Time-scale and noise optimality in self-organized critical adaptive networks

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Self-Organized Criticality (SOC)

- Adaptive networks with simple local rules can self-organize.
  (Bassler, Bornholdt, Christensen, Gross, Levina, Meisel, Rohlf,...)

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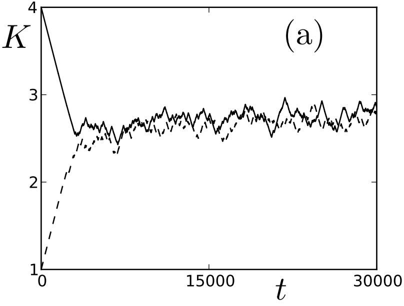

Figure : SOC example. K = average connectivity. C = frozen fraction.

Modified Bornholdt-Rohlf Boolean Network

1. nodes $v_i(t) \in \{\pm 1\}$, directed edges $e_{ij}(t) \in \{-1, 0, +1\}$.
2. dynamical update rule ($t = 0$, random graph), define

$$
\begin{aligned}
f_i(t) &= \sum_j e_{ij}(t) v_j(t) + \mu v_i(t) + \sigma r_i, \;\; \vec{r} \sim \mathcal{N}(0,1) \\
v_i(t+1) &= \begin{cases} \mathrm{sgn}[f_i(t)] & \text{if } f_i(t) \neq 0, \\ v_i(t) & \text{if } f_i(t) = 0. \end{cases}
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3. T_v node dynamics steps, $T_a := \lfloor T_v/2 \rfloor$, measure activity

$$
A_i := \frac{1}{T_v - T_a} \left[\sum_{t=T_a}^{T_v-1} v_i(t) \right].
$$

4. topological update rule, choose one node i randomly

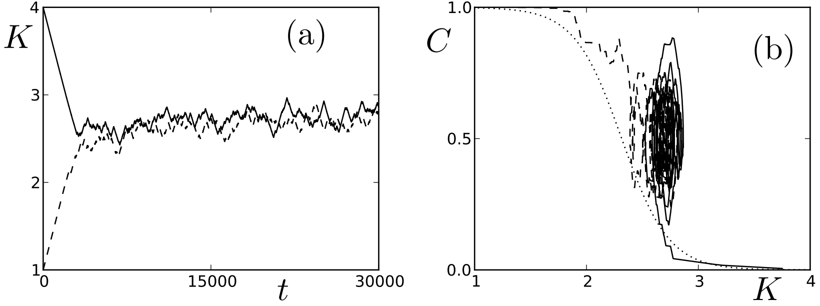
$$
\begin{aligned}
|A_i| &> 1 - \delta & \text{create an edge } e_{ij}(t) \neq 0, \\
|A_i| &\leq 1 - \delta & \text{delete an edge } e_{ij}(t) = 0.
\end{aligned}
$$

SOC Ingredients & Observations

- Large time scale separation $T_v = 1/\epsilon \gg 1$ needed

  topology dynamics $\leftrightarrow$ slow   node dynamics $\leftrightarrow$ fast.

- SOC is robust to small noise $0 < \sigma \ll 1$.



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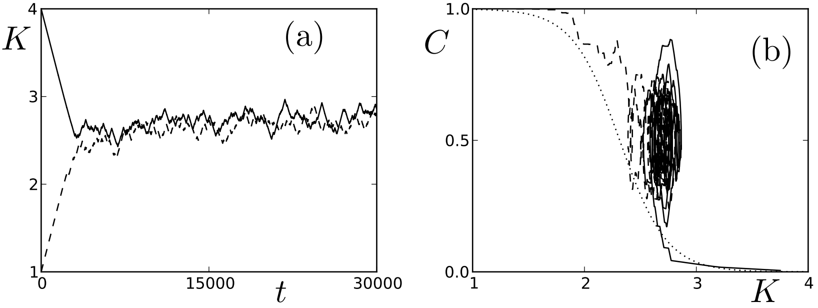

- Steady state near fast subsystem bifurcation point?!
- Information processing $\leftrightarrow$ perturbations $\leftrightarrow$ finite-time.

Are there optimal values of (ϵ, σ)?

$$\text{error} = E_X := \frac{2}{T} \sum_{t=0}^{T/2} |X(t) - \mathcal{X}_T|, \qquad \text{for } X \in \{K, C\}.$$

- $X_T := \langle X(t) \rangle_{[T/2, T]}$ where $\langle \cdot \rangle_{[T/2,T]}$ = time average.
- $\mathcal{X}$ average over 100 initial random graphs.

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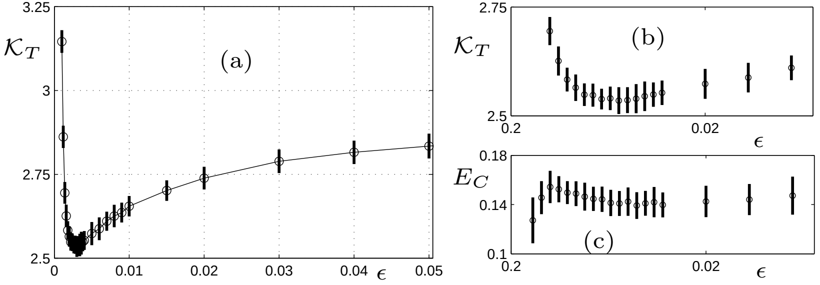

Figure : $T = 60000$ total topological dynamics steps.

Background for Models

Fast variables $x \in \mathbb{R}^m$, slow variables $y \in \mathbb{R}^n$, time scale separation $0 < \epsilon \ll 1$.

$$\left\{ \begin{array}{lcr} \frac{dx}{dt} = x' & = & f(x,y) \\ \frac{dy}{dt} = y' & = & \epsilon g(x,y) \end{array} \right. \quad \overset{\epsilon t = s}{\longleftrightarrow} \quad \left\{ \begin{array}{lcccr} \epsilon \frac{dx}{ds} & = & \epsilon \dot{x} & = & f(x,y) \\ \frac{dy}{ds} & = & \dot{y} & = & g(x,y) \end{array} \right.$$

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fast subsystem

slow subsystem

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fast subsystem — slow subsystem

- $C := \{f = 0\}$ = critical manifold = equil. of fast subsystem.
- C is normally hyperbolic if $D_x f$ has no zero-real-part eigenvalues.
- **Fenichel's Theorem:** Normal hyperbolicity $\Rightarrow$ "nice" perturbation.

Non-Monotonicity and Optimality in ϵ? - A Model...

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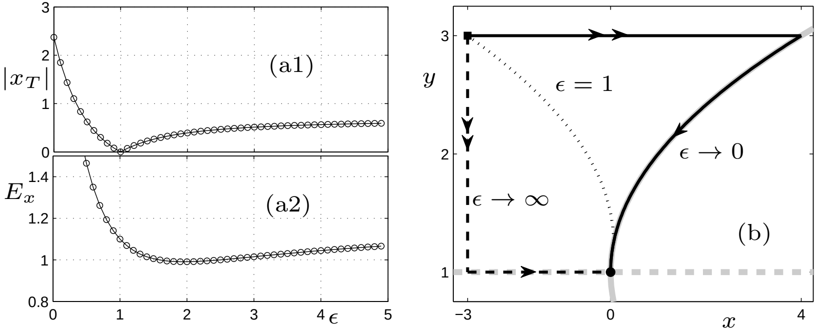

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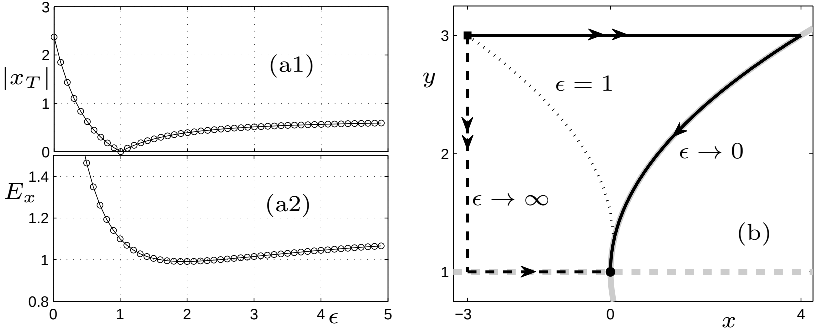
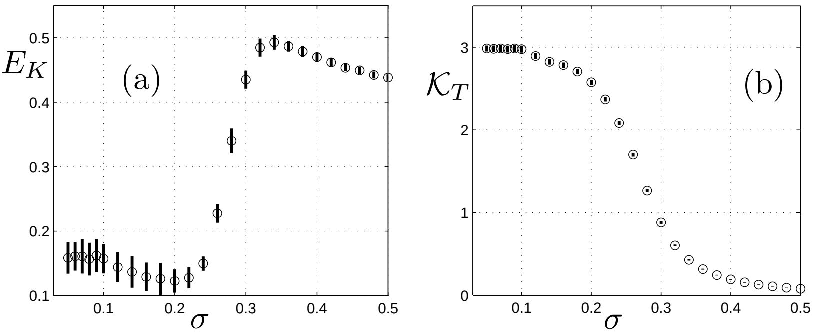


Important (new?) concept - time-scale resonance (TR).

Back to the Bornholdt-Rohlf Model... and Noise



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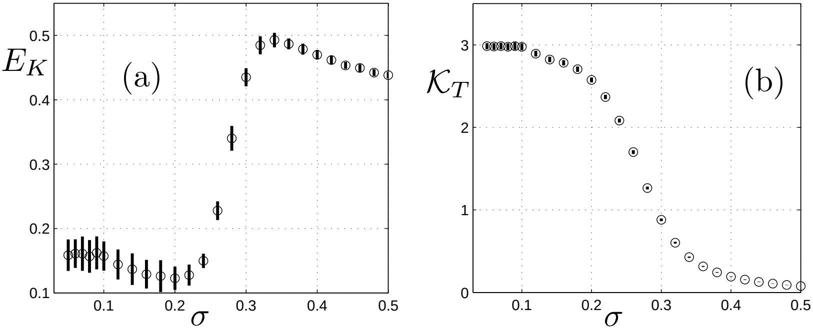

- Non-monotone error, small noise $\rightarrow$ noise optimality.
- SOC tipping, large noise $\rightarrow$ noise-induced phase transition.

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- Second throught: No, since we have SOC steady state.

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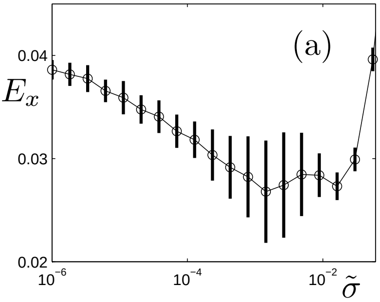

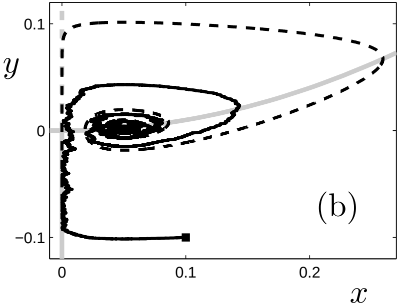

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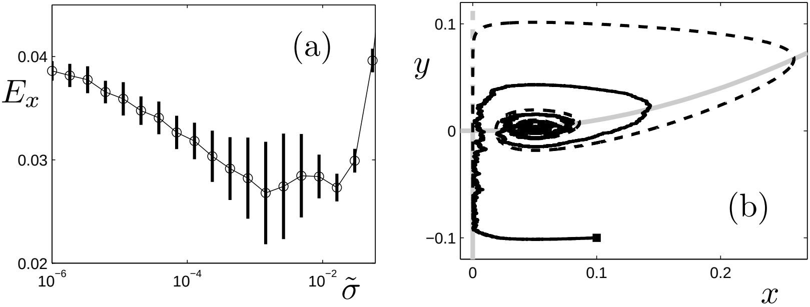

Important concept - steady-state stochastic resonance (SSR).

Overview & Conclusions

Major Effects:

- Time-scale resonance (TR) upon varying ϵ.
- Steady-state stochastic resonance (SSR) upon varying σ.
- Noise-induced phase transition can destroy SOC.

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- Simple models to capture effects in adaptive networks.
- Natural control parameters to self-optimize/control SOC.
- Fast subsystem bifurcation (cf. excitable systems).

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Thank you for your attention.