# Minotaur's labyrinth in complex networks

# & Explosive percolations

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Mathematical Physics of Complex Networks:
From Graph Theory to Biological Physics

At MPI, Dresden, May 14-18, 2012

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- Call for Nomination for Boltzmann Medal
- Steering Committee meeting at KIAS, Seoul : July 7-9, 2012

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- Boltzmann medalist nomination : September 1, 2012
- Early Bird Registration: February 28, 2013
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- Abstract Submission: March 31, 2013
- Abstract Submission Acceptance Notice: May 15. 2013

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# Minotaur's labyrinth in SF networks

**-- Random walks effectively trapped at local hubs**

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With Sungmin Hwang and D.-S. Lee

Mathematical Physics of Complex Networks:
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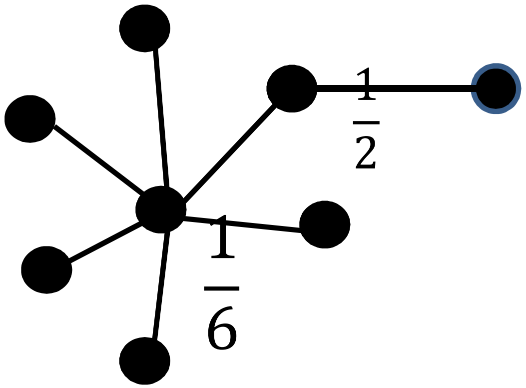
# Random walks on a scale-free network

$$p_{is}(t) = \sum_{j \in nn(i)} \frac{1}{k_j} p_{js}(t-1)$$

Probability that a RWer occupies at node *i* at time *t*, starting from node *s* at *t*=0.

$$p_{is}(t \to \infty) = \frac{k_i}{2L}$$

Noh and Rieger, PRL (2004)




$$P_o(t) = \frac{1}{N} \sum_{s=1} p_{ss}(t)$$

Purposes:

Probability to return to the origin $P_o(t) = \frac{1}{N}\sum_{s=1}^{N} p_{ss}(t)$

First passage time:

$p_{ss}(t) = ?$

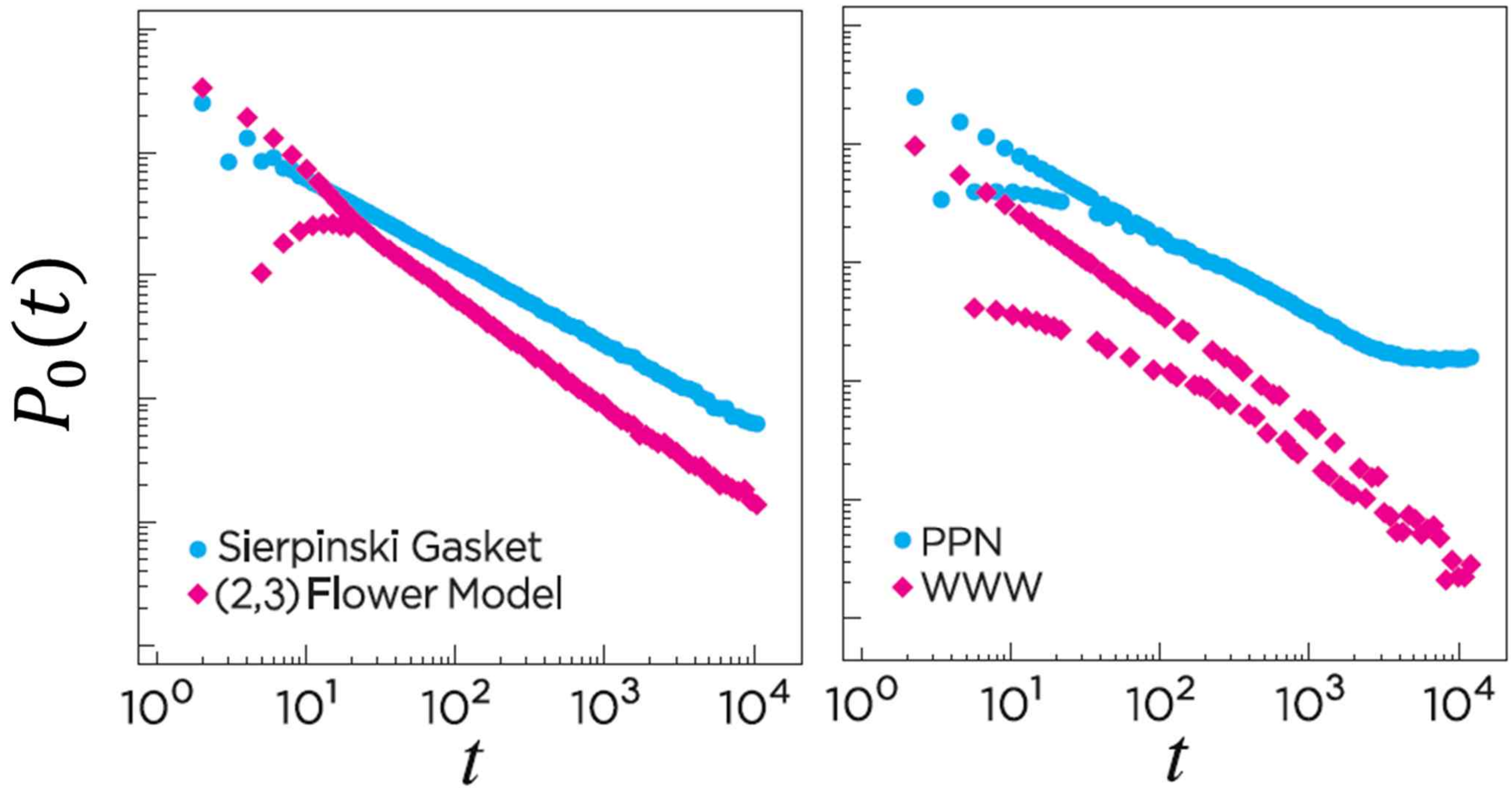
- First passage time distribution

- Mean first passage time

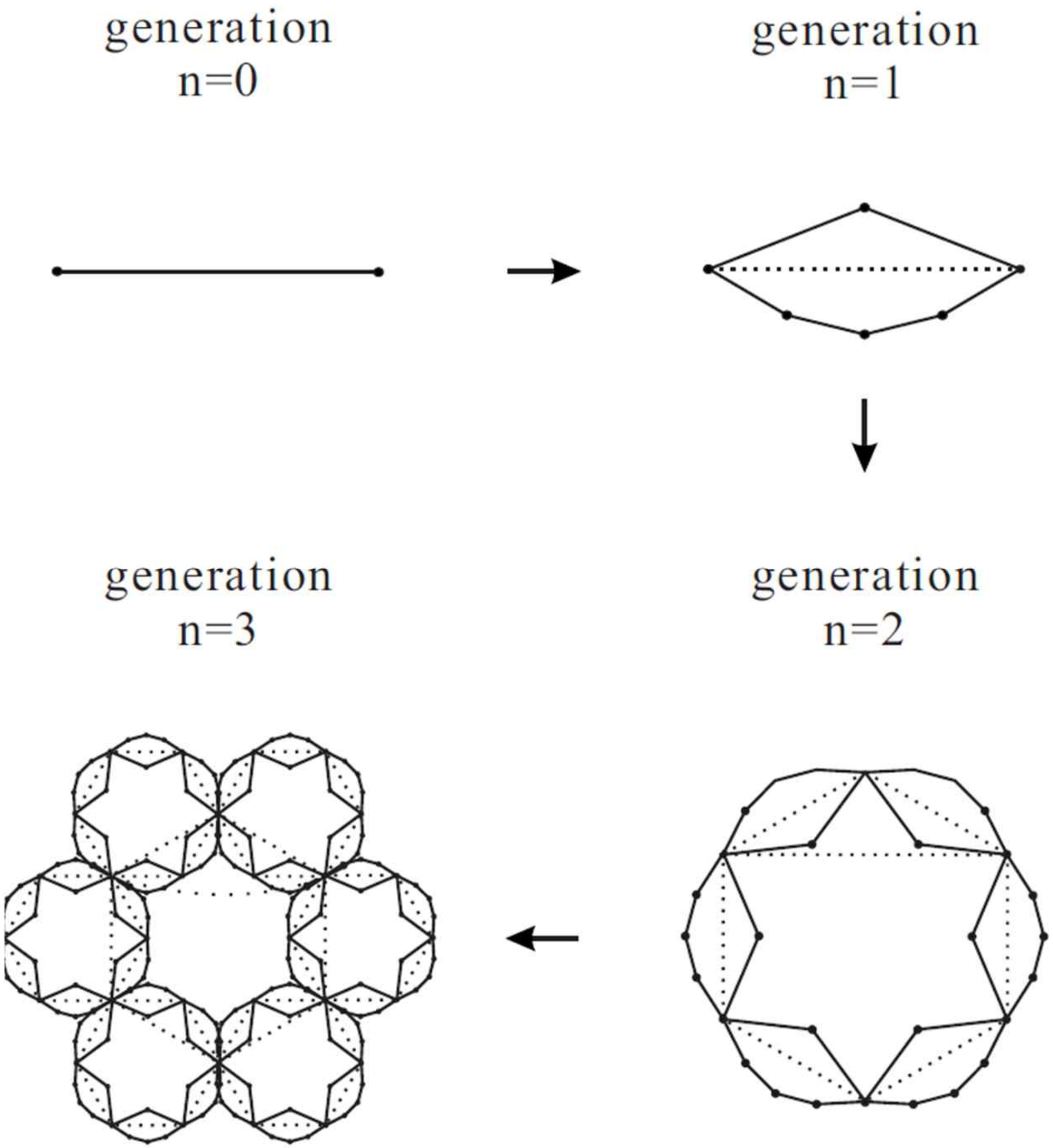
as a function of $d_s$ and $\gamma$. ➔ It shows crossover behaviors

- Many studies on these have been performed on deterministic SF nets,
- but not on un-deterministic networks, or
- asymptotic behaviors for some limited cases

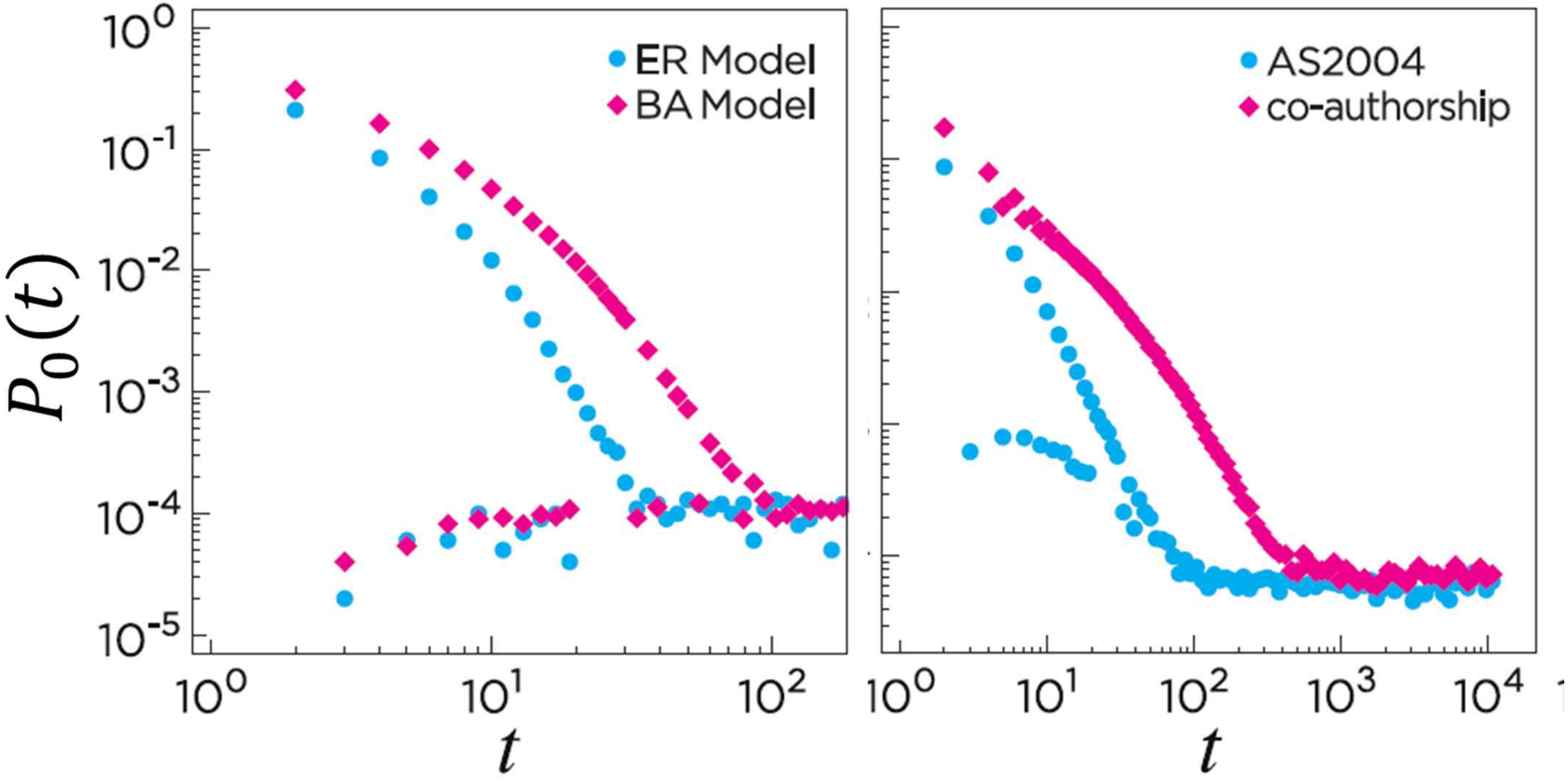
Probability to return to the origin $P_o(t) \sim t^{-d_s/2}$



# (2,4)-flower model



# Probability to return to the origin



# Probability to return to a given starting node *s*

$$p_{is}(t \to \infty) = \frac{k_i}{2L} \quad \Rightarrow \quad p_{is}(t) = \frac{\hat{k}_i(t)}{2\hat{L}(t)}$$

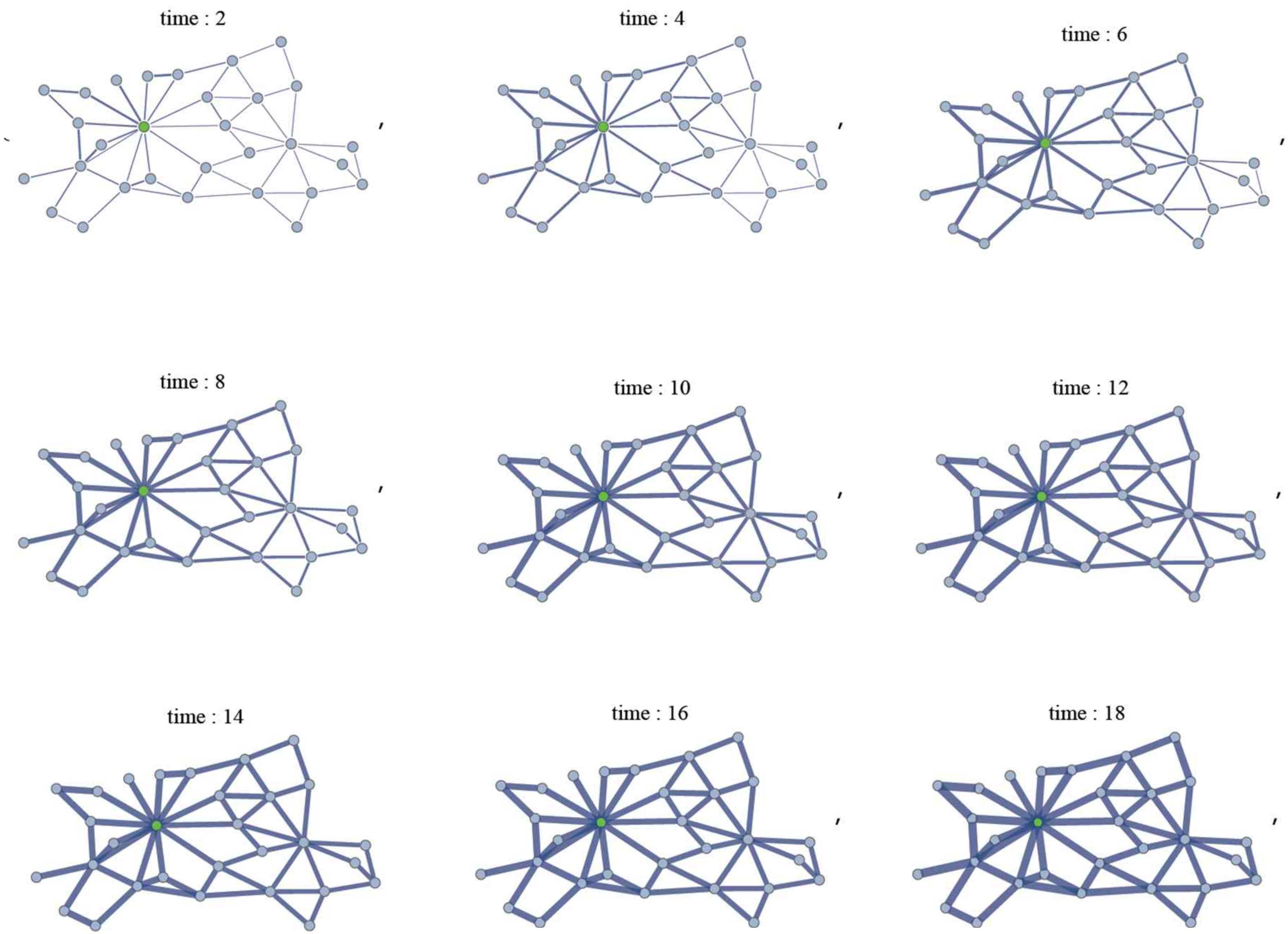
$$\widehat{k}_i(t) = \sum_{j \in nn(i)} \widehat{L}_{ij}(t)$$ Sum of the link accessibility from node j to i

$$\widehat{L}(t) = \sum_{i=1}^{N} \widehat{k}_i(t)/2$$ Number of accessed links

$$\widehat{L}(t) \simeq \frac{<k>}{2P_o(t-2)} \sim t^{d_s/2}$$ cf. $S(t) \sim t^{d_s/2}$ Number of distinct sites visited

$$\widehat{k}_h(t) \sim \widehat{L}(t)^{1/(\gamma-1)}$$ Similar to natural cutoff relation

$$\sim \ t^{d_s/2(\gamma-1)}$$


time : 2
time : 4
time : 6
time : 8
time : 10
time : 12
time : 14
time : 16
time : 18

# Probability to return-to-origin in random SF nets

$$\hat{k}_h \sim \begin{cases} t^{d_s/2(\gamma-1)} & \text{for } t \ll t_x \\ k_h & \text{for } t \gg t_x \end{cases} \qquad t_x \sim k_h^{2(\gamma-1)/d_s} \sim L^{2/d_s}$$

$$p_{hh}(t) = \frac{\hat{k}_h(t)}{2\hat{L}(t)} \sim \begin{cases} t^{-d_s^{(\text{hub})}/2} & \text{for } t \ll t_x, \\ \frac{k_h}{2L} & \text{for } t \gg t_x, \end{cases}$$

$$\boxed{d_s^{(\text{hub})} = d_s \frac{\gamma - 2}{\gamma - 1}}$$

$$\hat{k}_s \sim \begin{cases} t^{d_s/2(\gamma-1)} & \text{for } t \ll t_c(s) \\ k_s & \text{for } t \gg t_c(s) \end{cases} \qquad t_c(s) \sim k_s^{2(\gamma-1)/d_s}$$

$$\hat{k}_s \sim \begin{cases} t^{d_s/2(\gamma-1)} & \text{for } t \ll t_c(s) \\ k_s & \text{for } t \gg t_c(s) \end{cases} \qquad t_c(s) \sim k_s^{\,2(\gamma-1)/d_s}$$

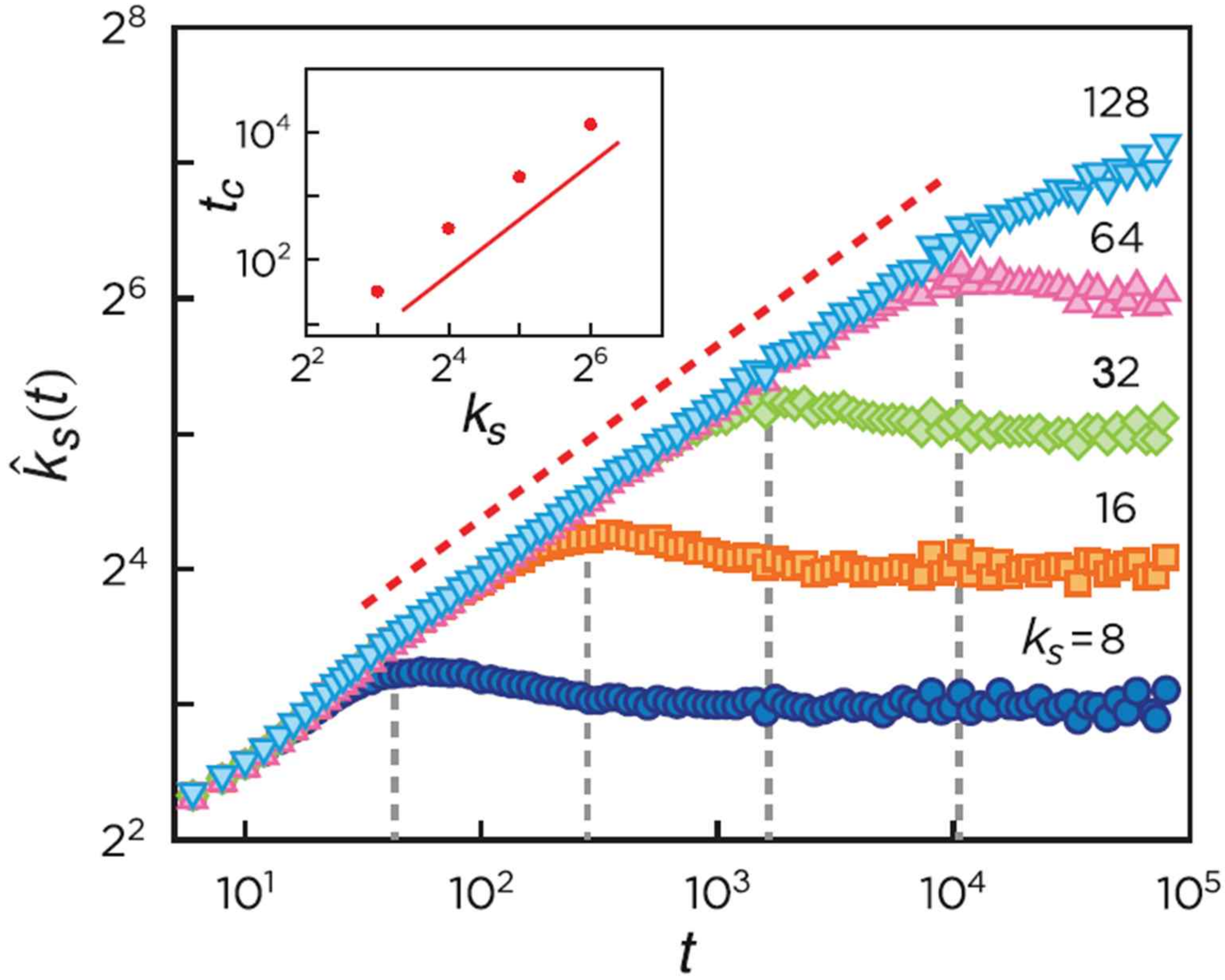
$$p_{ss}(t) \sim \begin{cases} t^{-d_s^{(\text{hub})}/2} & \text{for } t \ll t_c(s), \\ k_s t^{-d_s/2} & \text{for } t_c(s) \ll t \ll t_{\times}, \\ \frac{k_s}{2L} & \text{for } t \gg t_{\times}. \end{cases}$$

$$\boxed{d_s^{(\text{hub})} = d_s \frac{\gamma - 2}{\gamma - 1}}$$

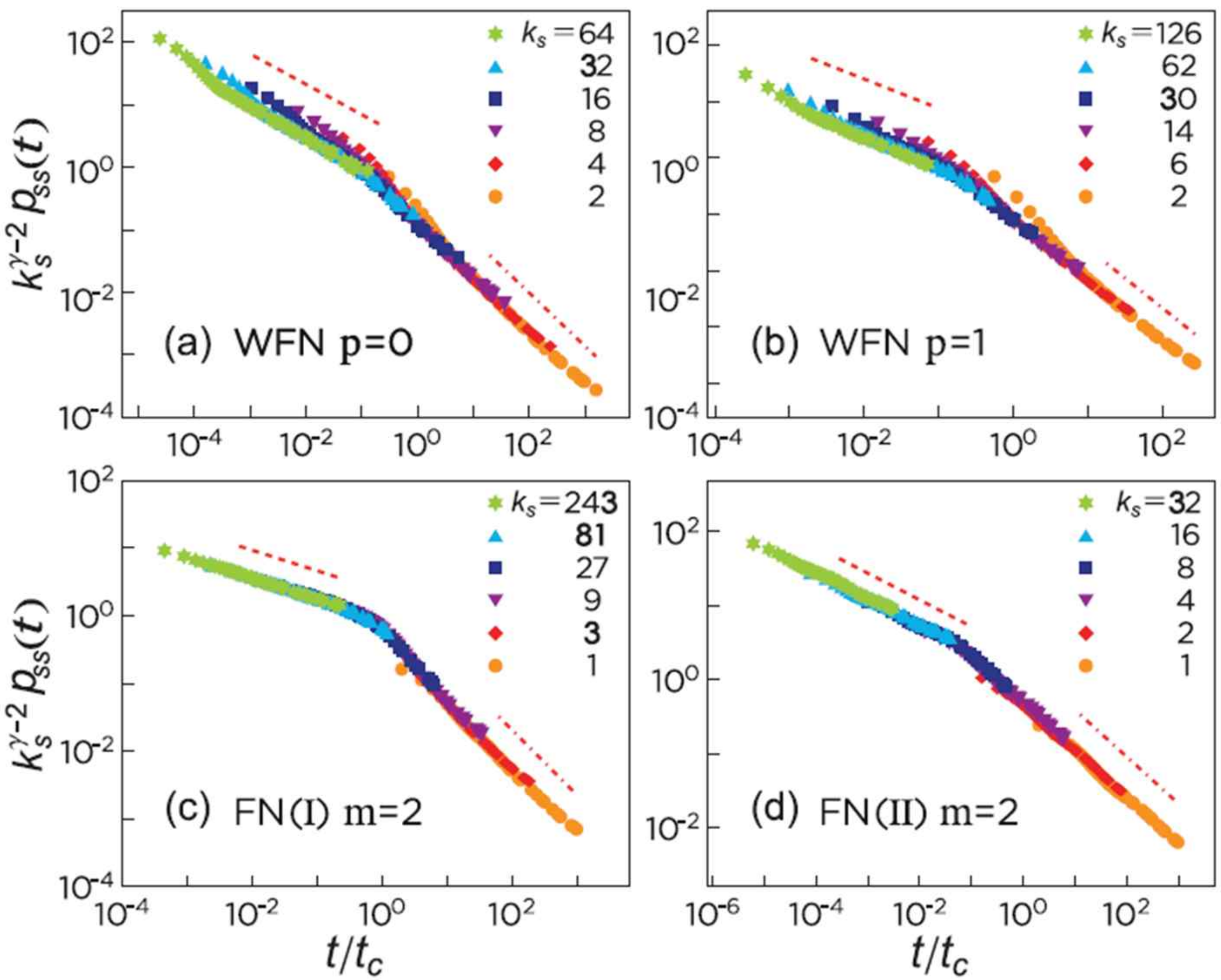
when $\gamma \to 2$, $d_s^{(\text{hub})} \to 0$, and $p_{ss}(t) \to$ const. during $t_c(s)$.

Random walks are trapped at local hubs, Minotaur's labyrinth.

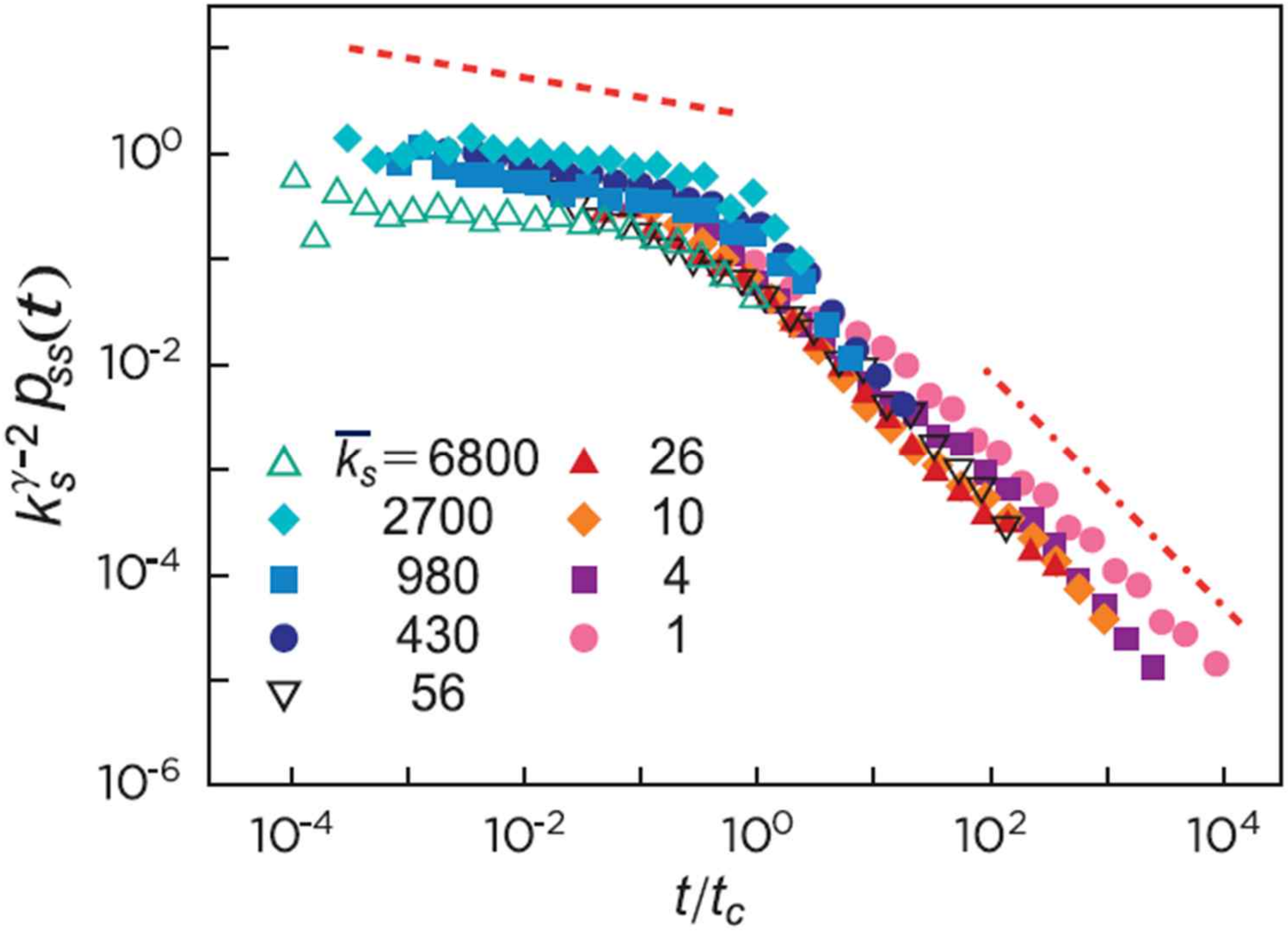
# Effective degree of starting node vs time



# Probability to return to the origin on model nets



# Probability to return to the origin on the WWW



# First passage time distribution for RWs

$$F_m(t) = \sum_{s=1}^{N} \frac{k_s}{2L} F_{m\,s}(t)$$

FPT probability for RWs starting from *s* to *m*

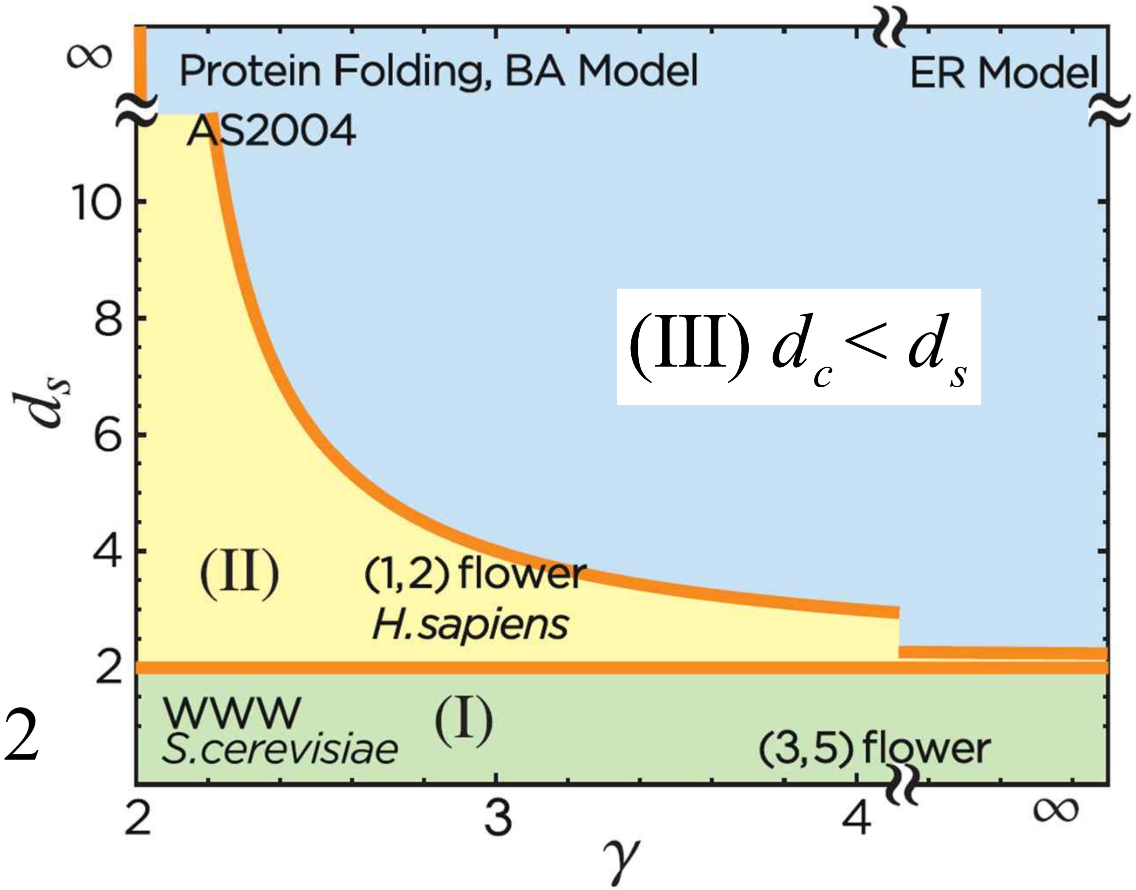
Using the renewal equation,

$$p_{m\,s}(t) = \delta_{m\,s}\delta_{t0} + \sum_{t'=0}^{t} F_{m\,s}(t')p_{m\,m}(t-t')$$

$$\mathcal{F}_m(z) = \frac{k_m\, z}{2L\,(1-z)} \frac{1}{\mathcal{R}_m(z)}$$

# Phase diagram in $(d_s, \gamma)$ space

(II) $2 < d_s < d_c = \dfrac{2(\gamma - 1)}{(\gamma - 2)}$




(I) $d_s < 2$

(I) $d_s < 2$

$1 \ll t \ll t_c(k_m)$

$$F_m(t) \sim \frac{k_m}{2L} t^{-(1-d_s^{(h)}/2)}$$

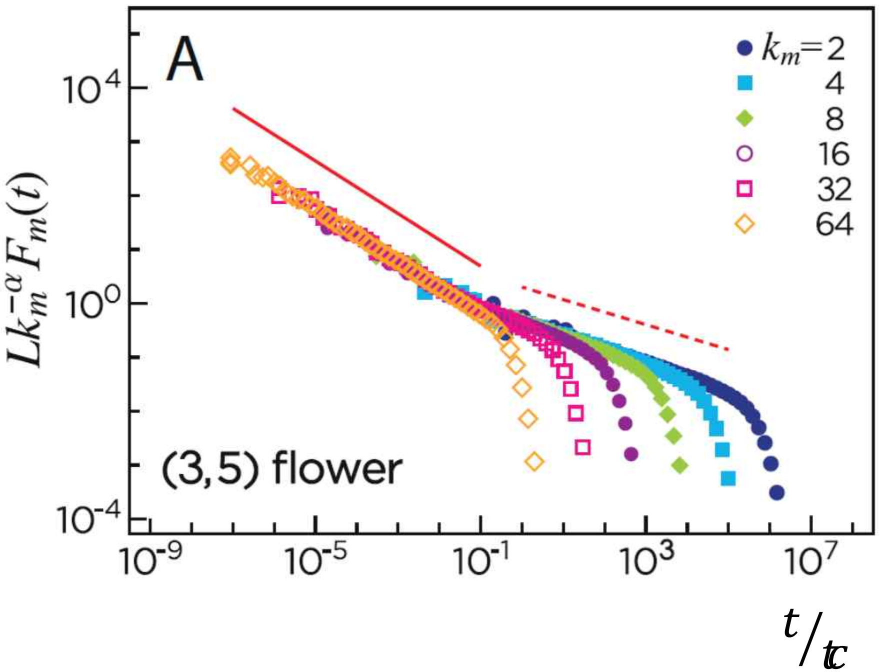
$$d_s^{(h)} = d_s \frac{\gamma - 2}{\gamma - 1}$$

$t_c(k_m) \ll t \ll t_x$

$$F_m(t) \sim \frac{1}{2L} t^{-(1-d_s/2)}$$

$t \gg t_x$

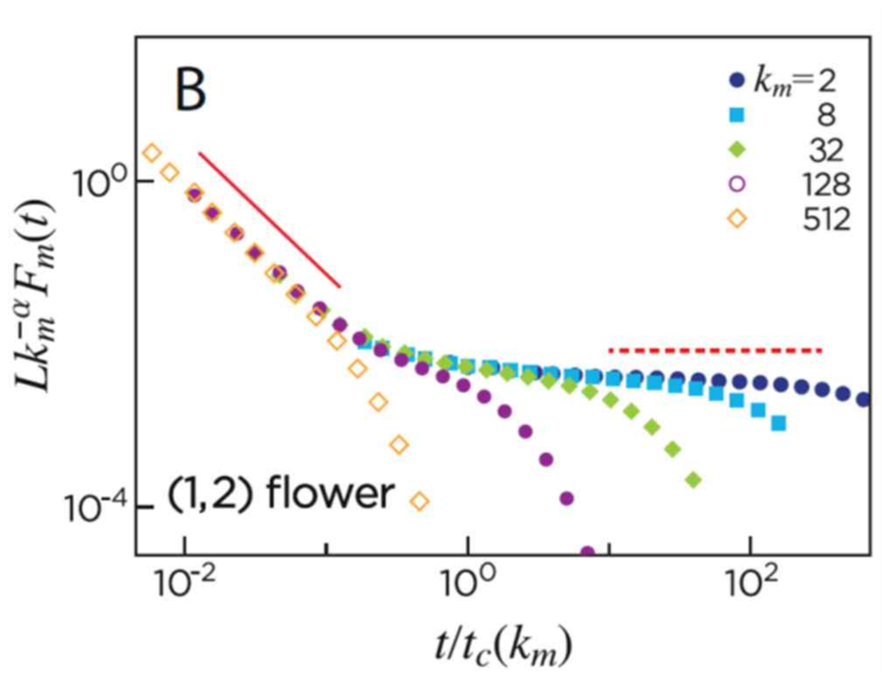
$$F_m(t) \sim N^{-d_s/2} e^{-t/N^{d_s/2}}$$



(II) $2 < d_s < d_c = \frac{2(\gamma-1)}{(\gamma-2)}$

$1 \ll t \ll t_c(k_m)$

$F_m(t) \sim \frac{k_m}{2L} t^{-(1-d_s^{(h)}/2)}$




$t_c(k_m) \ll t$

$F_m(t) \sim N^{-1} k_m^{\alpha} e^{-t/Nk_m^{-\alpha}}$

$\alpha = (1 - \frac{2}{d_s})(\gamma - 1)$

(III) $d_c < d_s$

$$F_m(t) \sim N^{-1} k_m e^{-t/N k_m^{-1}} \qquad \text{for any } t$$

# Mean First Passage Time

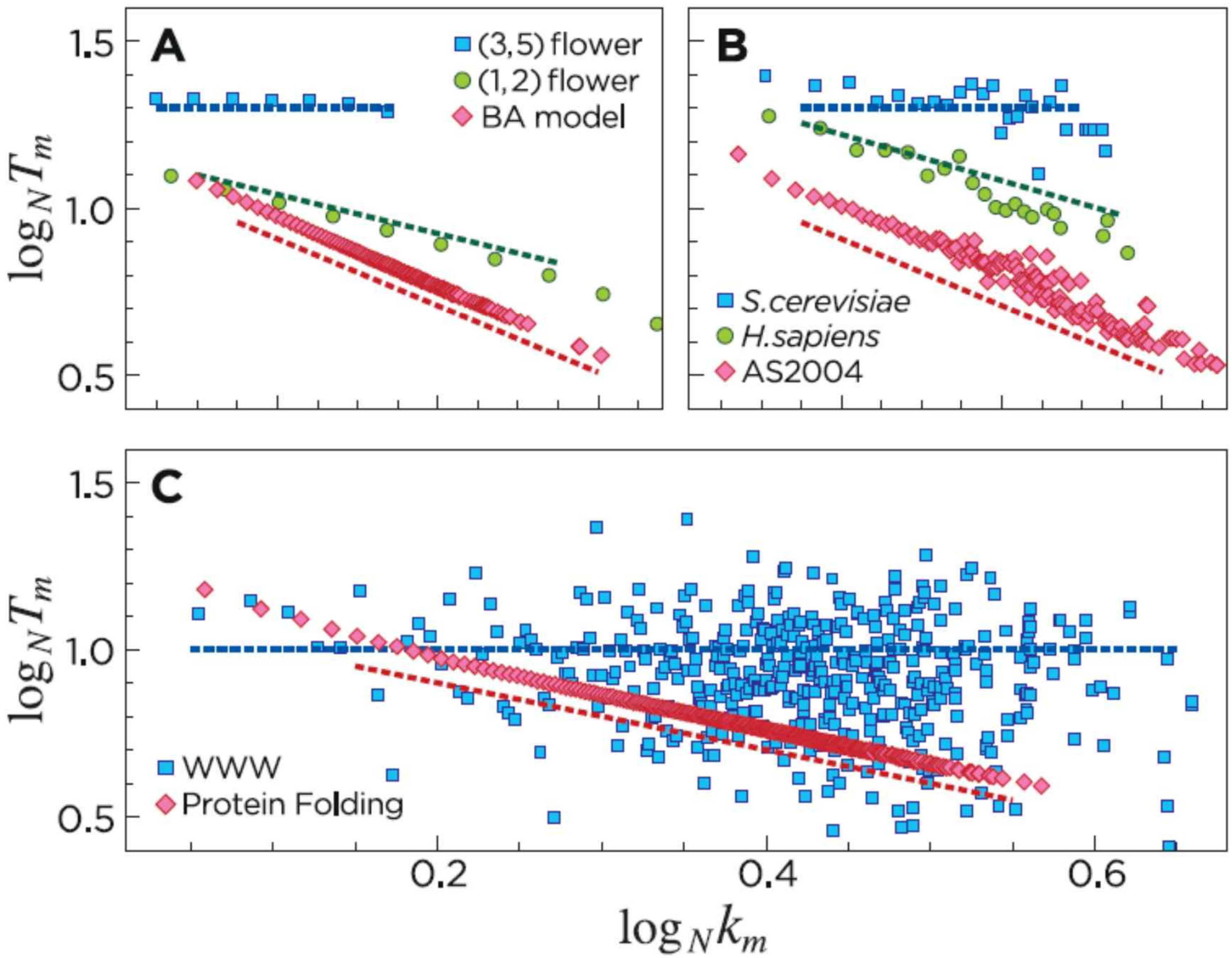
$$\mathcal{F}_m(z) = \frac{k_m z}{2L(1-z)} \frac{1}{\mathcal{R}_m(z)}$$

$$T_m = \frac{\partial}{\partial z}\mathcal{F}_m(z)\Big|_{z=1} \approx \frac{2L}{k_m}\mathcal{R}^*_m(1) + 1$$

$$= \frac{2L}{k_m}\sum_{t=0}^{\infty}(R_m(t) - R_m(\infty)) + 1.$$

$$T_m \approx \frac{2L}{k_m}\int_1^{t_x}[R_m(t) - R_m(\infty)]dt$$

$$\sim \begin{cases} N^{2/d_s} & \text{(I)} \quad d_s < 2, \\ Nk_m^{-\alpha} & \text{(II)} \quad 2 < d_s < d_c, \\ Nk_m^{-1} & \text{(III)} \quad d_s > d_c, \end{cases}$$



A
(3,5) flower
(1,2) flower
BA model
B
S.cerevisiae
H.sapiens
AS2004
C
WWW
Protein Folding
1.5
1.0
0.5
0.2
0.4
0.6
$\log_N T_m$
$\log_N k_m$

## Conclusions

1. Probability to return to the origin has been studied in diverse scale-free networks

2. First passage time problems have been studied in diverse scale-free networks

Complete analytic formulae for those quantities including crossover behavior over time are derived in terms of spectral dimensions, $d_s, \gamma, k_s, k_m,$ and $N$.

# Suppression effect on explosive percolations

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With Y.S. Cho

Mathematical Physics of Complex Networks:
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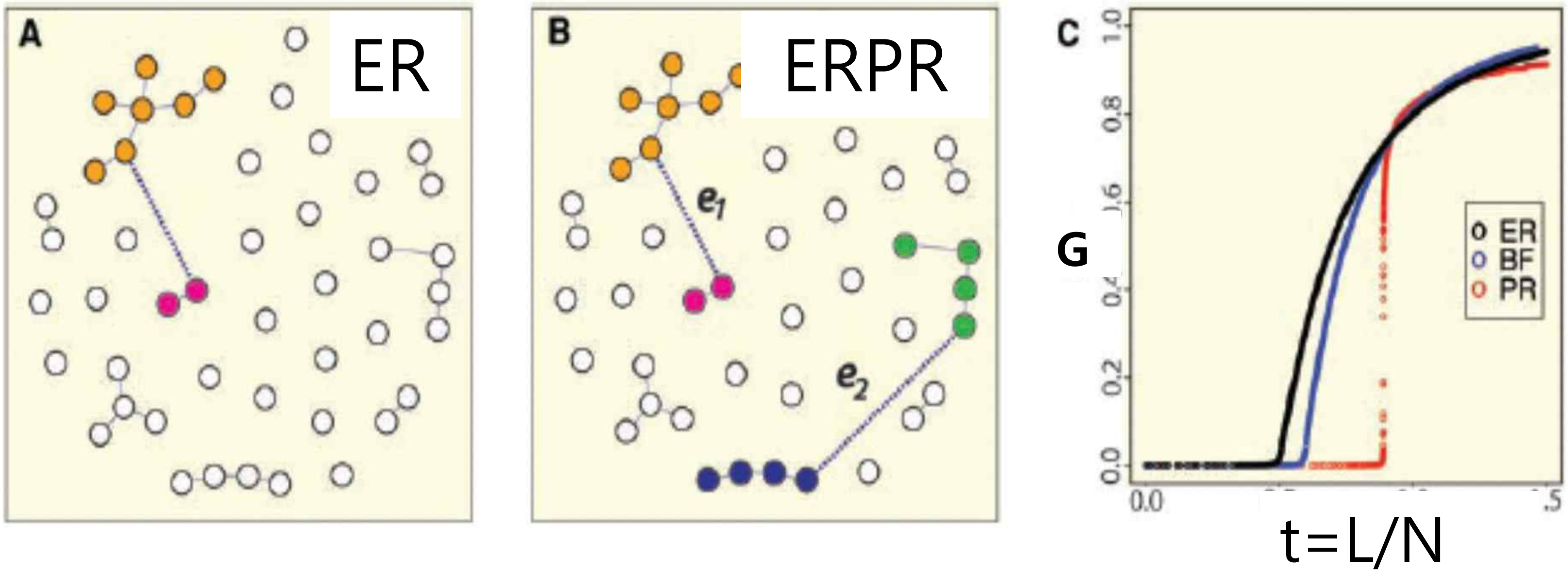
# 1. Background




1) The number of nodes is fixed as *N*.

2) Edges are added one by one to the system between two nodes randomly chosen at each time step.

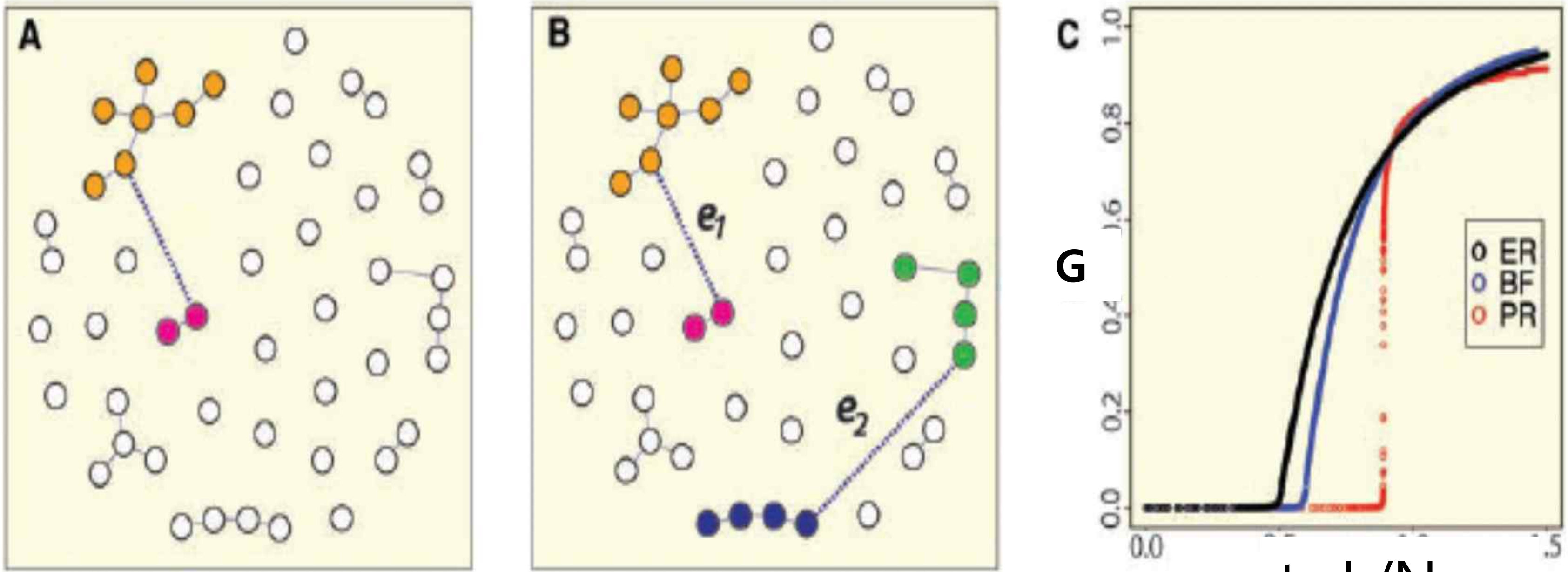
→ Percolation transition at $t_c = L_c/N = 1/2$

→ Continuous transition

# Achlioptas process

Achlioptas et al, Science (2009,3)

ER ERPR




t=L/N

1. Pick up two edge candidates randomly.
2. Calculate the product of two-cluster sizes:
   By $e_1$, 7*2=14 vs. by $e_2$, 4*4=16 ➔ $e_1 < e_2$ (product rule)
3. Then, $e_1$ is attached, and $e_2$ is discarded.

➔ Growth of large clusters is suppressed.
➔ Percolation transition point is delayed.

ERPR

# 2. Goal

Is the explosive percolation transition continuous or discontinuous ?

1) Achlioptas et al, **Explosive percolation transition**, Science (2009,3).
2) Many others.

1) R.A. da Costa, S.N. Dorogovtsev, A.V. Goltsev, J.F.F. Mendes **Explosive Percolation Transition is Actually Continuous,** PRL 105, 255701 (2010).
2) P. Grassberger, C. Christensen, G. Bizhani, S.-W. Son, M. Paczuski, **Explosive percolation is continuous, but with unusual finite size behavior,** PRL 106, 225701 (2011).
3) O. Riordan and L. Warnke, **Explosive percolation is continuous**, Science 333, 322 (2011).
4) H.K. Lee, B.J. Kim, and H. Park, **Continuity of the explosive PT**, PRE 84, 020101 (2011).

# *Avoiding Small Subgraphs in Achlioptas Processes*

**Michael Krivelevich,[1,*] Po-Shen Loh,[2,†] Benny Sudakov[3,‡]**

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[2] *Department of Mathematics, Princeton University, Princeton, New Jersey 08544; e-mail: ploh@math.princeton.edu*

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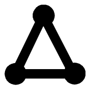
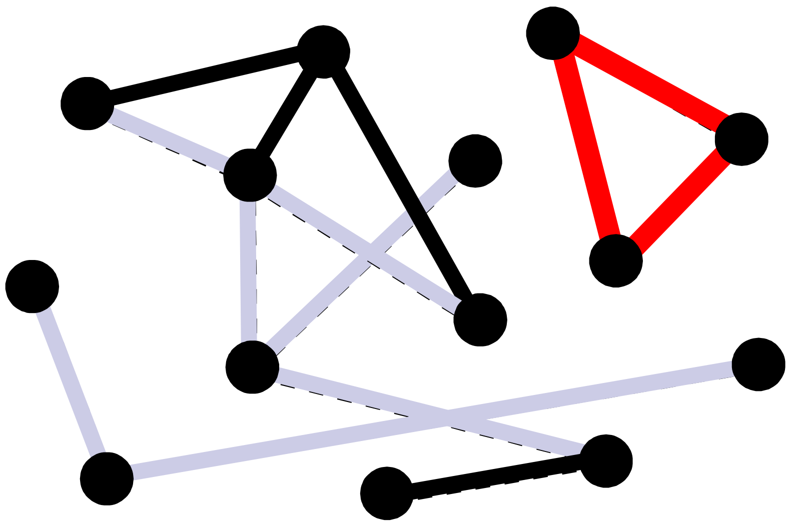
*DOI 10.1002/rsa.20254*

From the Achlioptas slide

# Introduction

- Achlioptas process:
  - start with the empty graph on **n** vertices
  - in each step **r** edges are chosen uniformly at random (among all edges never seen before)
  - select one of the **r** edges that is inserted into the graph, the remaining **r – 1** edges are discarded
- Goal: Avoid creating a copy of some fixed graph **F**

F = , r = 2



How long can we avoid **F** by this freedom of choice?

From the Achlioptas slide

# Introduction

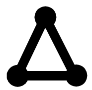
- $N_0 = N_0(F, r, n)$ is a threshold:

There is a strategy that Avoids creating a copy of F with probability 1-o(1)

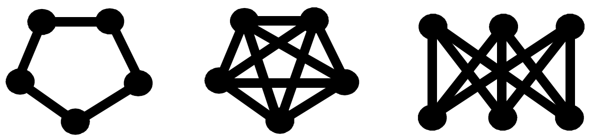
Every strategy will be forced to create a copy of F with probability 1-o(1)

N /$N_0$ $\quad$ $N_0=N_0(F, r, n)$ $\quad$ N {$N_0$ $\quad$ N = N(n) = # steps

$n^1$ $\quad$ $n^{1.2}$ $\quad$ $n^{1.286...}$ $\quad$ $n^{1.333...}$ $\quad$ $\frac{1}{r}\binom{n}{2} \asymp n^2$

r=1 $\quad$ F =  , r=2 $\quad$ r=3 $\quad$ r=4

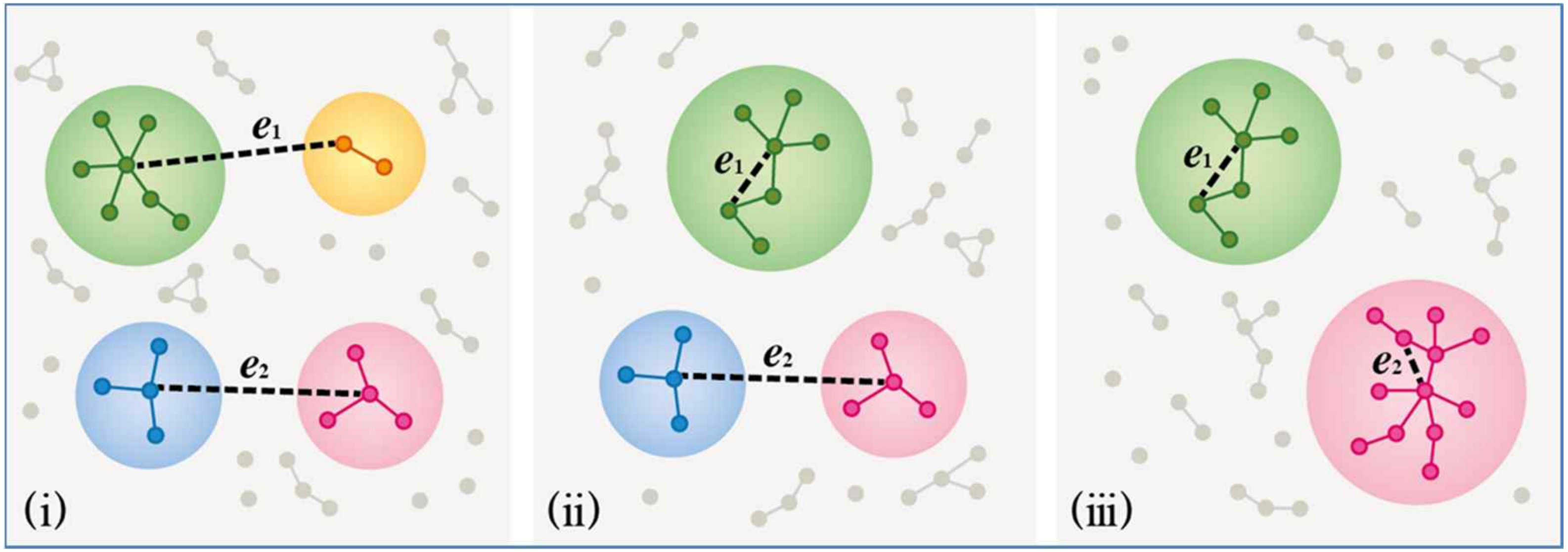
If F is a cycle, a clique or a complete bipartite graph with parts of equal size, an explicit threshold function is known. (Krivelevich, Loh, Sudakov, 2007+)



- ✓ The Achlioptas process (AP):
  the dynamics avoiding the formation
  of a given pattern in evolution of graph.

- ✓ The percolation model following the AP:
  the target pattern is giant component.
  Thus, **the dynamics has to be proceeded to avoid the formation of a giant cluster.**

# 3. Classification of edge candidates

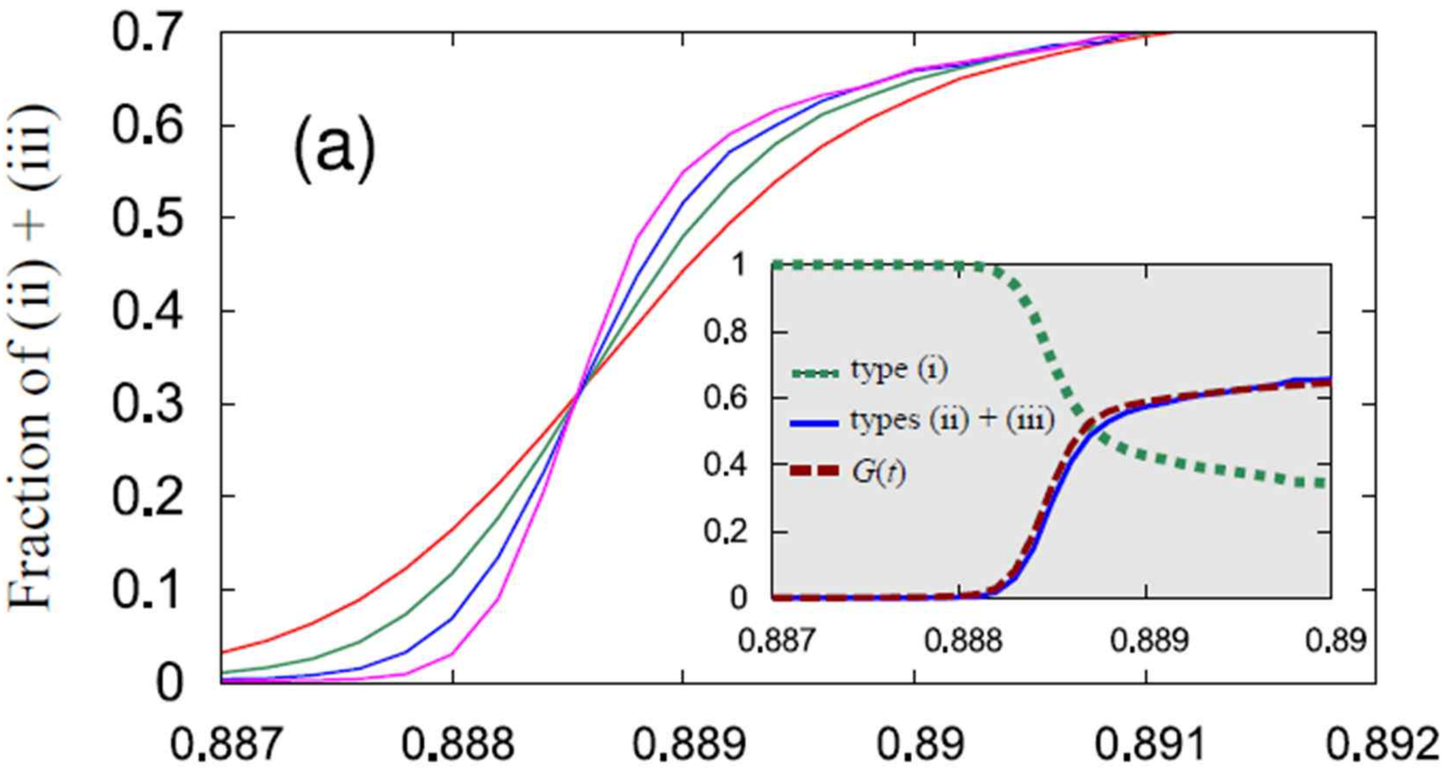



Inter-cluster edges

Inter-cluster edge
+
Intra-cluster edge

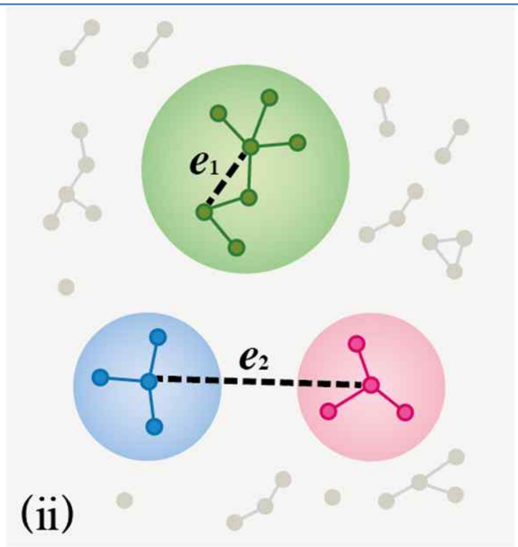
Intra-cluster edges

# Fraction of type (ii) & (iii)




t=L/N

# 4. Model Variants (Product Rule)




For the case (ii)

ERPR-A (original rule)

$S_1^2=7^2$ vs. $S_{2a}*S_{2b}=4*4=16$

➔Take $e_2$

But $e_1$ is desirable

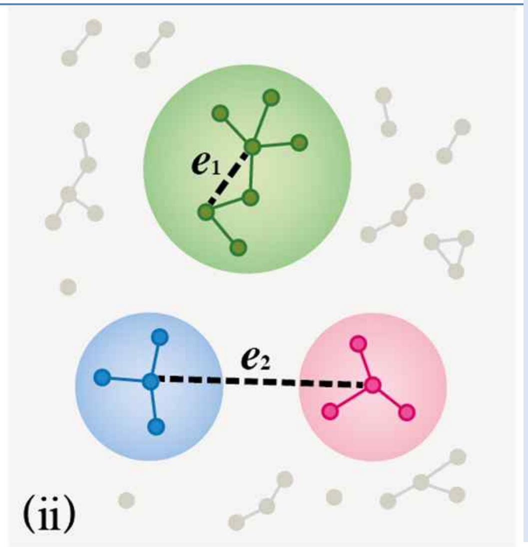
ERPR-B

➔Take $e_1$ (Absolutely)
Cluster size unchanged

ERPR-C

Case (ii) is excluded.

# Model Variants (Sum Rule)




For the case (ii)

ERSR-A

$2S_1=2*7$ vs. $S_{2a}+S_{2b}=4+4=8$

➔Take $e_2$

But $e_1$ is desirable

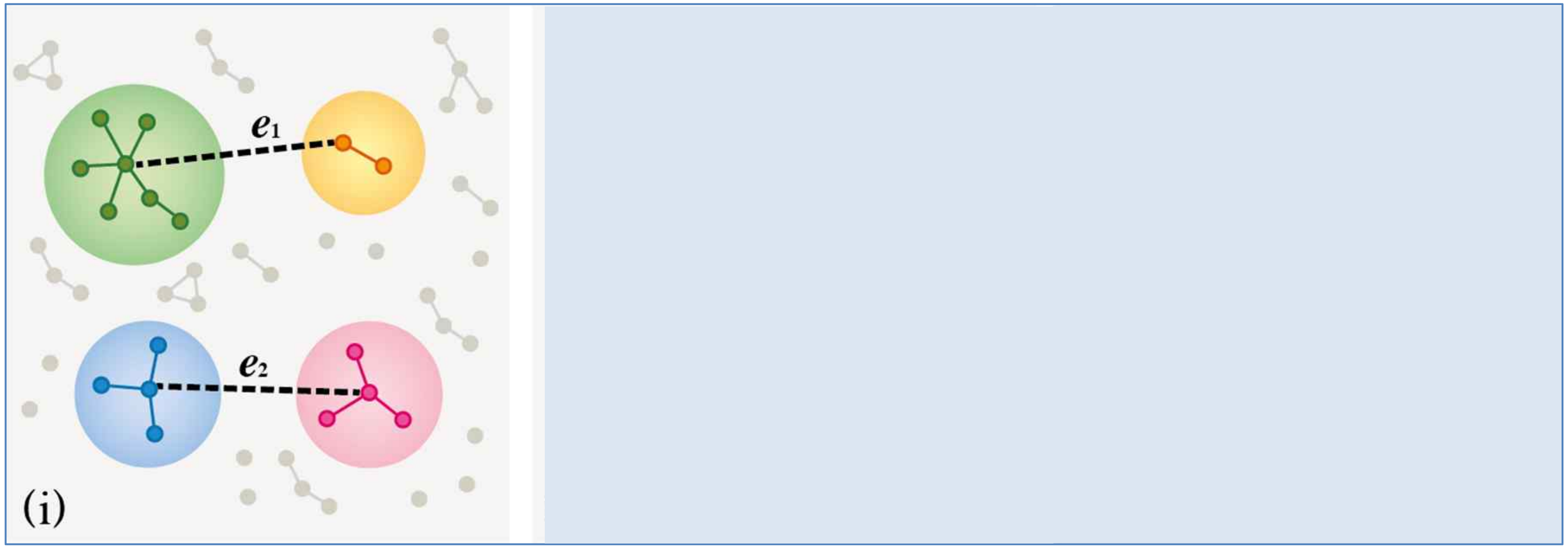
ERSR-B

➔Take $e_1$ (Absolutely)
Cluster size unchanged

ERSR-C

Case (ii) is excluded.

# 5. Intrinsic fault of product rule



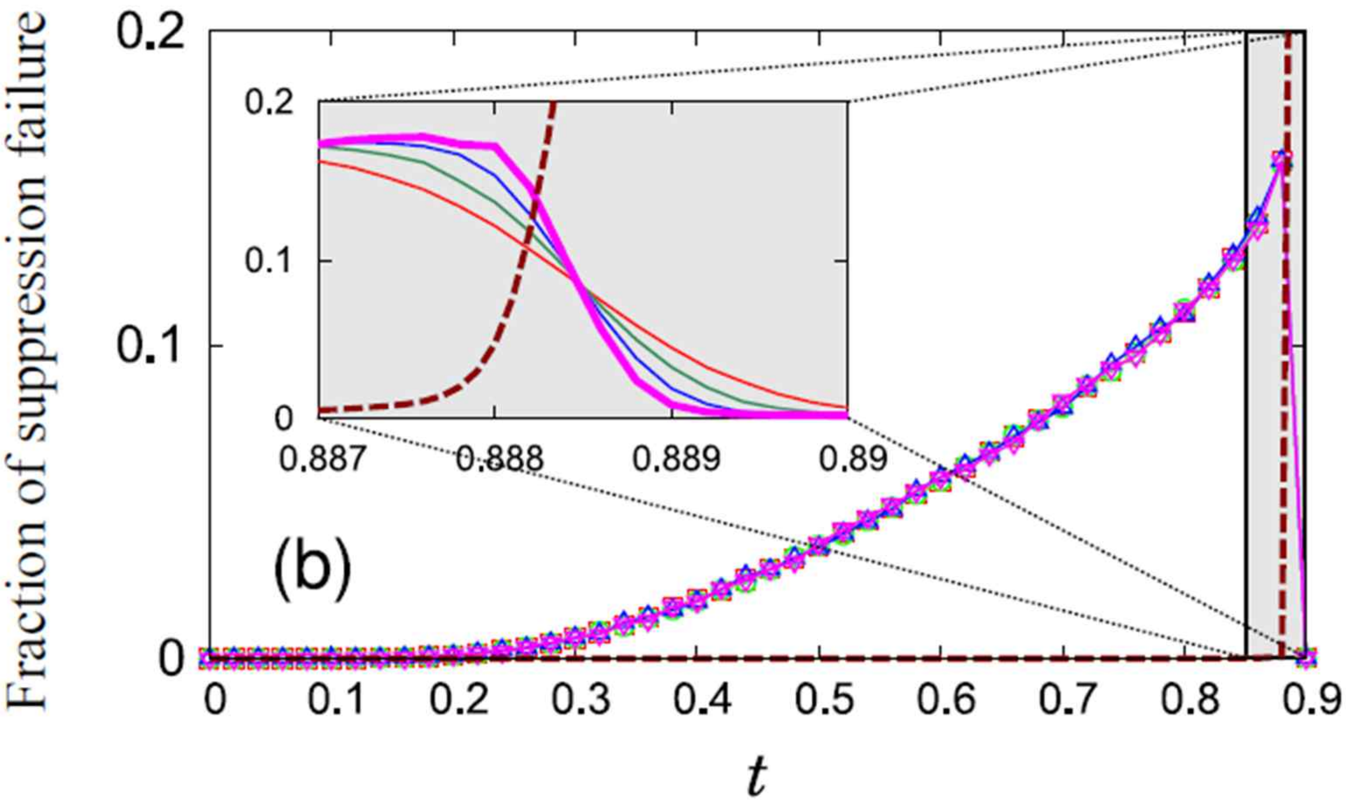

For the case (i)

$S_{1a}*S_{1b}=7*2=14$ vs.
$S_{2a}*S_{2b}=4*4=16$

$e_1$ was taken in PR.

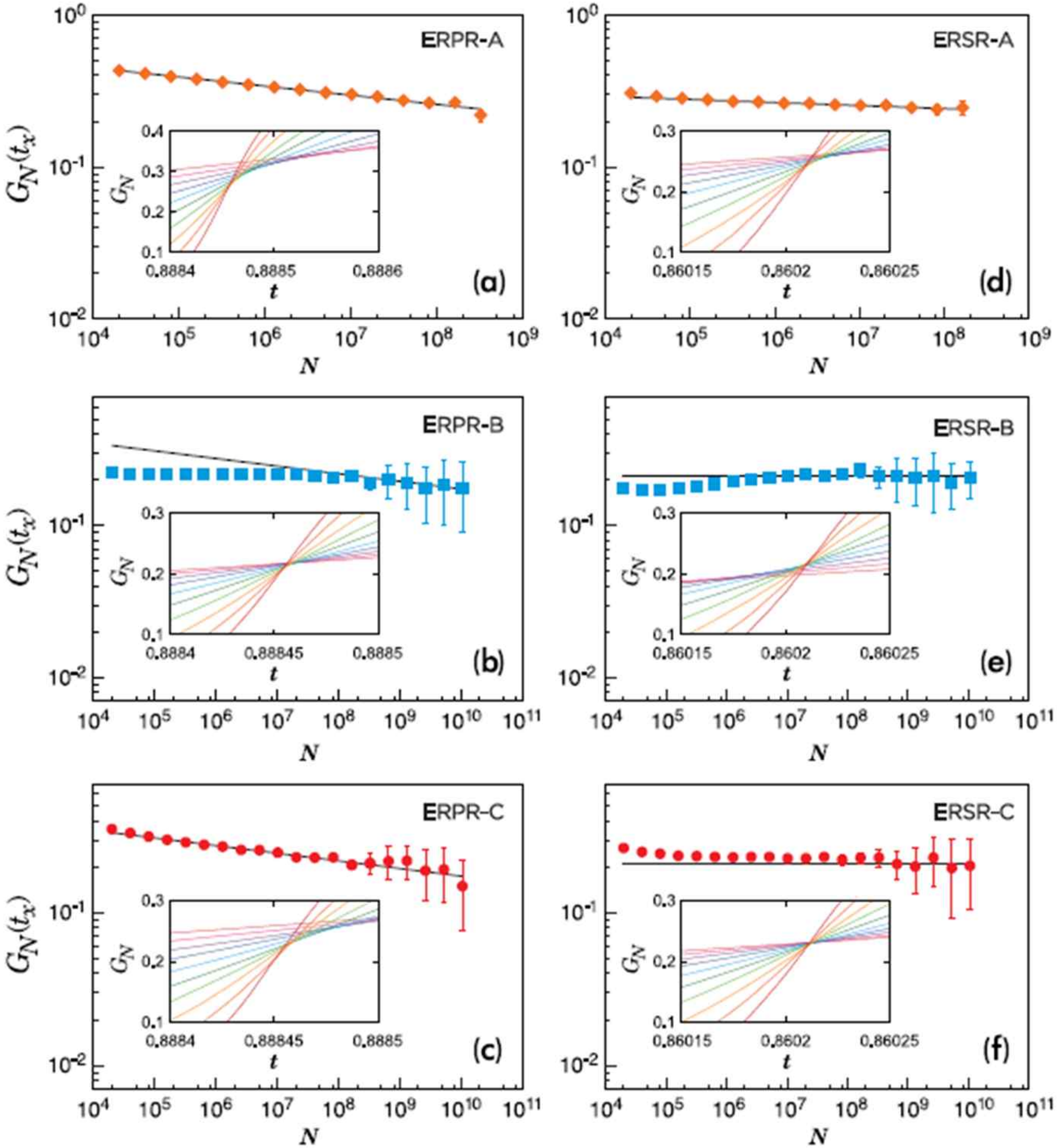
$S_{1a}+S_{1b}=7+2=9$ vs.
$S_{2a}+S_{2b}=4+4=8$

➔ $e_2$ has to be taken

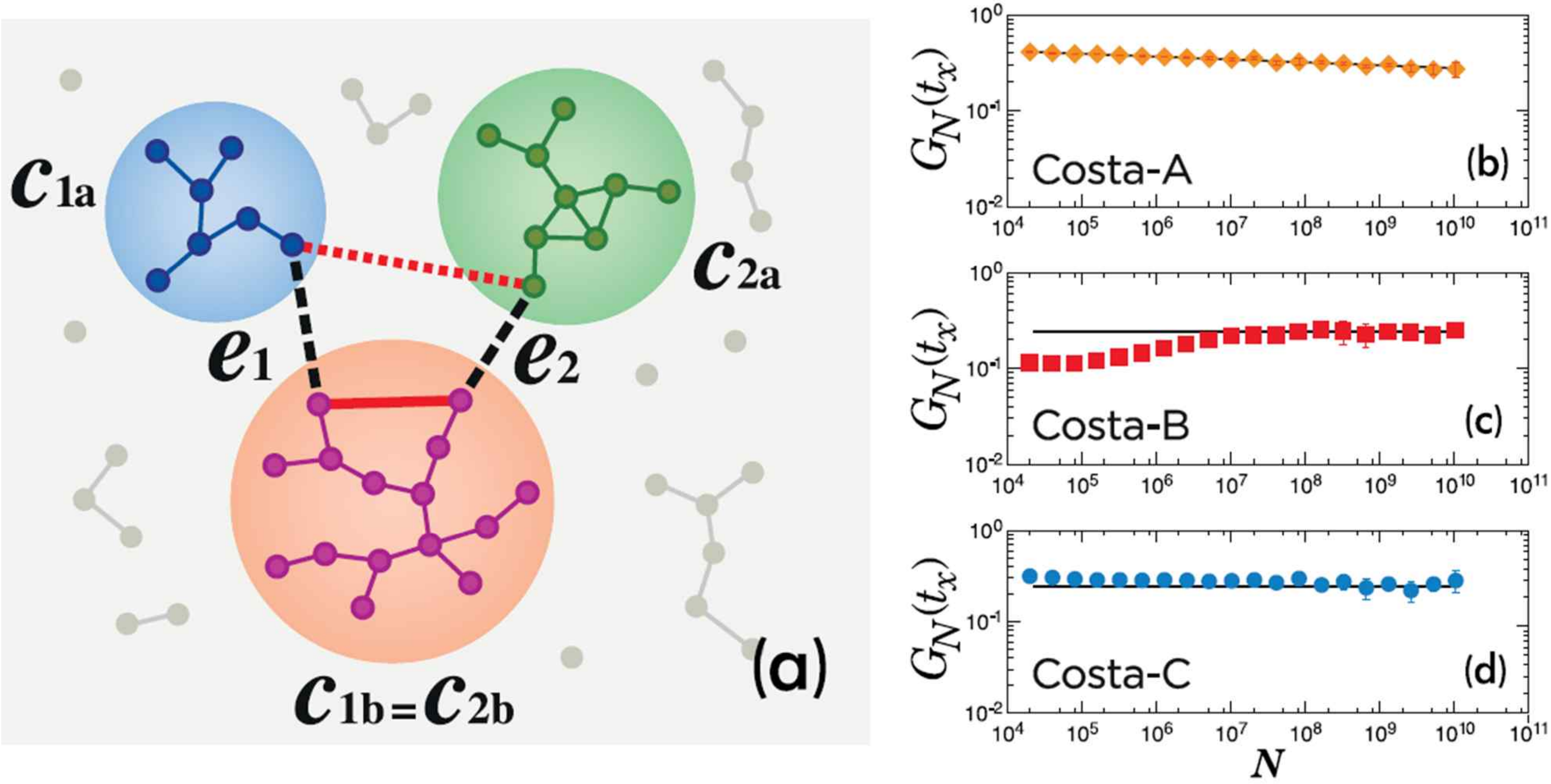


Fraction of suppression failure
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
t
(b)
0.2
0.1
0
0.887
0.888
0.889
0.89

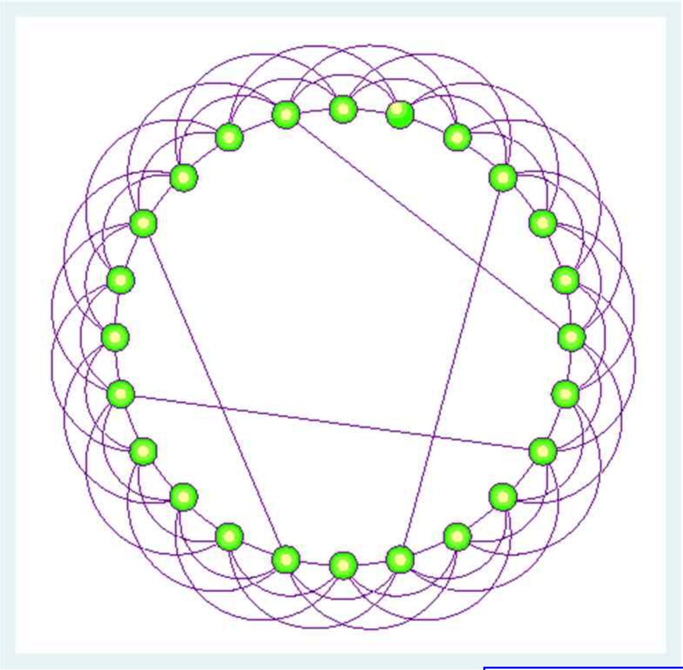
# 6. Results



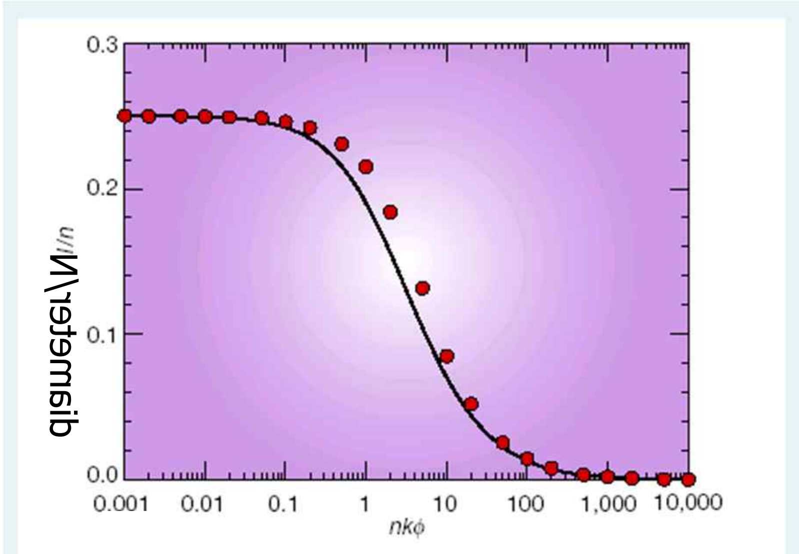
# 7. da Costa, Dorogovtsev, Goltsev, & Mendes model



# Small-world network model by Watts & Strogatz



[Strogatz 1998]




average number of shortcuts

MBER 15 PHYSICAL REVIEW LETTERS

**Small-World Networks: Evidence for a Crossover Picture**

Marc Barthélémy* and Luís A. Nunes Amaral
er for Polymer Studies and Department of Physics, Boston University, Boston, Massachusett
(Received 8 December 1998)

Addition or rewiring of p=1/N fraction of links changes to the SW network

# Conclusions

1. Size-dependent behavior of the order parameter is sensitive to the dynamic rules.

2. This makes it hard to reach a conclusion (discontinuous or continuous transition) based on numerical data.

3. Comparison between randomness in choosing edge candidates and suppression strength should to be made analytically. The difference should be compared with the order of time delayed due to the addition of intra-cluster edges.