The Impact of Time Delays in Network Synchronization in a Noisy Environment

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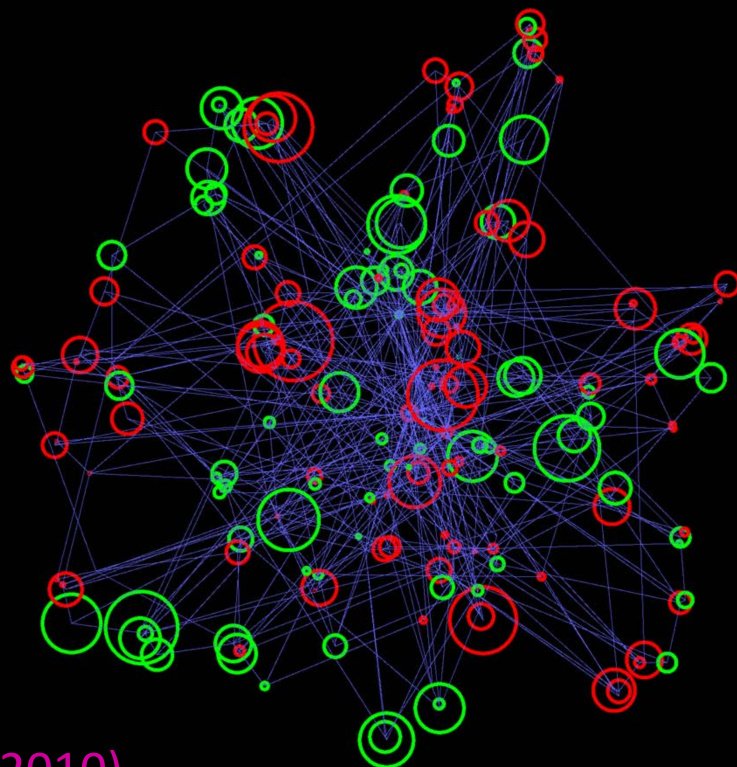

Phys. Rev. Lett. **105,** 068701 (2010)
http://arxiv.org/abs/1209.4240 (2012)

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Hutchinson model
(logistic growth with delay in population dynamics)

$\tau > 0$

$$\partial_t N(t) = rN(t)\left[1 - \frac{N(t-\tau)}{K}\right]$$

population size

intrinsic growth rate

carrying capacity

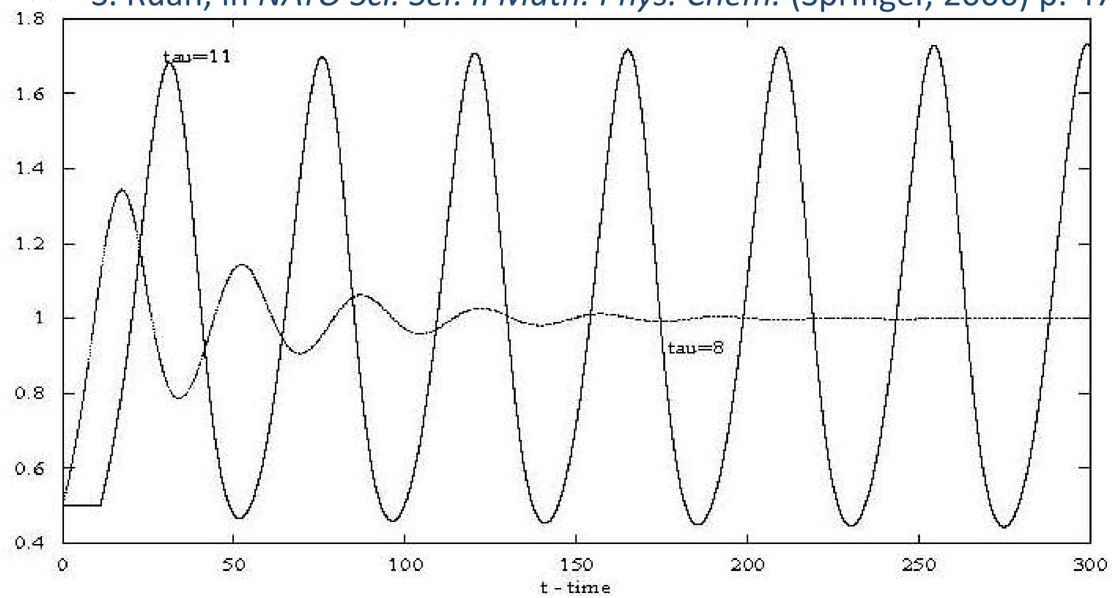
$$(N^* = 0), \quad N^* = K$$

$$N(t) = K + x(t)$$

$$\partial_t x(t) = -rx(t-\tau)$$

stability of $N^* = K$:
$r\tau < \pi/2$

S. Ruan, in *NATO Sci. Ser. II Math. Phys. Chem.* (Springer, 2006) p. 477



Hutchinson (1948); Maynard Smith (1971); R.M. May (1973)

Synchronization/Coordination in Coupled Systems

- individual units or agents (represented by static or mobile nodes) attempt to adjust their local state variables (e.g., pace, load, alignment, coordination) in a decentralized fashion.

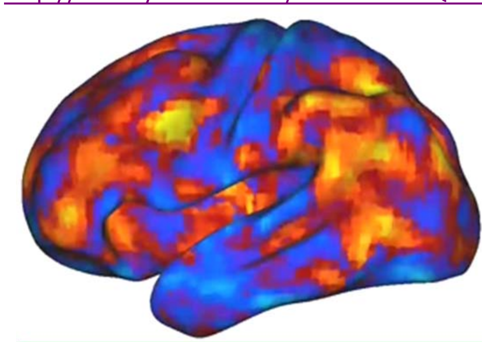
  Craig Reynolds (1987); Vicsek *et al*. (1995); Cavagna *et al*. (2010).

- nodes interact or communicate only with their local neighbors in the network, possibly to improve global performance or coordination.

- nodes react (perform corrective actions) to the information or signal received from their neighbors possibly with some time lag (as result of finite transmission, queuing, processing, or execution delays)
- Applications: autonomous coordination, unmanned aerial vehicles, microsatellite clusters, sensor and communication networks, load balancing, flocking, distributed decision making in social networks

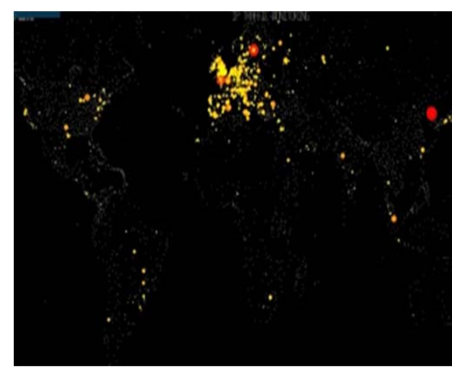
flocking birds

http://www.youtube.com/watch?v=VaQ66lDZ-08



spontaneous brain activity (fMRI)
(Justin Vincent; http://martinos.org/~vincent/)

http://www.youtube.com/watch?v=6AmSpHxnKm8



IP activity
(Zeus load balancer)

Synchronization/Coordination in a Noisy Environment with Time Delays

$$\partial_t h_i(t) = -\sum_j C_{ij}[h_i(t-\tau) - h_j(t-\tau)] + \eta_i(t)$$

network/coupling strength (C_{ij}) delay (τ) noise ($\eta_i(t)$)

spread or width:
(measure of de-coordination):

$$\langle w^2(t)\rangle = \left\langle \frac{1}{N}\sum_{i=1}^{N}[h_i(t) - \bar{h}(t)]^2 \right\rangle$$

$$\partial_t \boldsymbol{h}(t) = -\boldsymbol{\Gamma}\,\boldsymbol{h}(t-\tau) + \boldsymbol{\eta}(t)$$

network Laplacian:

$$\Gamma_{ij} = \delta_{ij} C_i - C_{ij}$$

$$\partial_t \tilde{h}_k(t) = -\lambda_k \tilde{h}_k(t-\tau) + \tilde{\eta}_k(t)$$

$$0 = \lambda_0 \le \lambda_1 \le \lambda_2 \le \ldots \le \lambda_{N-1} = \lambda_{\max}$$

$$\langle w^2(t)\rangle = \frac{1}{N}\sum_{k=1}^{N-1}\langle \tilde{h}_k^2(t)\rangle$$

Coordination, Noise, Time Delay

$$\partial_t h(t) = -\lambda h(t-\tau) + \eta(t)$$

$$\langle \eta(t)\eta(t')\rangle = 2D\delta(t-t')$$

characteristic equation: $(h(t) = ce^{st})$

$$g(s) \equiv s + \lambda e^{-\tau s} = 0$$

$$s_\alpha = s_\alpha(\lambda, \tau), \quad \alpha = 1,2,\ldots$$

infinitely many relaxation "rates", $\{s_\alpha\}$, for $\tau > 0$

$$\langle h^2(t)\rangle = \sum_{\alpha,\beta} \frac{-2D[1-e^{(s_\alpha+s_\beta)t}]}{g'(s_\alpha)g'(s_\beta)(s_\alpha+s_\beta)}$$

synchronizability: $\langle h^2(\infty)\rangle < \infty$

synchronizability condition:

$$\mathrm{Re}(s_\alpha) < 0 \quad \forall \alpha$$

$$\lambda\tau < \pi/2$$

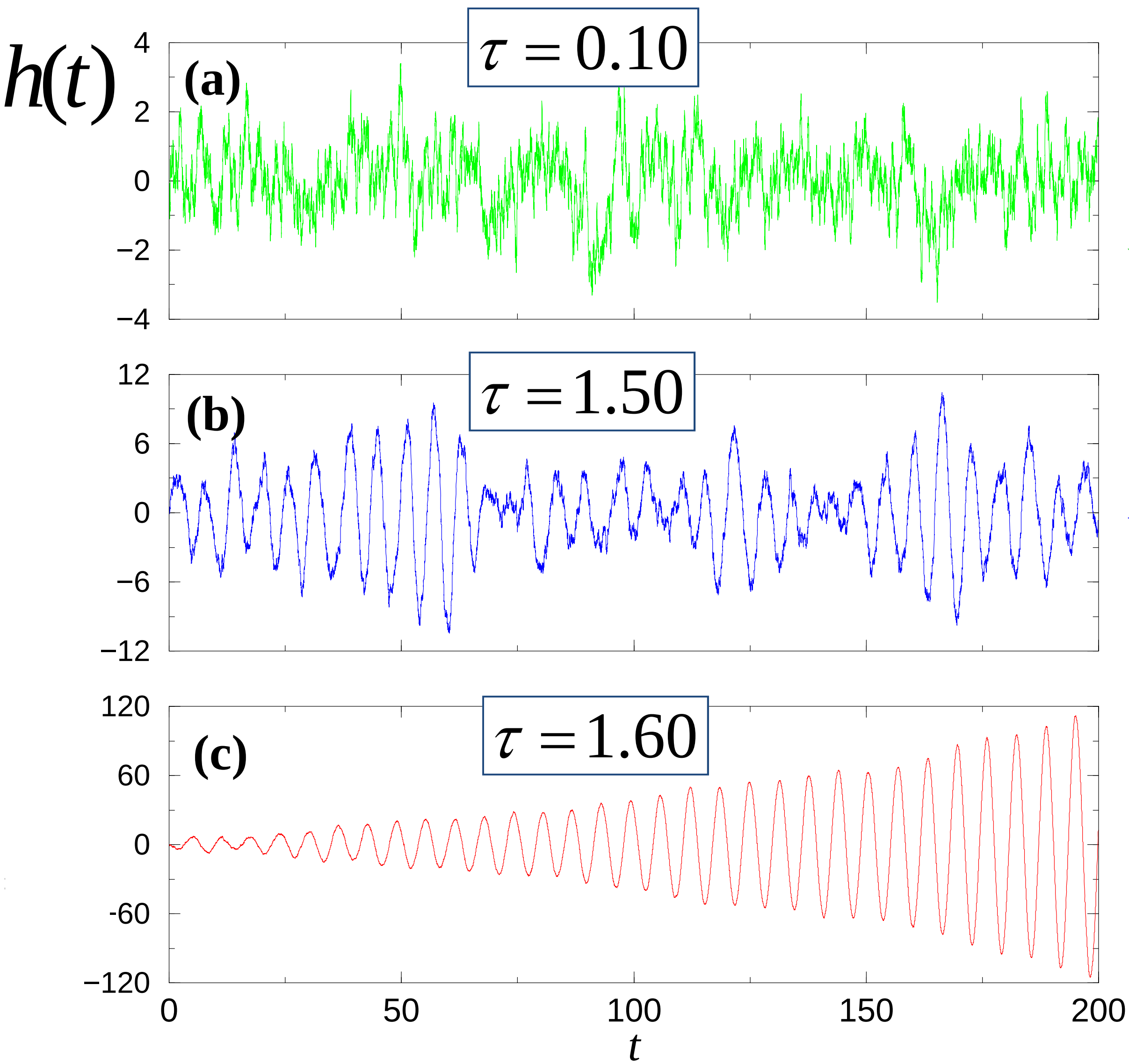
Frisch & Holme (1935); Hayes (1950); Hutchinson (1948); Maynard Smith (1971); R.M. May (1973)

Coordination, Noise, Time Delay

$$(\lambda\tau)_c = \pi/2$$

$$\lambda = 1, D = 1, dt = 0.01$$

$$\tau_c = \pi/2 \approx 1.57$$



$\lambda\tau < \frac{1}{e}$

$\frac{1}{e} < \lambda\tau < \frac{\pi}{2}$

$\frac{\pi}{2} < \lambda\tau$

Coordination, Noise, Time Delay

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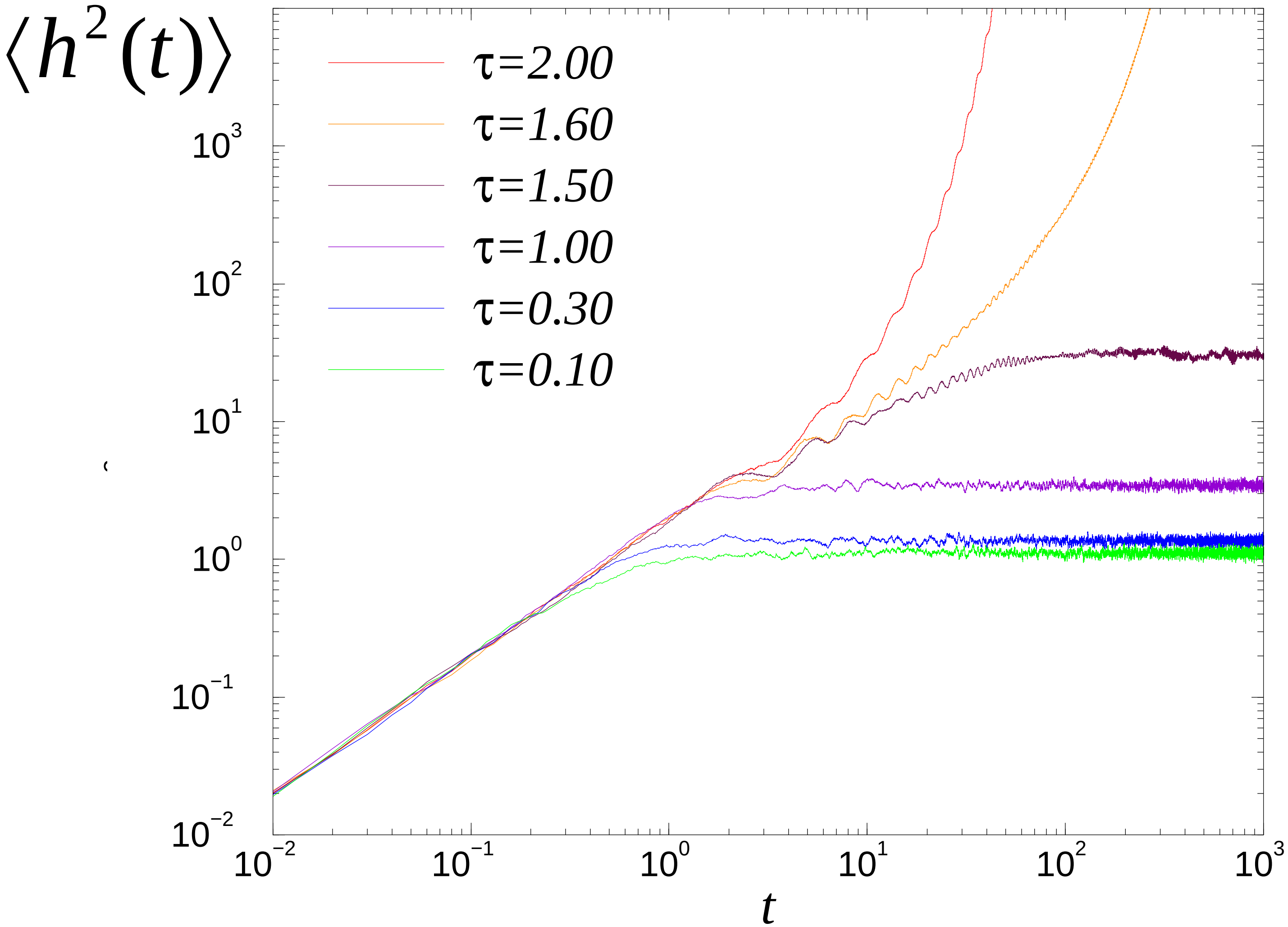
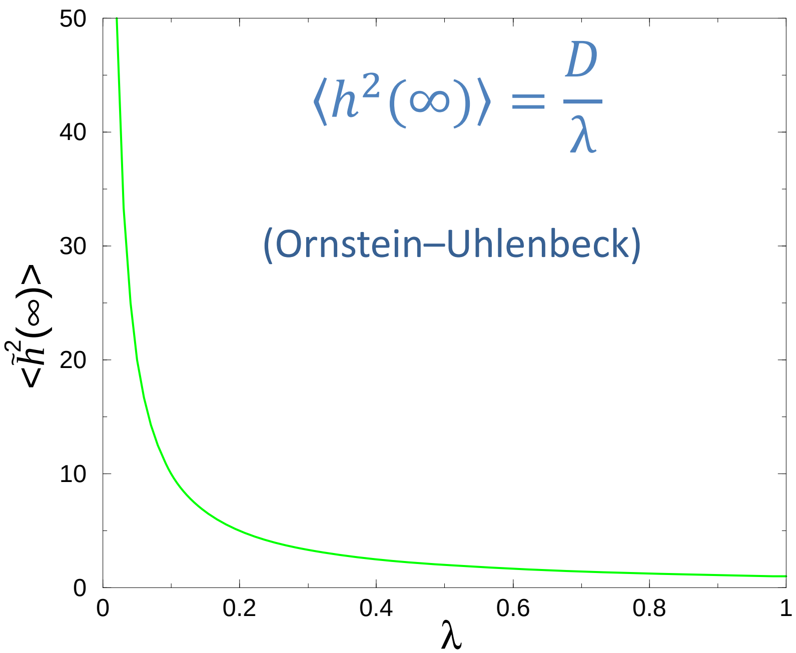


Coordination, Noise, Time Delay

$$\partial_t h(t) = -\lambda h(t-\tau) + \eta(t)$$

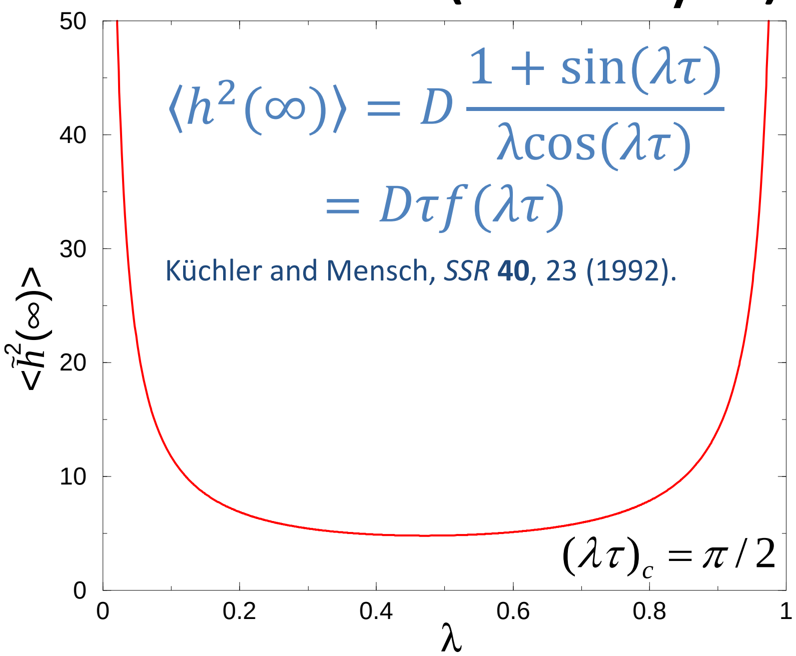
$$\langle \eta(t)\eta(t') \rangle = 2D\delta(t-t')$$

$\tau = 0$



***monotonically decreasing** function of the coupling* $\boldsymbol{\lambda}$

$\tau > 0 \quad (\tau = \pi/2)$



***non-monotonic** function of the coupling* $\boldsymbol{\lambda}$

Implications for Networks:

$$\left\langle w^2(t) \right\rangle = \frac{1}{N} \sum_{k=1}^{N-1} \left\langle \tilde{h}_k^2(t) \right\rangle = \frac{D\tau}{N} \sum_{k=1}^{N-1} f(\lambda_k \tau)$$

Hunt *et al.*, *PRL* (2010)

Synchronizability and Coordination:

$$\langle \tilde{h}_k^2(\infty) \rangle < \infty \quad \forall k$$

$$\lambda_k \tau < \pi / 2 \quad \forall k$$

$$\lambda_{\max} \tau < \pi / 2$$

Olfati-Saber and Murray (2004)
(deterministic consensus problems)

Limitations of Network Synchronization

Simple example: ***unweighted graphs***

largest degree — largest eigenvalue of the network Laplacian

$$\frac{N}{N-1}k_{\max} \leq \lambda_{\max} \leq 2k_{\max}, \qquad \lambda_{\max} = O(k_{\max})$$

Fiedler (1973); Anderson and Morley (1985); Mohar (1991)

$k_{\max}\tau < \pi/4$: sufficient for synchronizability/stability

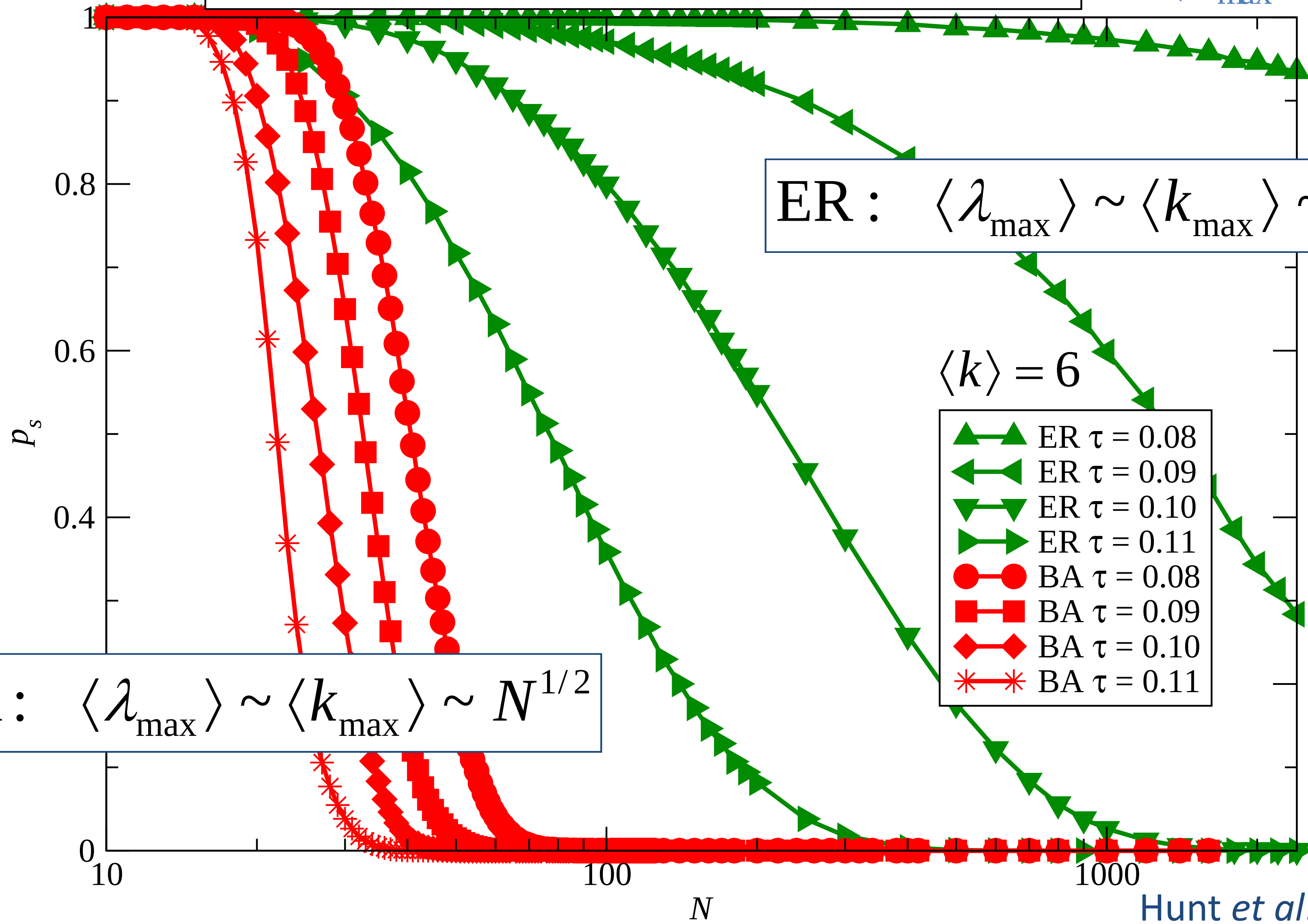
$k_{\max}\tau > \pi/2$: synchronization/stability breaks down

- networks with potentially large degrees can be extremely vulnerable to intrinsic network delays while attempting to synchronize, coordinate, or balance their tasks, load, etc.

Limitations of Network Synchronization

heterogeneous vs. homogeneous unweighted random graphs

$p^s(\tau,N)$ fraction of synchronizable networks $(\lambda_{max}\tau < \pi/2)$



$$\text{ER}: \quad \langle \lambda_{max} \rangle \sim \langle k_{max} \rangle \sim \ln(N)$$

$$\text{BA}: \quad \langle \lambda_{max} \rangle \sim \langle k_{max} \rangle \sim N^{1/2}$$

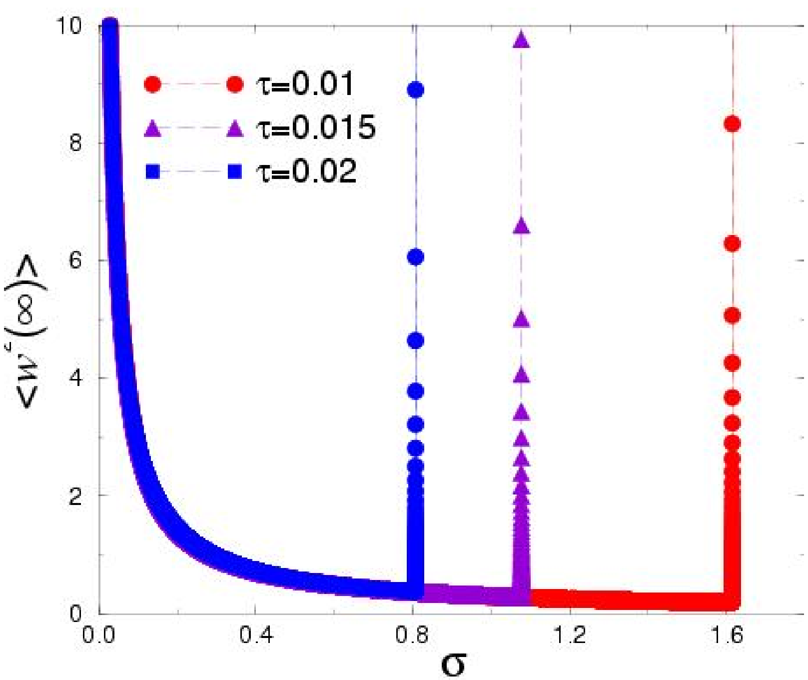
Hunt *et al.*, *PRL* (2010)

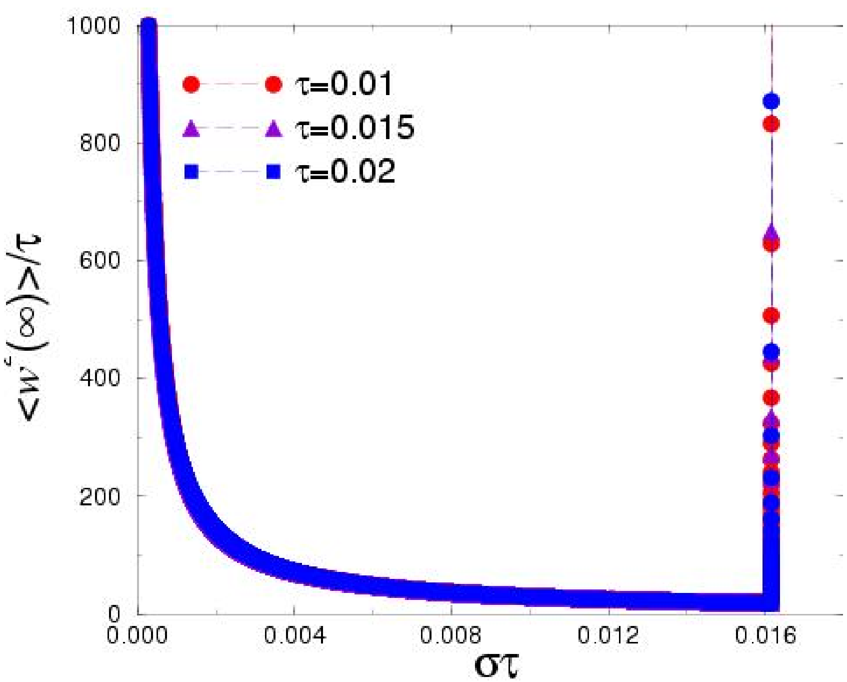
Scaling in the Synchronizable Regime

$$C_{ij} = \sigma A_{ij} \quad \rightarrow \quad \lambda_k' = \sigma \lambda_k$$

$$\left\langle w^2(\infty) \right\rangle_{\sigma,\tau} = \frac{D\tau}{N} \sum_{k=1}^{N-1} f(\sigma \lambda_k \tau) = \tau g(\sigma\tau)$$

$$\frac{\left\langle w^2(\infty) \right\rangle_{\sigma,\tau}}{\tau} = g(\sigma\tau)$$





BA network, N=1000, $\langle k \rangle \approx 6$

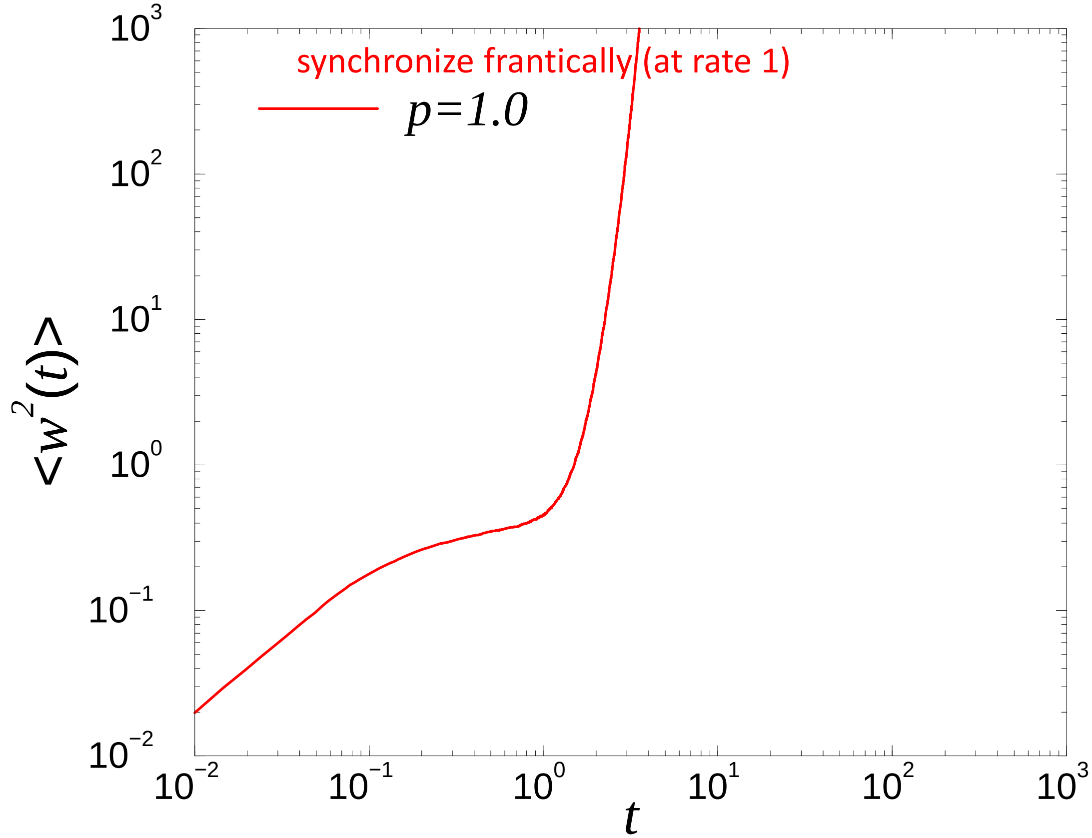
Trade-Offs

$$\partial_t h_i(t) = -p(t)\sum_j C_{ij}[h_i(t-\tau) - h_j(t-\tau)] + \eta_i(t)$$

BA network, N=200, $\langle k \rangle \approx 6$

$$\lambda_{max}\tau \approx 1.2\pi/2$$

Synchronization rate: p



Hunt *et al.*, *PRL* (2010)

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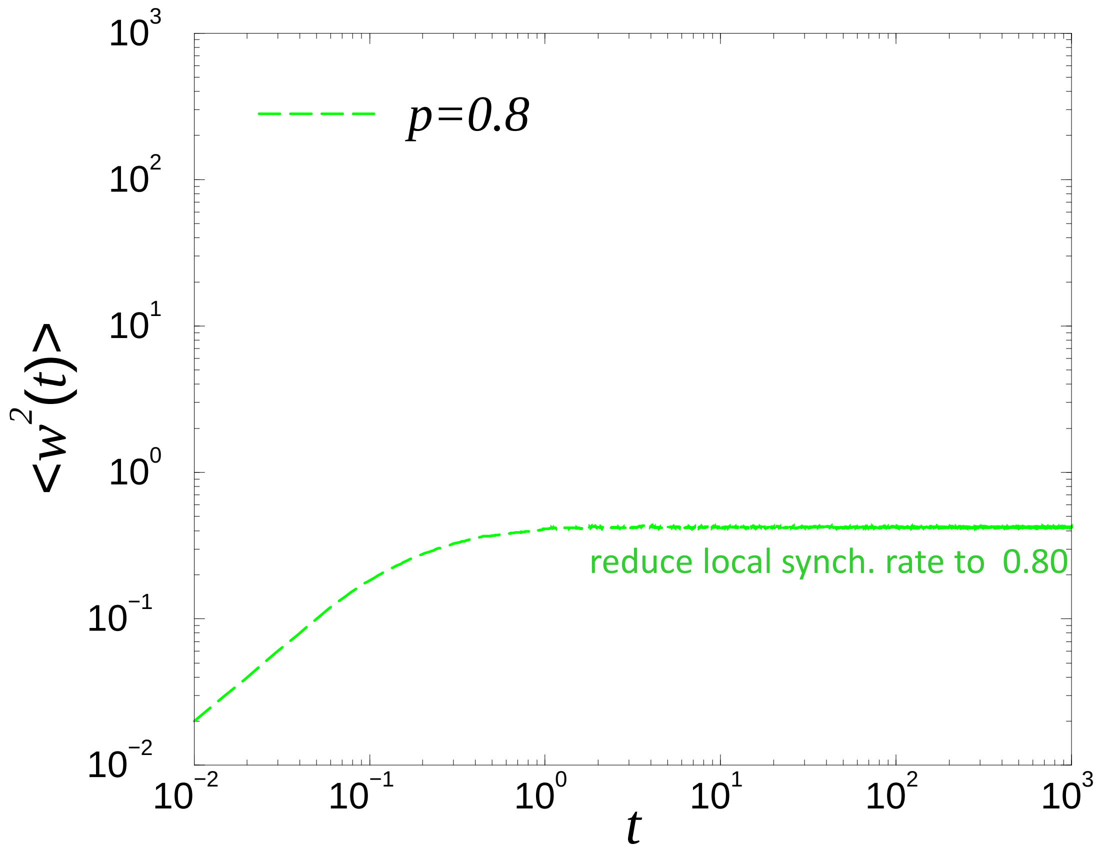

Hunt *et al.*, *PRL* (2010)

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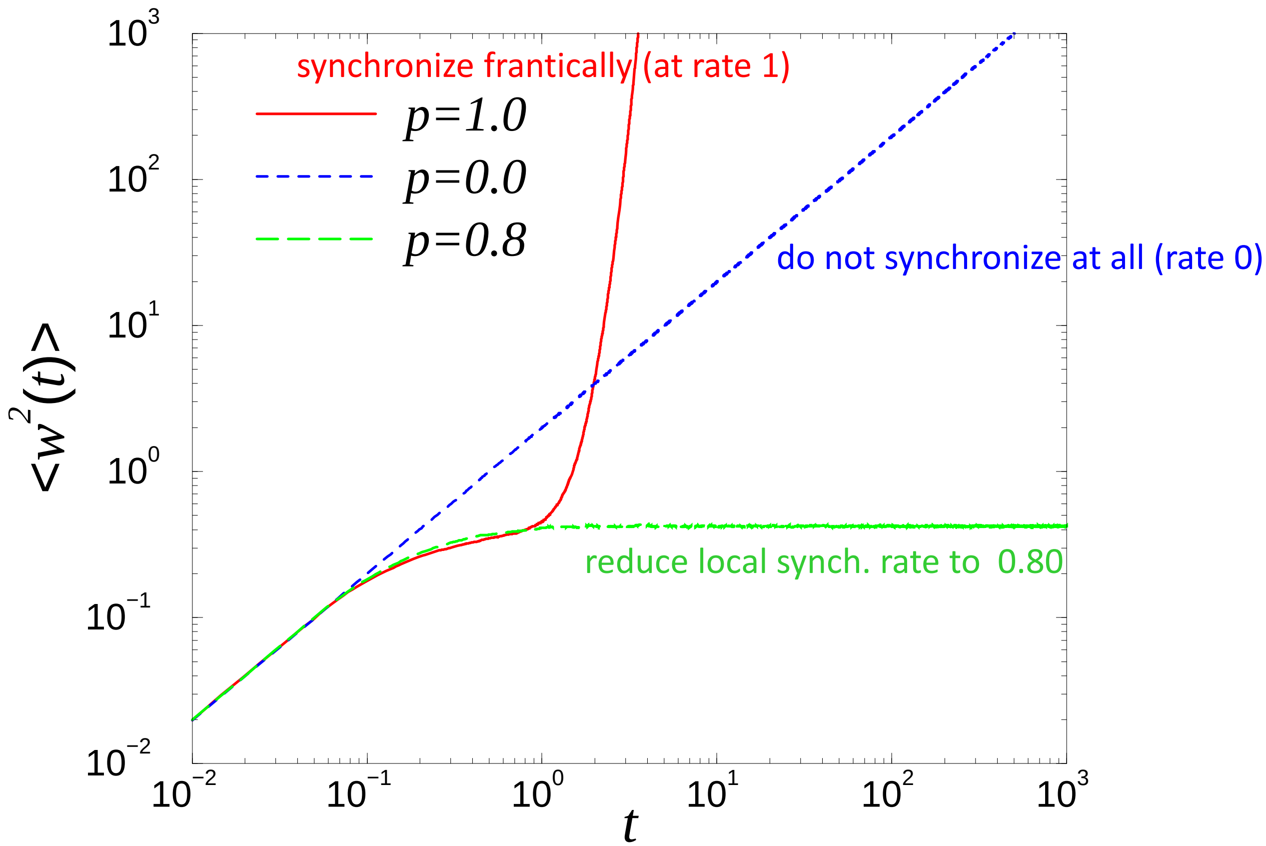


- reducing the local synch. frequency can stabilize the system

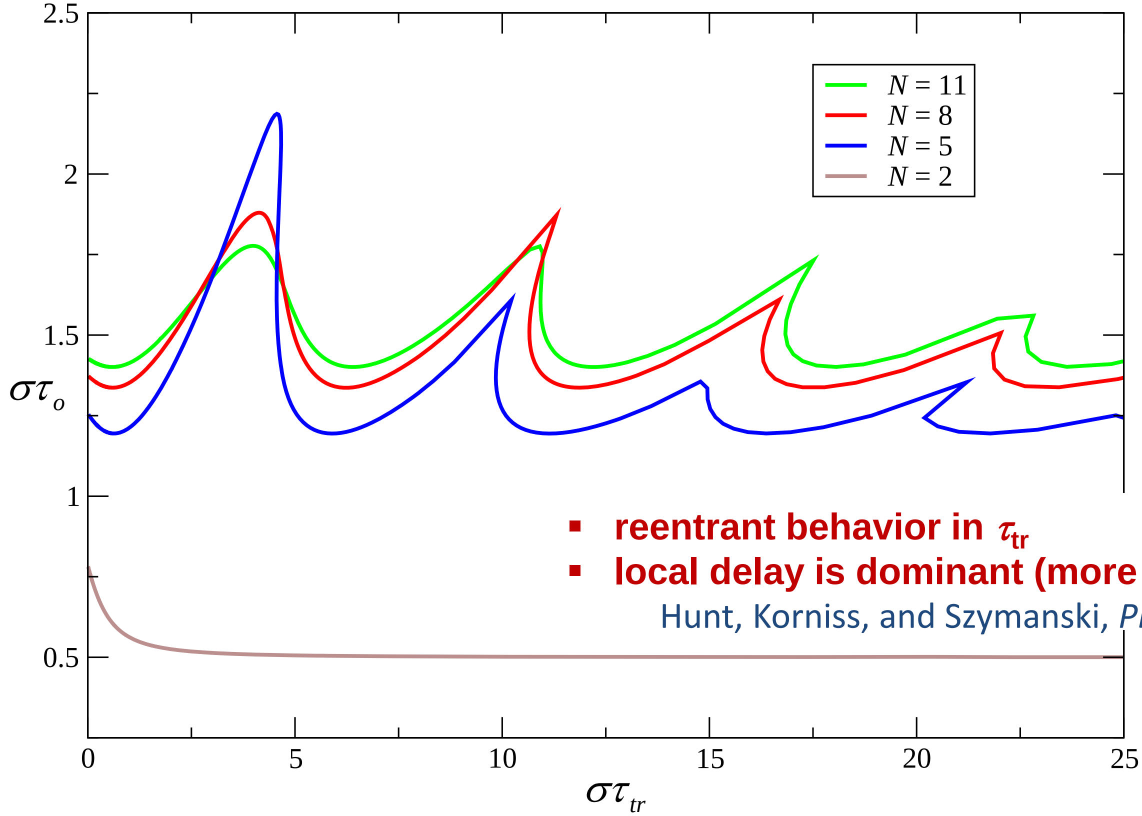
(in fact, ***even no synchronization at all is better than "over-synchronization":*** power-law divergence *vs* exp. divergence of the fluctuations with time)

Hunt *et al.*, *PRL* (2010)

Coordination with Multiple Time Delays

Complete Graph with *N* nodes ("normalized"):

$$\partial_t h_i(t) = -\frac{\sigma}{N-1}\sum_{j\neq i}[h_i(t-\tau_o) - h_j(t-\tau_o-\tau_{tr})] + \eta_i(t)$$



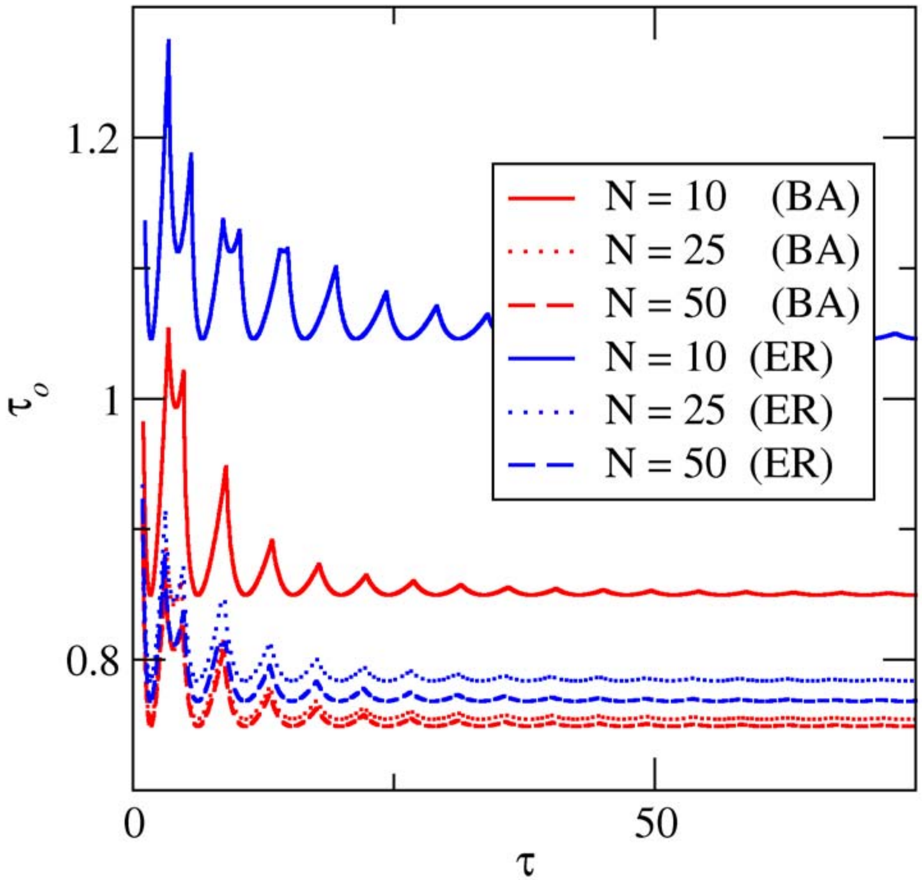
- **reentrant behavior in τ_{tr}**
- **local delay is dominant (more harmful)**

Hunt, Korniss, and Szymanski, *PLA* (2011).

Hunt *et al.* (2012)

Coordination with Multiple Time Delays

$$\partial_t h_i(t) = -\frac{\sigma}{k_i}\sum_j A_{ij}[h_i(t-\tau_o) - h_j(t-\tau)] + \eta_i(t)$$



τ_o : local delays (reaction, decision, execution)

$\tau = \tau_o + \tau_{tr}$: local delays + transmission, queuing delays

- **local delay is dominant (more harmful)**

Hunt *et al.* (2012)

Global vs. Local Weighted Coupling

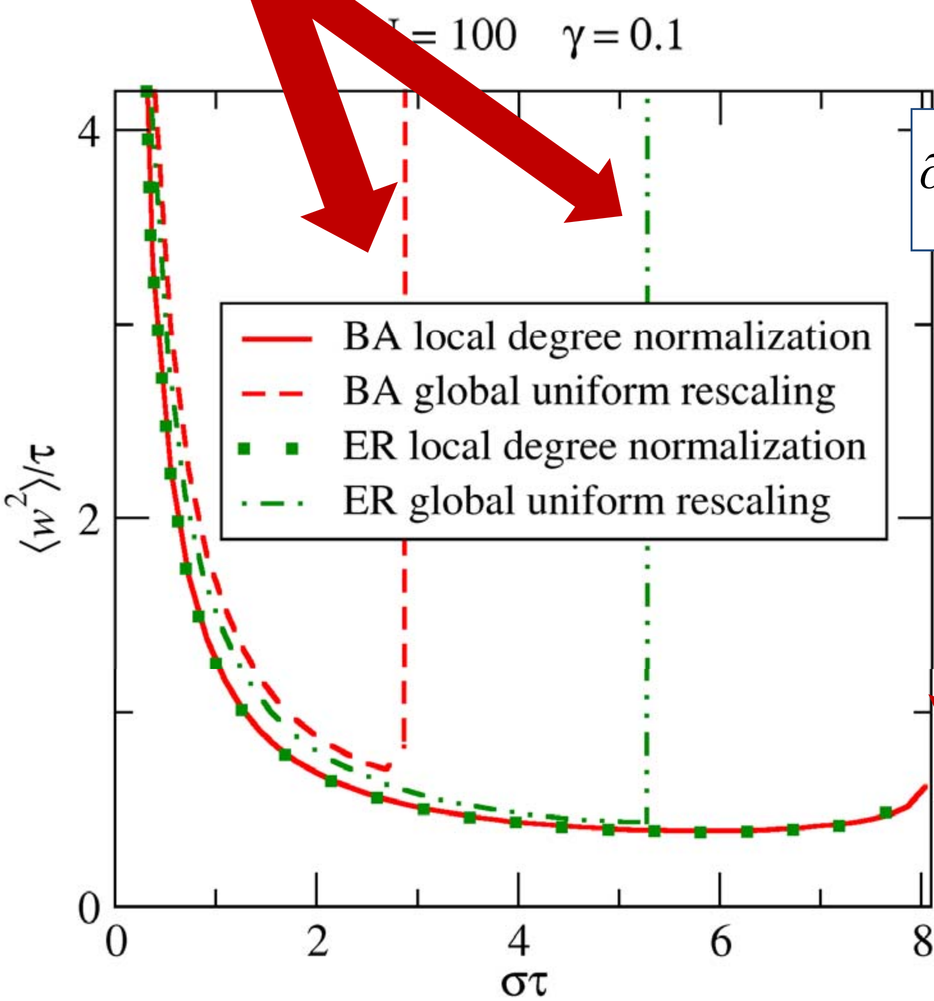
$$\tau_o = \gamma\tau$$

$$\partial_t h_i(t) = -\frac{\sigma}{\langle k \rangle}\sum_j A_{ij}[h_i(t-\gamma\tau) - h_j(t-\tau)] + \eta_i(t)$$

identical total interaction cost:

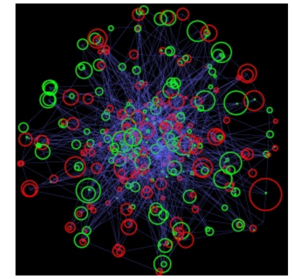
$$\sum_{i,j}\frac{\sigma}{\langle k \rangle}A_{ij} = \sum_{i,j}\frac{\sigma}{k_i}A_{ij} = \sigma N$$

$$\partial_t h_i(t) = -\frac{\sigma}{k_i}\sum_j A_{ij}[h_i(t-\gamma\tau) - h_j(t-\tau)] + \eta_i(t)$$

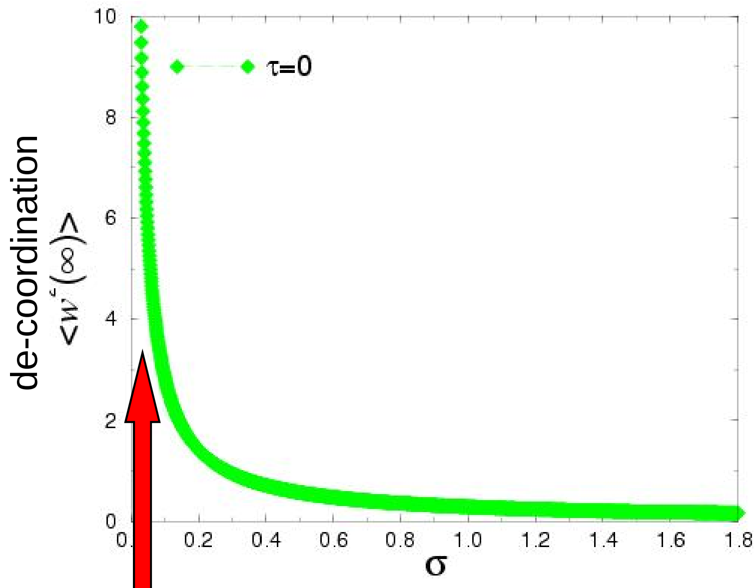


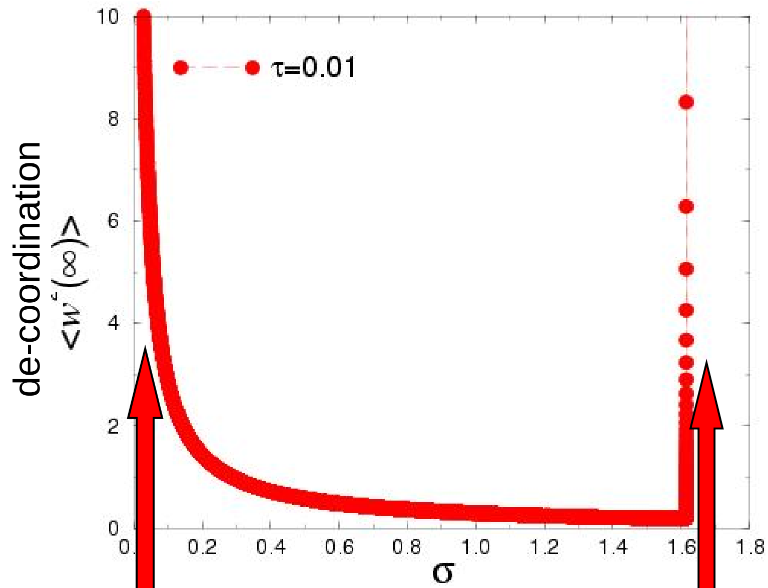
Hunt *et al.* (2012)

The Impact of Time Delays in Information and Communication Networks



- nodes/individuals constantly react to endogenous and exogenous information: coordination/agreement/consensus/alignment
- they react to the information or signal received from their neighbors possibly with some time lag τ (as result of finite transmission, decision, or execution delays)





network connectivity or communication frequency

low connectivity /
no communication

low connectivity /
no communication

high connectivity /
"too much communication"

Hunt et al., *PRL* **105 ,** 068701 (2010)

Summary

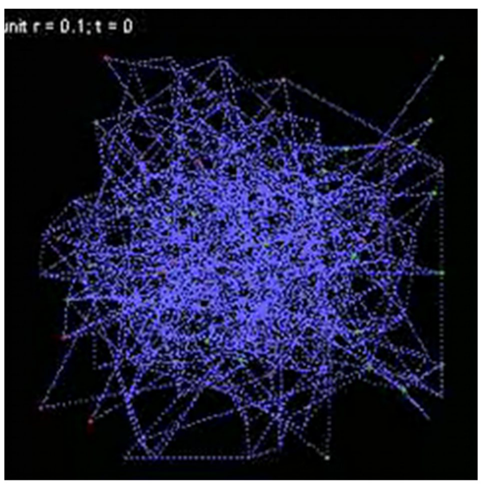
- Delays can destroy synchronization/coordination in networks
- Networks with large hubs can be particularly vulnerable in this regard
- Too much communication can cause more harm than good
- On the other hand, understanding the fundamental scaling properties of the underlying fluctuations (in particular the ones associated with the largest-eigenvalue mode) can guide optimization and trade-offs to control and to reduce these large fluctuations
- Currently studying the effects of heterogeneously distributed time delays $\{\tau_{ij}\}$

  D. Hunt, B.K. Szymanski, G. Korniss, http://arxiv.org/abs/1209.4240 (2012).

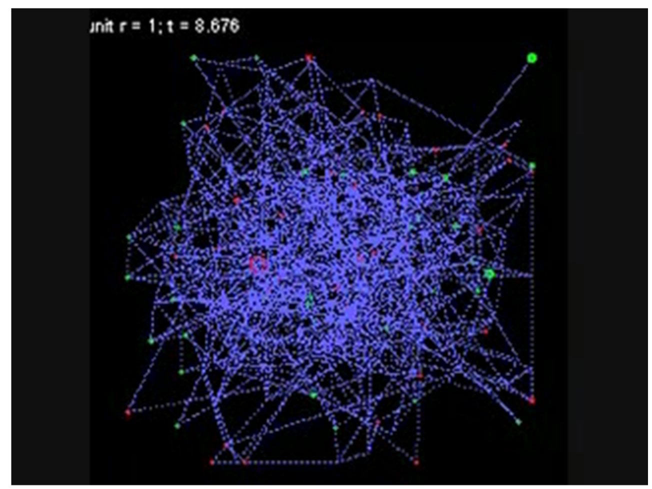
  D. Hunt, G. Korniss, and B.K. Szymanski, *Phys. Lett. A* **375**, 880 (2011).

  D. Hunt, G. Korniss, and B.K. Szymanski, *Phys. Rev. Lett.* **105,** 068701 (2010).

$\tau = 0$



$\tau > \tau_c$



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