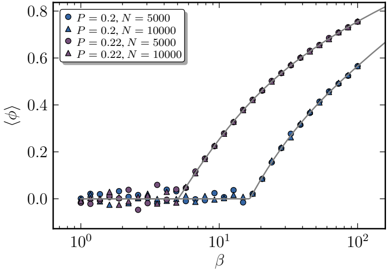
Random topology Segregated core

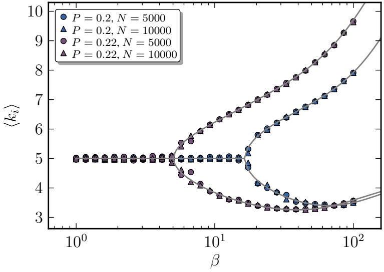


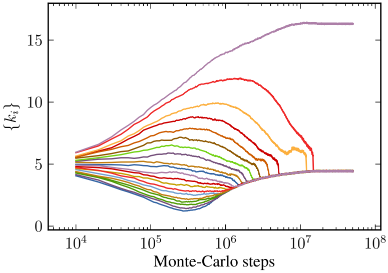
T. P. Peixoto, PRE 85, 041908 (2012)

Dynamical robustness against noise

Boolean networks and gene regulation



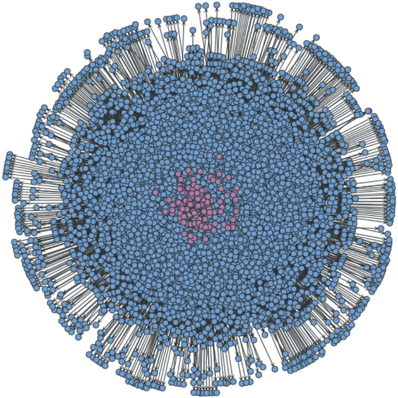




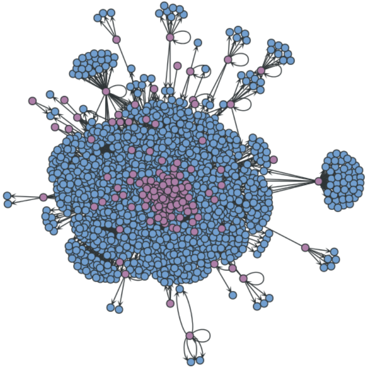
T. P. Peixoto, PRE 85, 041908 (2012)

Dynamical robustness against noise

Boolean networks and gene regulation



Saccharomyces cerevisiae



Escherichia coli

T. P. Peixoto, PRE 85, 041908 (2012)

Conclusion

- Stochastic blockmodels are useful.
- Robust topologies are simple!