Mesoscale Structures and the Laplacian Spectra of Random Geometric Graphs

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Graph Spectra

Graph Laplacian, $\mathcal{L} = \mathcal{L}(G) = D-A$

$$\ell_{ij} = \begin{cases} k_i & \text{if } i = j,\ k_i \text{ is the degree of node } i \\ -1 & \text{if } i \neq j \text{ and } i \text{ and } j \text{ are connected} \\ 0 & \text{otherwise} \end{cases}$$

What connections can be made between a network's structure and it spectrum?

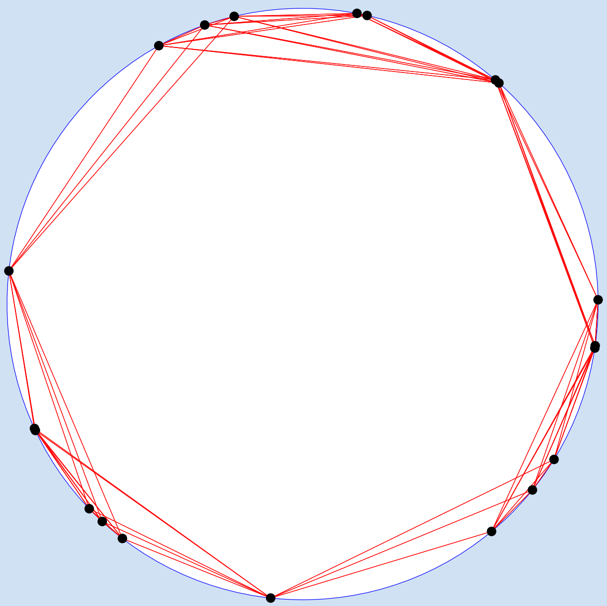
- The multiplicity of $\lambda = 0$ is equal to the number of components of G.
- The sum of the eigenvalues is twice the number of edges.
- Some dynamical properties of the graph can be related to eigenvalues of the Laplacian

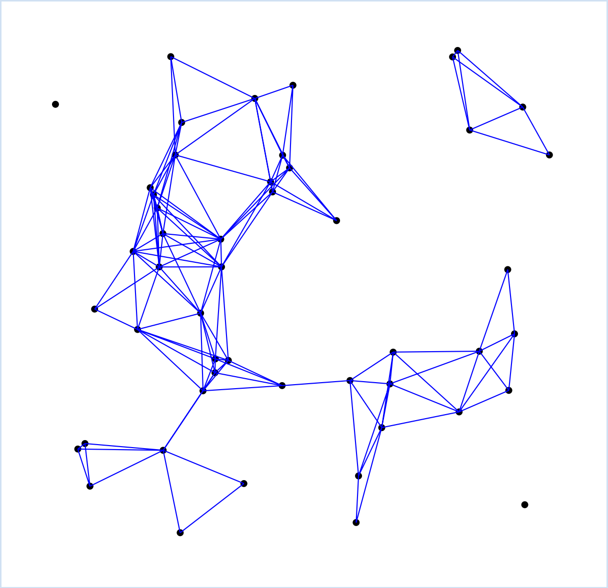
In Random Geometric Graphs, certain mesoscale structures corresponding to graph automorphisms produce integer eigenvalues and account for a third of the entire spectrum.

Random Geometric Graphs

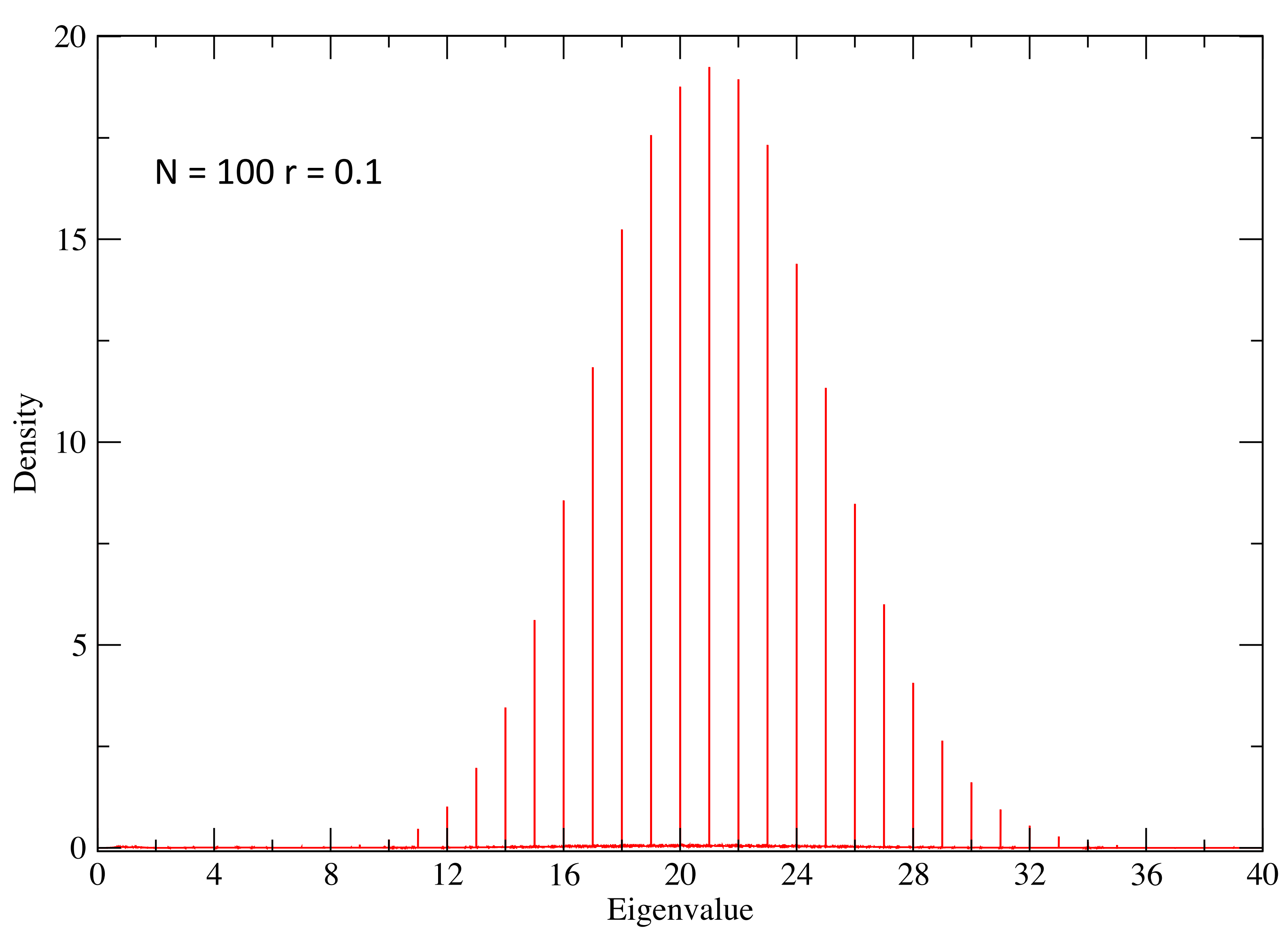
Nodes are placed randomly in space according to a specified probability distribution and connected if their distance is less than a certain connectivity threshold, r.

Random Geometric Graphs are used to model wireless networks, transportation grids, and biological processes.

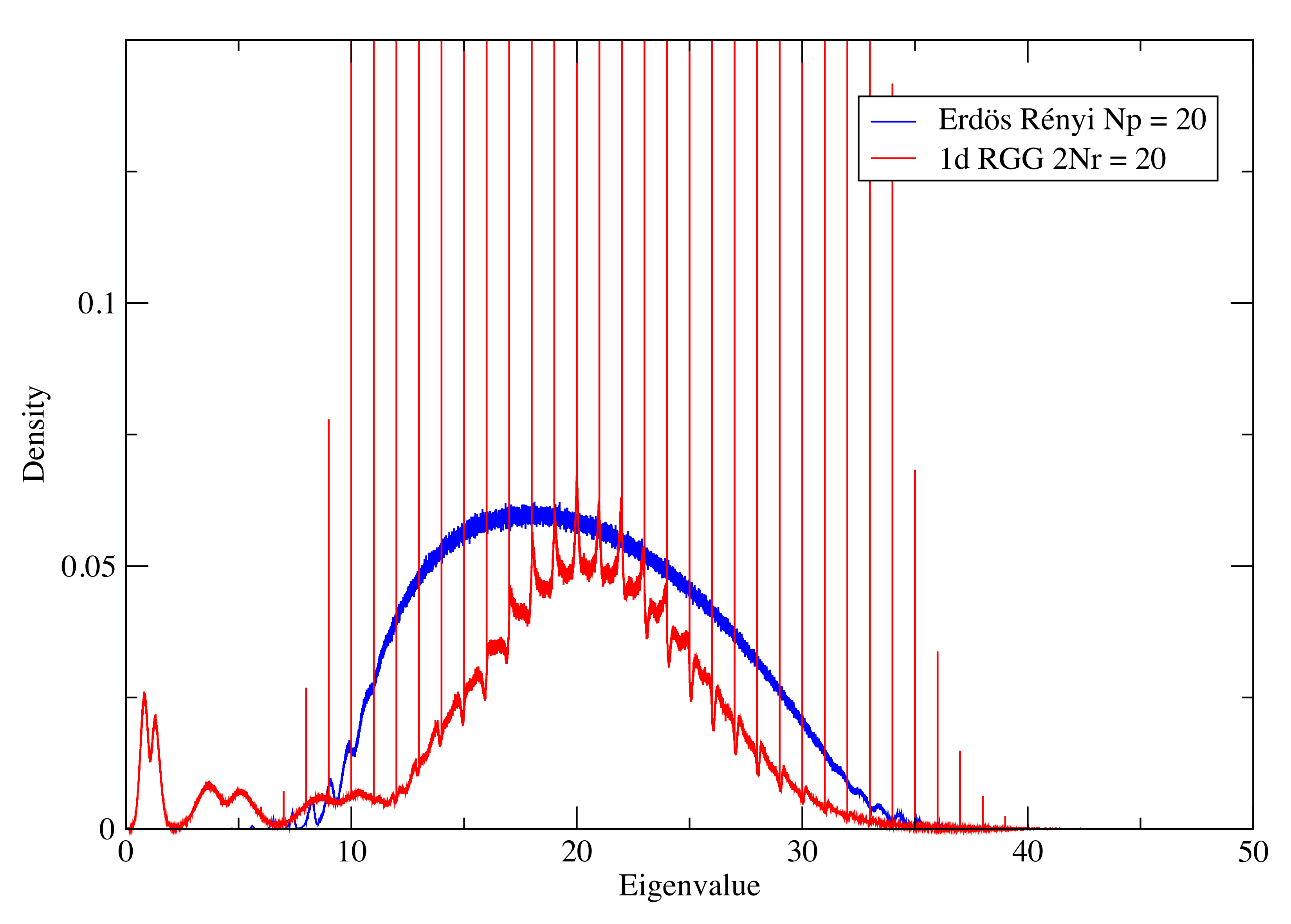




The Laplacian Spectrum

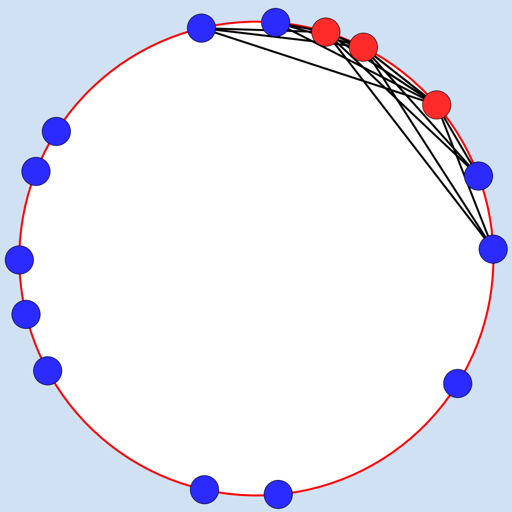


The Laplacian Spectrum



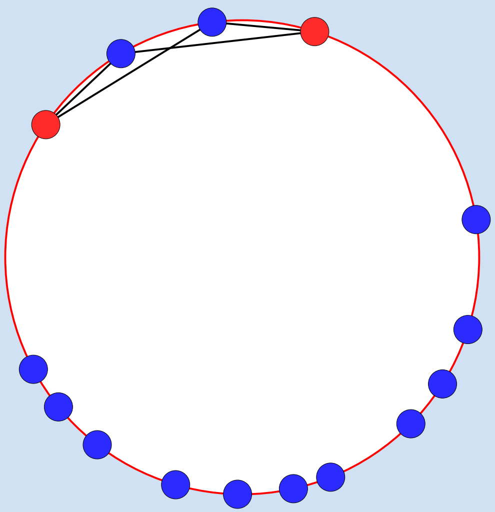
Orbits Produce Integer Eigenvalues

Mesoscale structures consisting of nodes that share the same neighbors produce integer eigenvalues.



Type I

n nodes, of degree k , connected to each other that have the same k-n+1 neighbors will give λ = k+1 with multiplicity $\geq$ n-1



Type II

n nodes, NOT connected to each other, that have the same k neighbors will give λ = k with multiplicity $\geq$ n-1

R. Grone, R. Merris, SIAM J. Discrete Math. Vol. 7, No. 2 (1994)

Orbits Produce Integer Eigenvalues

Consider an eigenvector $\mathbf{x}$ that is +1 on one node in the orbit $\mathbf{x}_A = 1$

-1 on another node in the orbit $\mathbf{x}_B = -1$

0 elsewhere

$$\mathcal{L}\mathbf{x} = \lambda\mathbf{x}$$

$$\begin{aligned}(L\mathbf{x})_A &= \sum_{i=1}^{N} L_{Ai}\mathbf{x}_i \\ &= k(1) + (-1)(-1) + 0 + \cdots + 0 \\ &= (k+1)\mathbf{x}_A\end{aligned}$$

$$\begin{aligned}(L\mathbf{x})_B &= \sum_{i=1}^{N} L_{Bi}\mathbf{x}_i \\ &= (-1)(1) + k(-1) + 0 + \cdots + 0 \\ &= (k+1)\mathbf{x}_B\end{aligned}$$

There are n-1 such independent and orthogonal eigenvectors.

These eigenvalues do not appear in the spectrum of the network's quotient graph.

Calculating the Contribution from Type I Orbits

$$E(k) = \frac{N(N-1)}{2}\int_0^R dx P(x) P_{k-2\in N_s}(x,R,N) P_{0\in N_{ex}}(x,R,N)$$

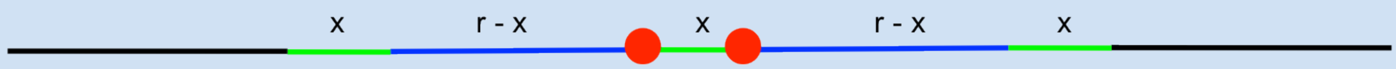
the probability that

two nodes are a distance x apart

k-2 nodes are in the shared region

no nodes are in the excluded region

Type I Orbits Account for a Third of the Spectrum



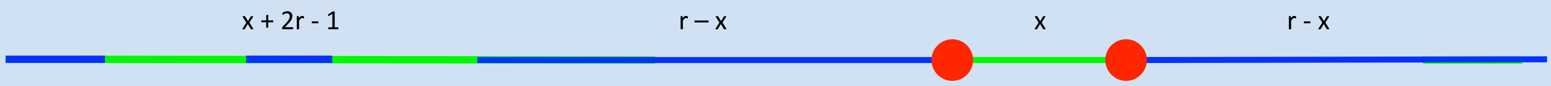
For r < ⅓, $$E(k) = \frac{N! r (2r)^{k-2} (1-2r)^{N-k}}{(k-1)!(N-k)!} {}_2F_1\left[1, k-N, k, \tfrac{r}{1-2r}\right]$$

For ⅓ < r ≤ ½, $$E(k) = \frac{N! (2r)^{k-2} (1-2r)^{N-k+1}}{(k-2)!(N-k+1)!} {}_2F_1\left[1, 2-k, N-k+2, \tfrac{1-2r}{r}\right]$$

In the limit of large N, Type I orbits account for a third of the spectrum.

$$\sum_{k=2}^{N} E(k) = \begin{cases} \frac{N}{3}\left[1-(1-3r)^{N-1}\right] & 0 \le r \le \frac{1}{3} \\ \frac{N}{3}\left[1-(6r-2)^{N-1}\right] & \frac{1}{3} \le r \le \frac{1}{2} \end{cases}$$

Eigenvalue Condensate



For ⅓ ≤ r ≤ ½ and 1 – 2r < x ≤ r, there is an extra contribution to the eigenvalue λ = N.

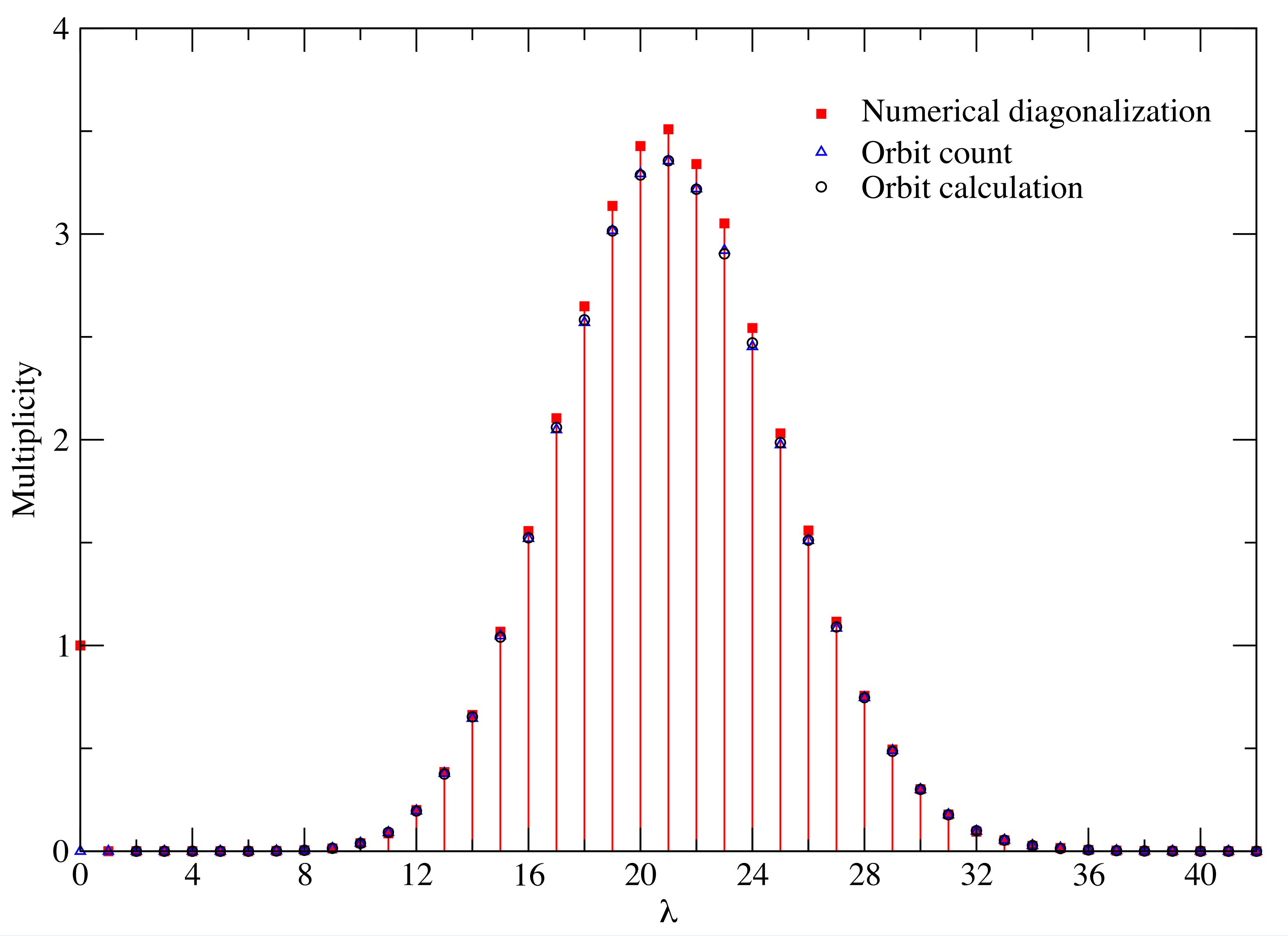
$$E^*(N) = N\left(2^{N-1} - 1\right)(3r-1)^{N-1}$$

In the limit of large N, with Nr constant, the contribution goes to zero.

However, in the limit r➔½ the eigenvalues collect at λ = N.

$$\lim_{r\to\frac{1}{2}} \frac{1}{N} E^*(N) = 1 - \left(\tfrac{1}{2}\right)^{N-1}$$

Analytical and Numerical Comparison



Conclusion

- In spatial networks the the discrete part of the Laplacian spectrum consists of a much larger fraction of the eigenvalues than in non-spatial networks.
- We have identified mesoscale structures that allow graph automorphisms and result in integer eigenvalues.
- We have analytically calculated the expected number of integer eigenvalues due to these structures.
- In the large scale limit in 1d RGGs, these eigenvalues account for a third of the spectrum.

This work is supported by NSF Grant No. DMR - 0908286