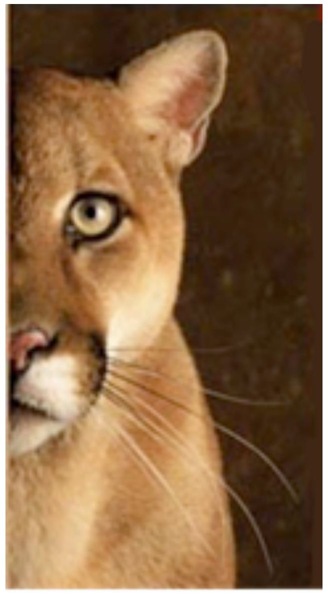
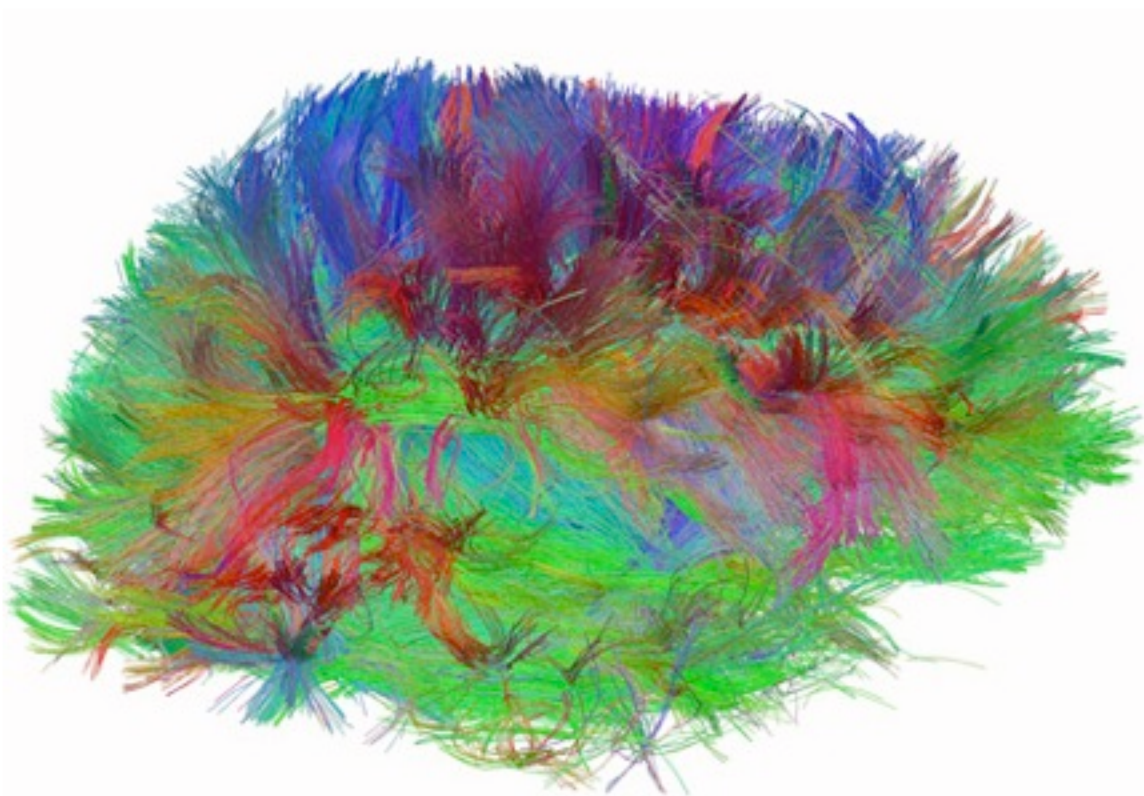
Structure of Correlations in Neuronal Networks

Krešimir Josić University of Houston

James Trousdale (UH), Yu Hu (UW)
Eric Shea-Brown (UW)

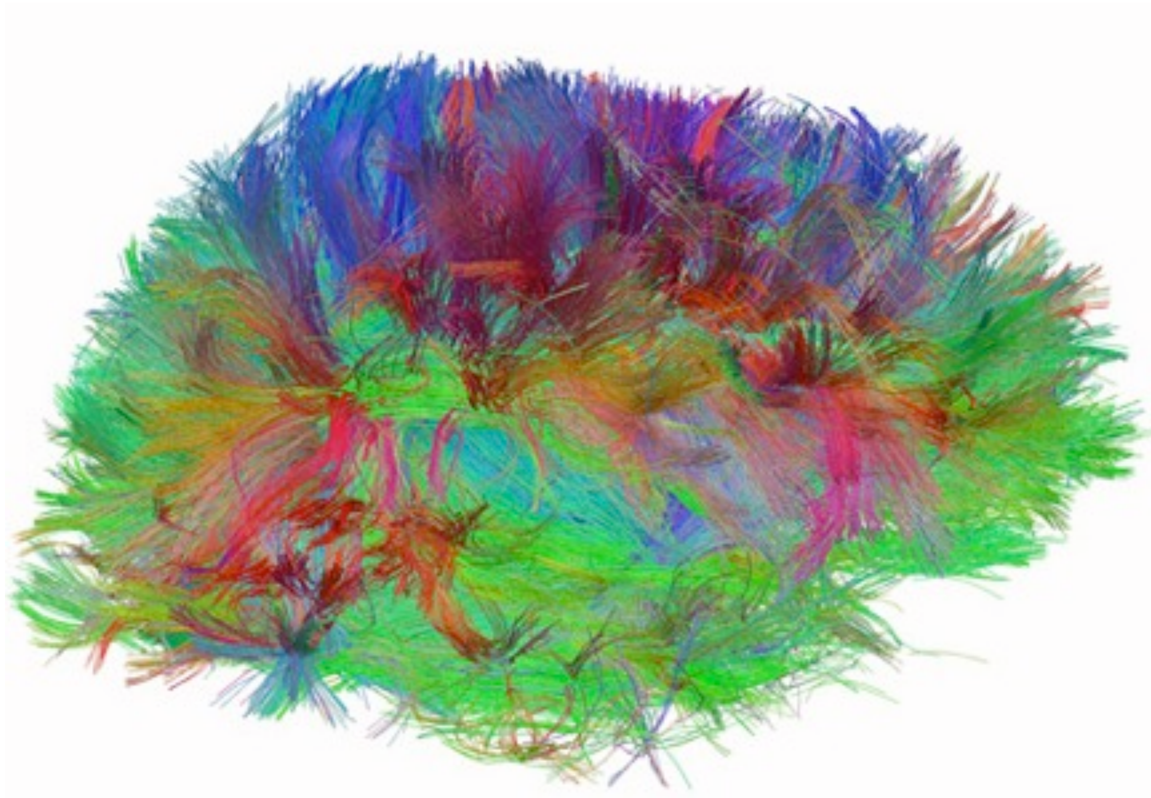


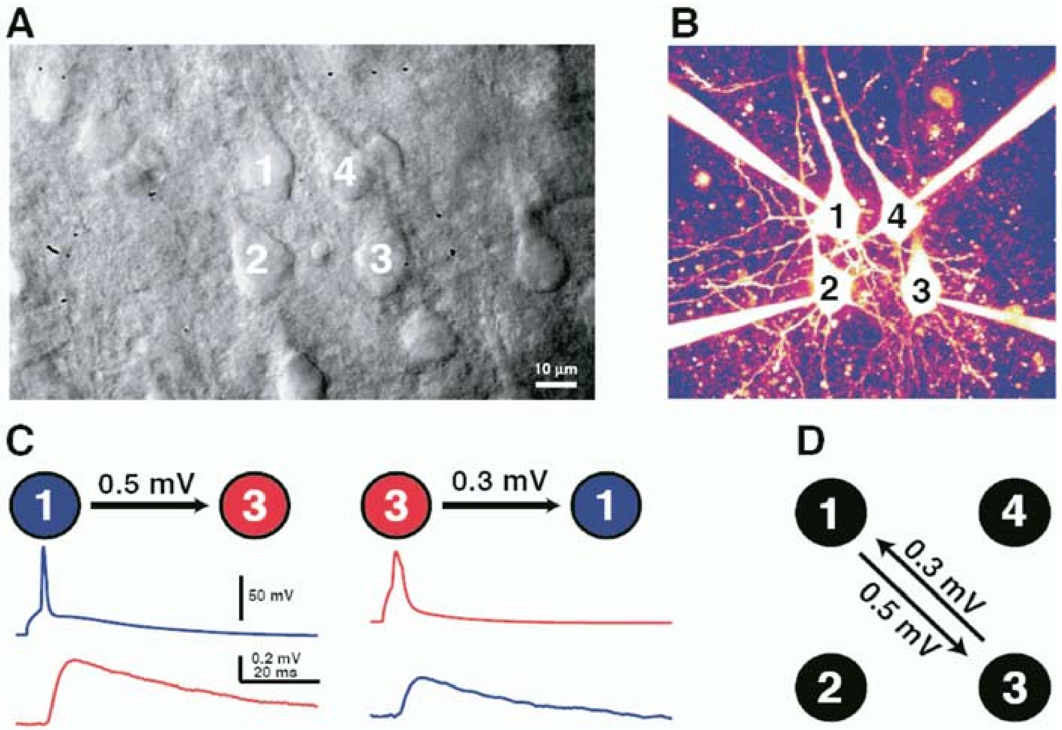
The Connectome



Van J. Wedeen, MGH/Harvard

The Connectome

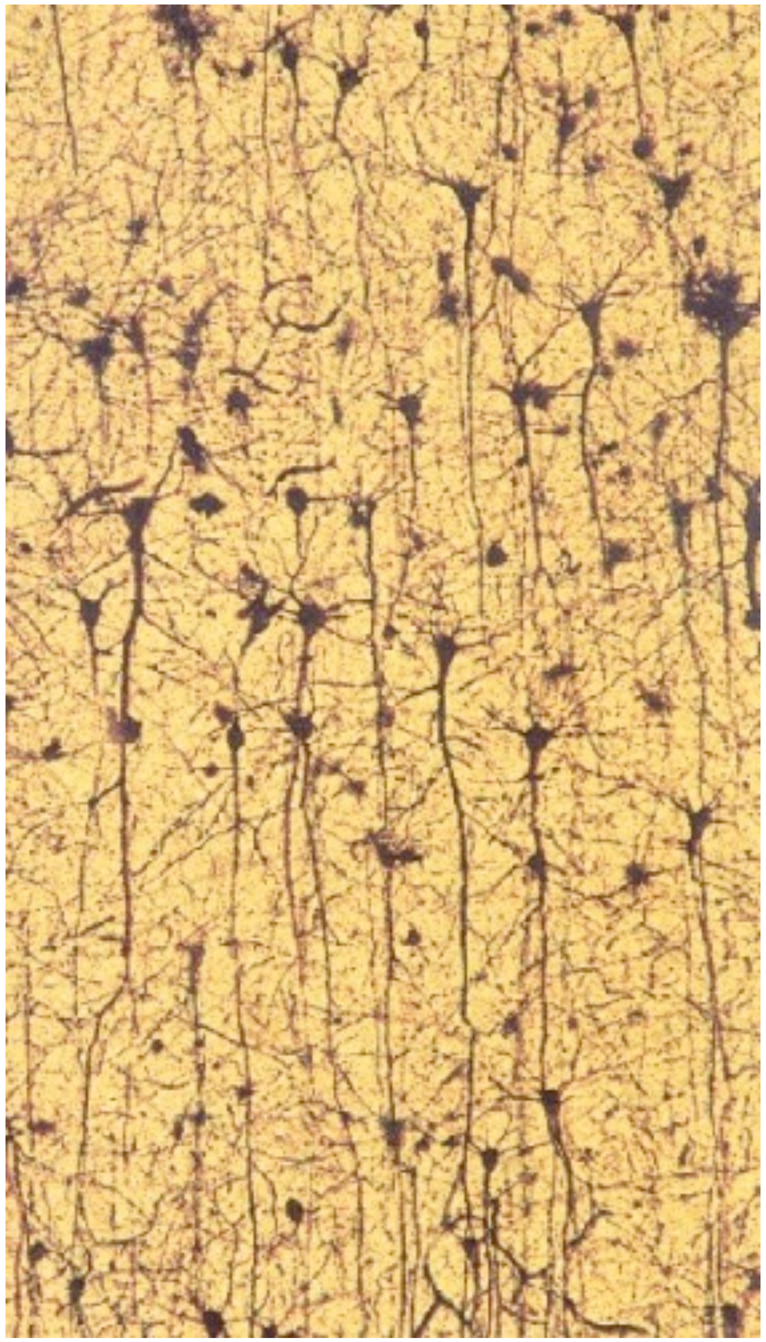


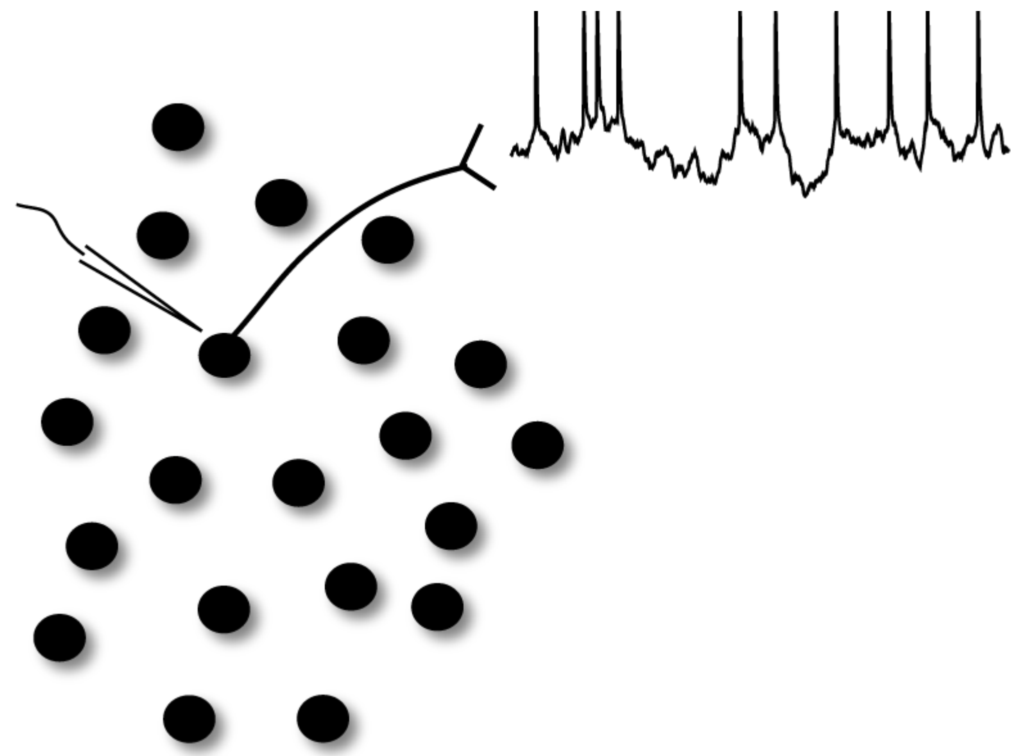


Van J. Wedeen, MGH/Harvard

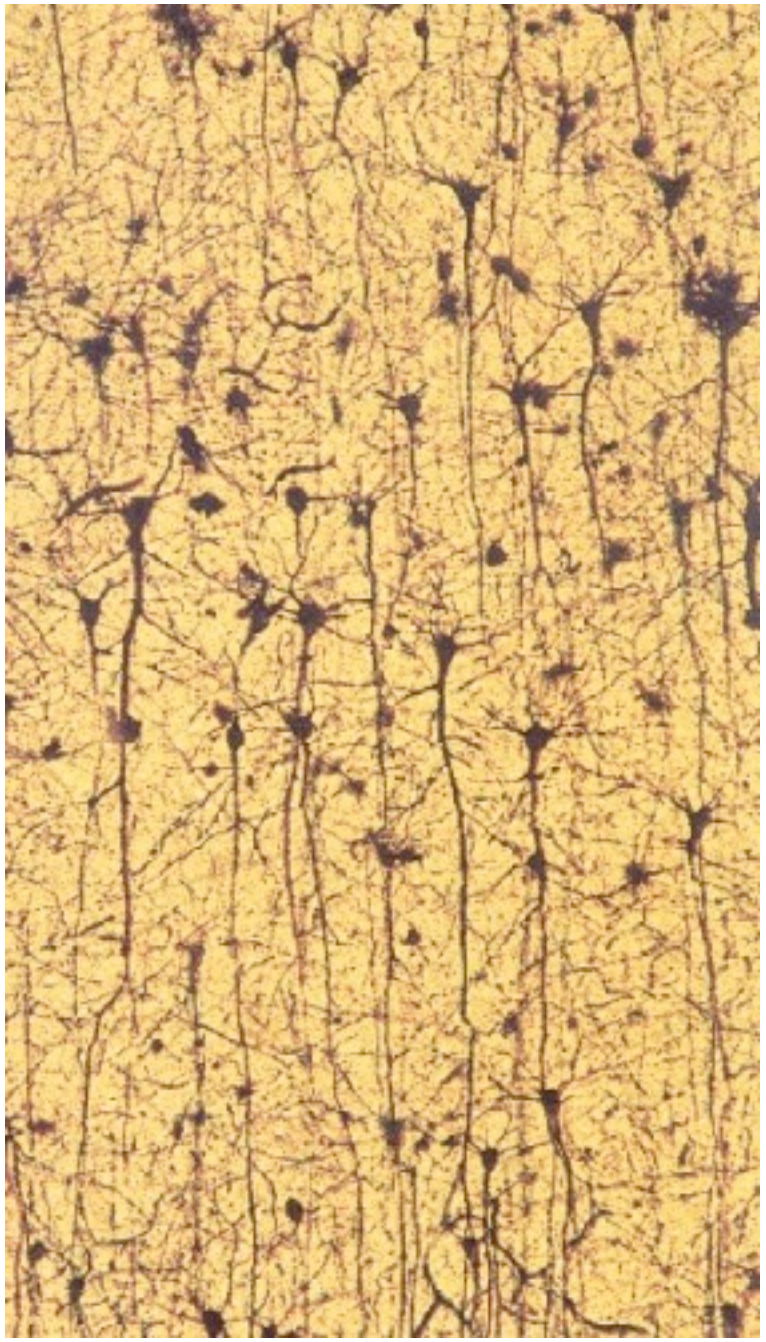
Song, et al. 2004

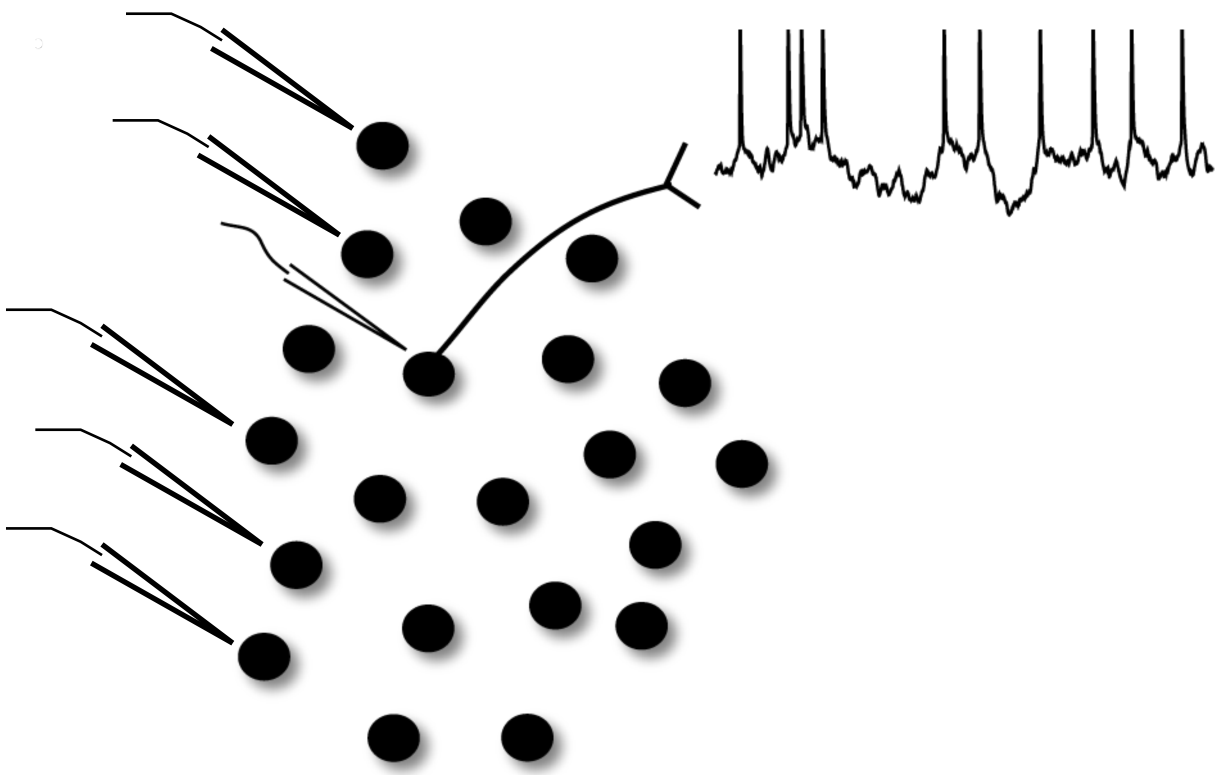
Recordings from neurons



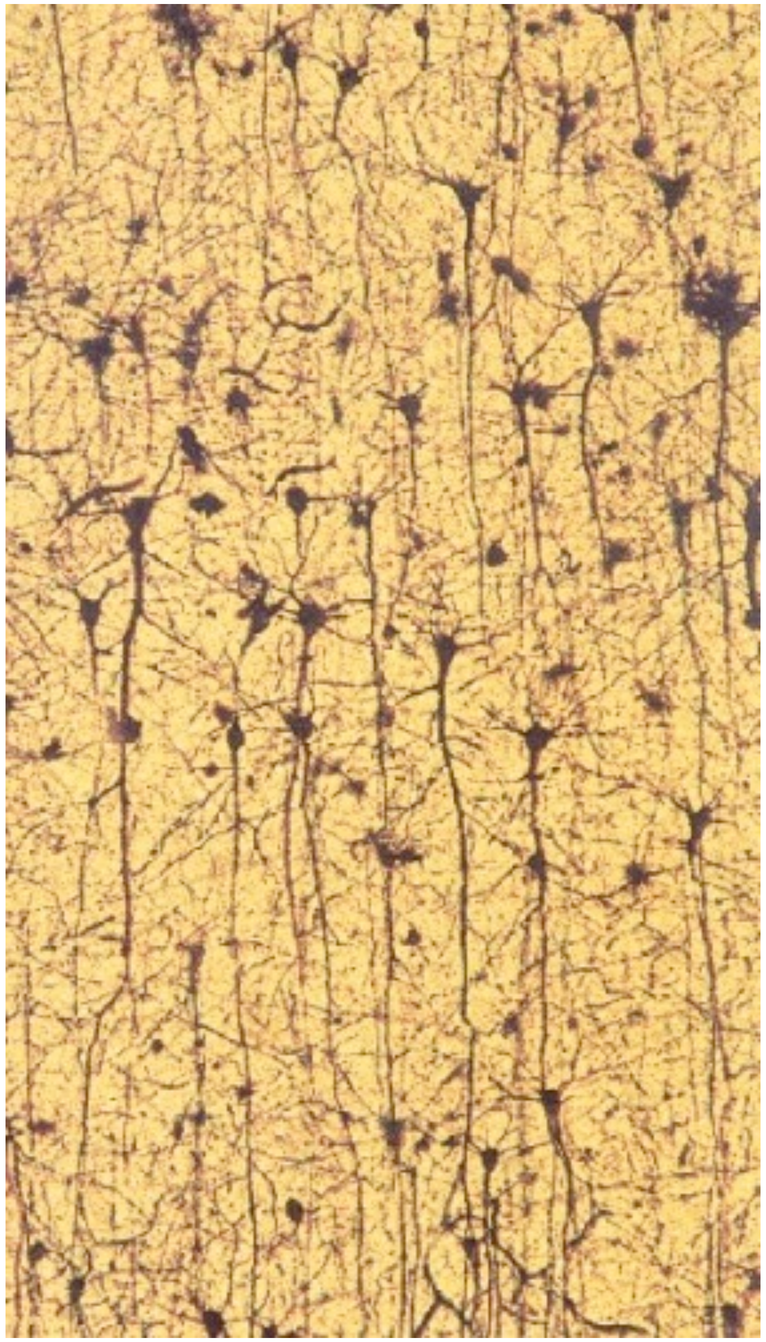


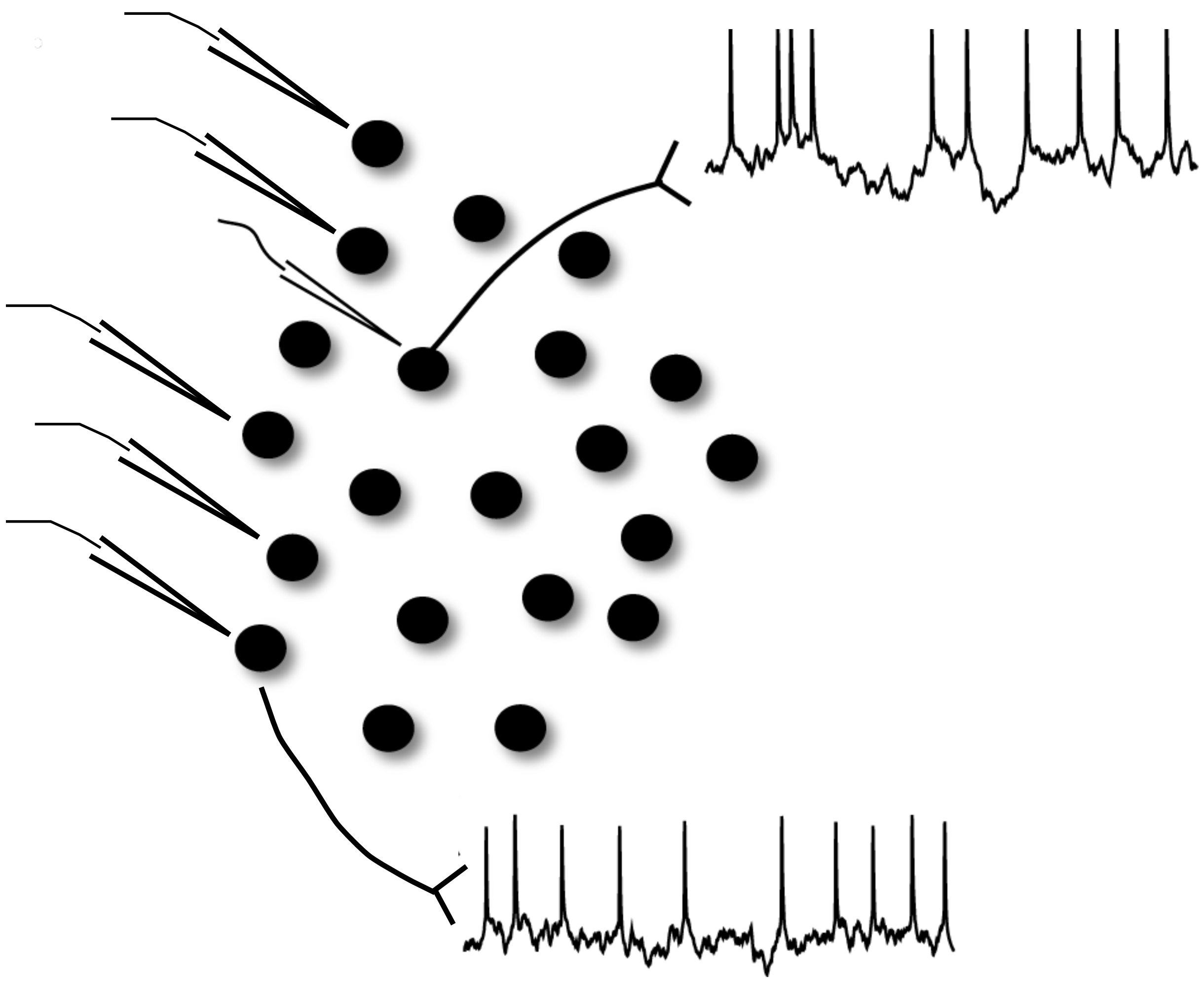
Recordings from neurons



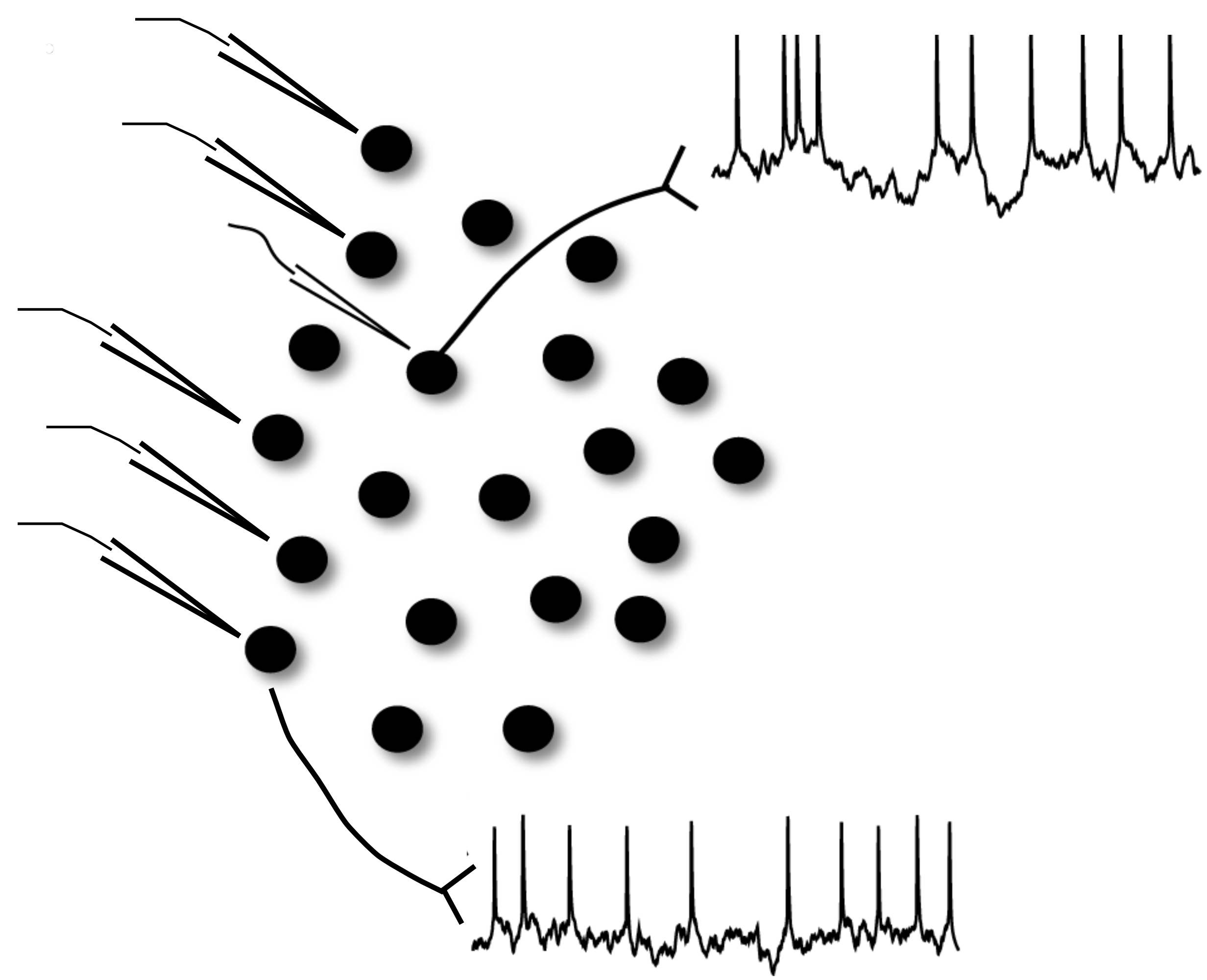


Recordings from neurons





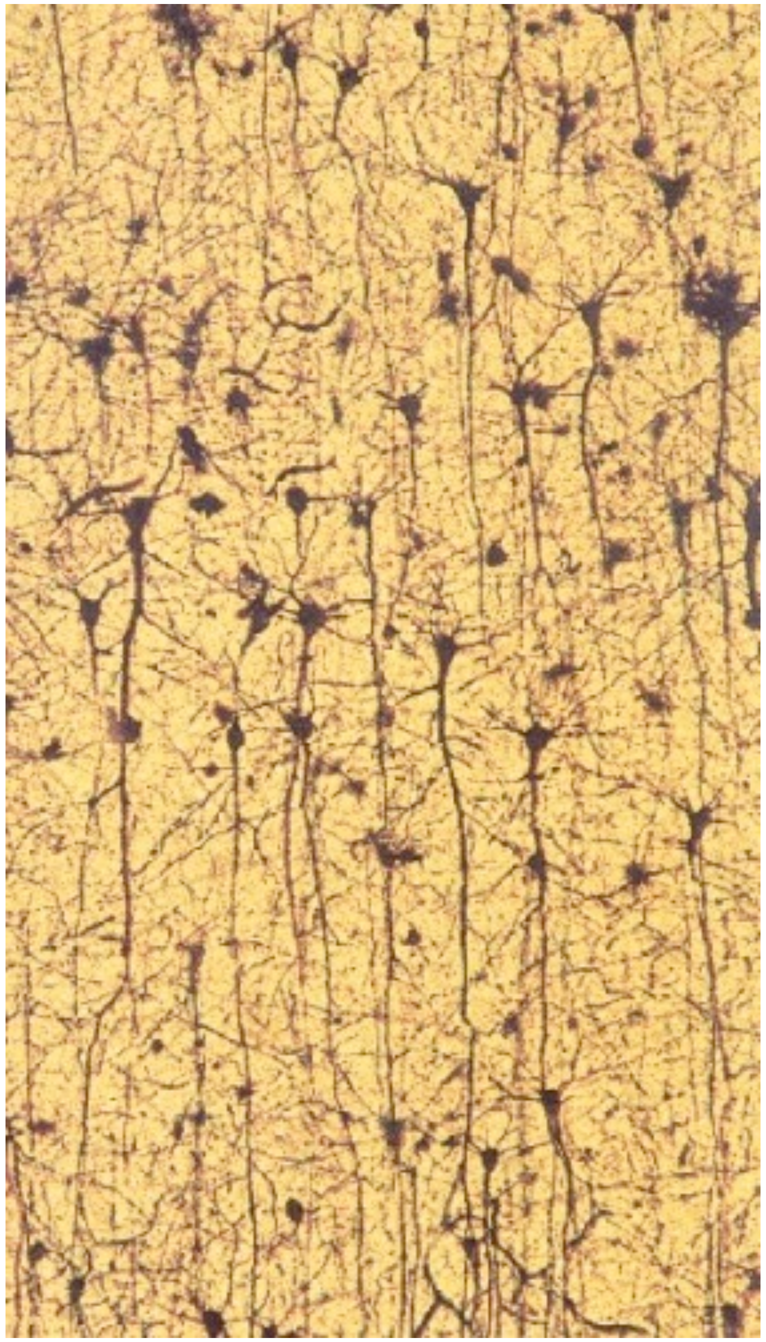
Recordings from neurons



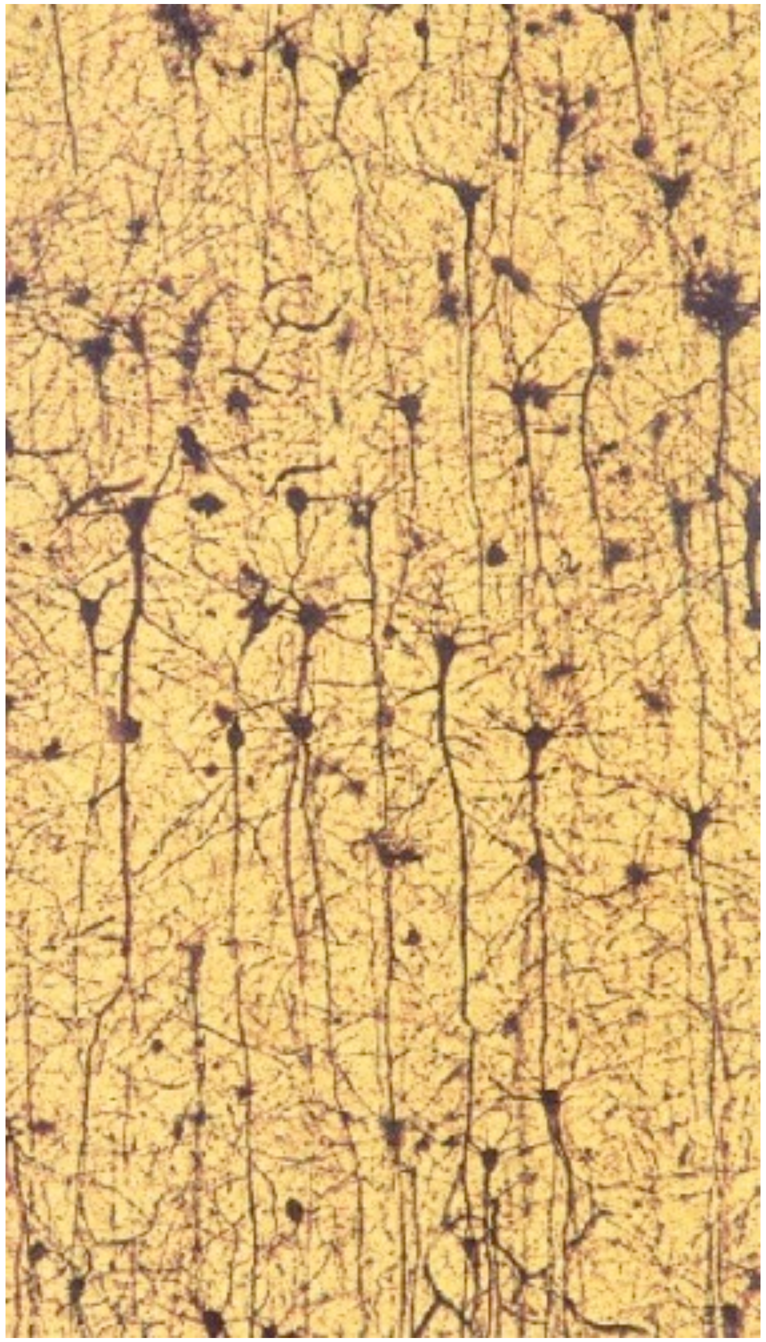
Are neuronal responses dependent or independent?

Tolias, Dragoi, Smirnakis, Angelaki, ...

Recordings from neurons

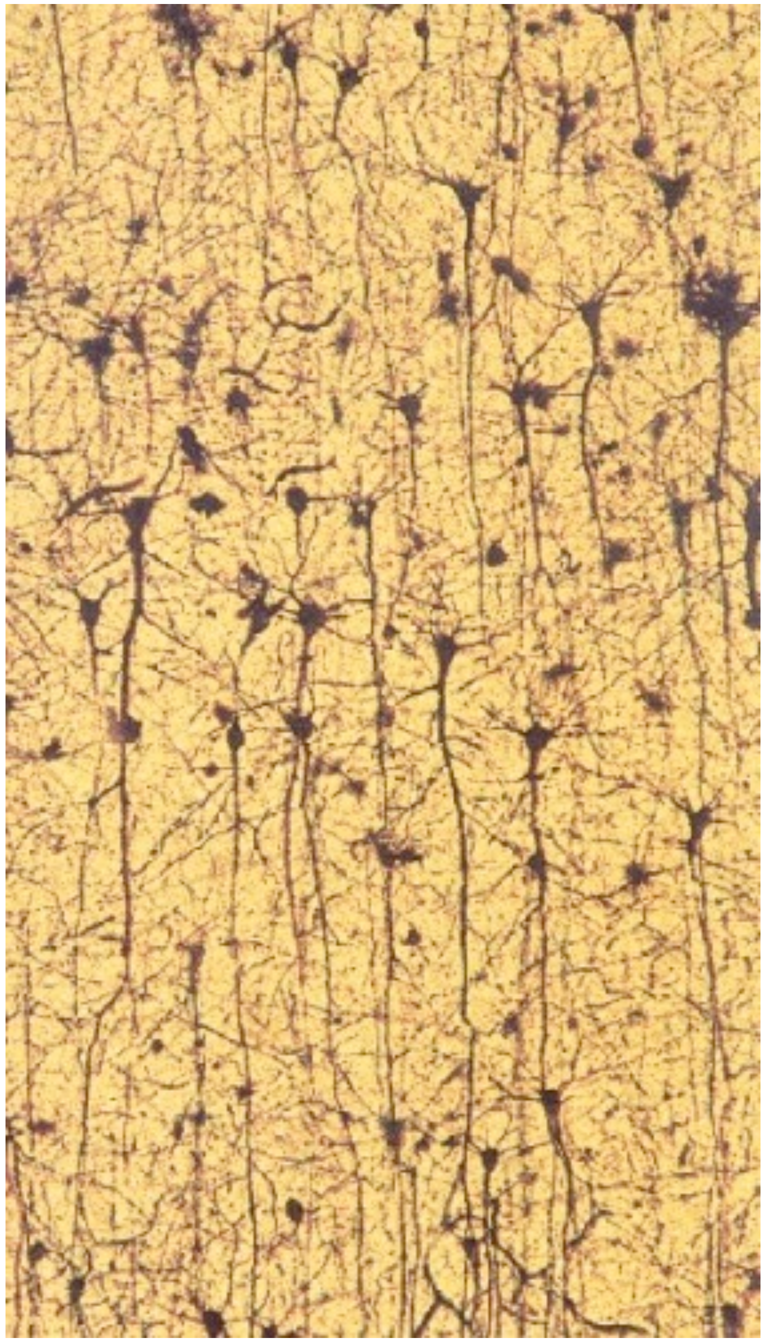


Recordings from neurons



How are structure and dynamics related in neuronal networks?

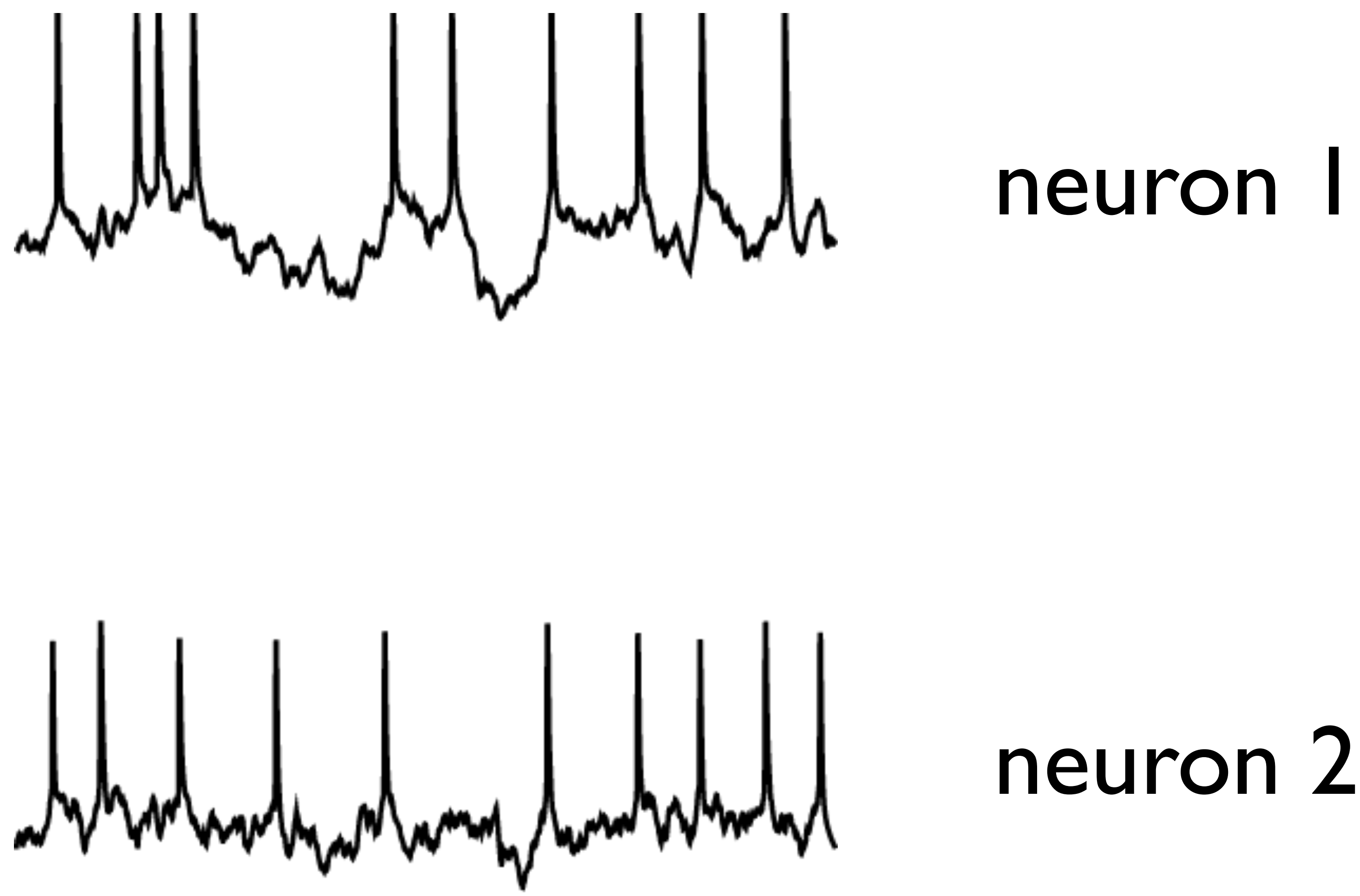
Recordings from neurons

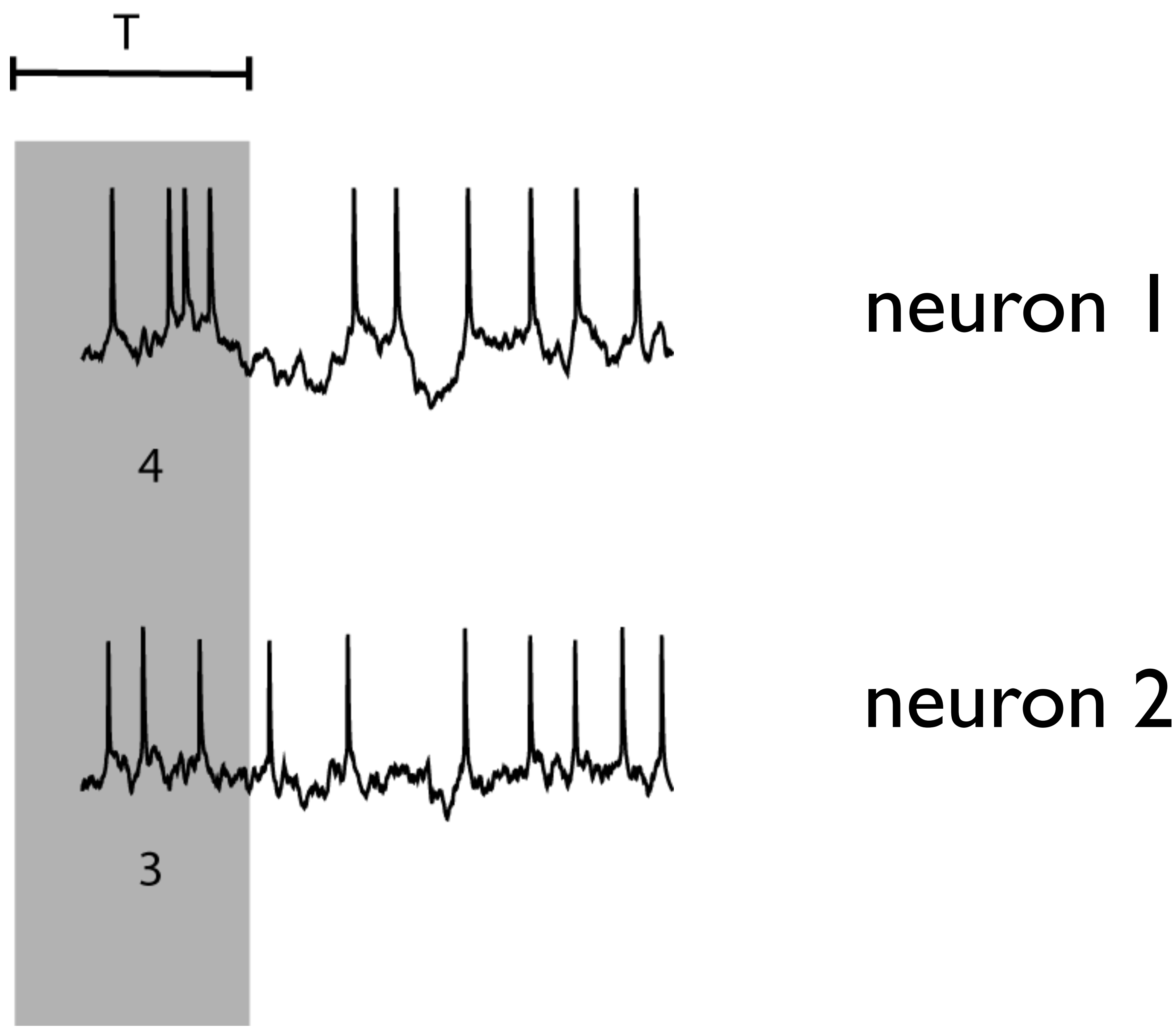


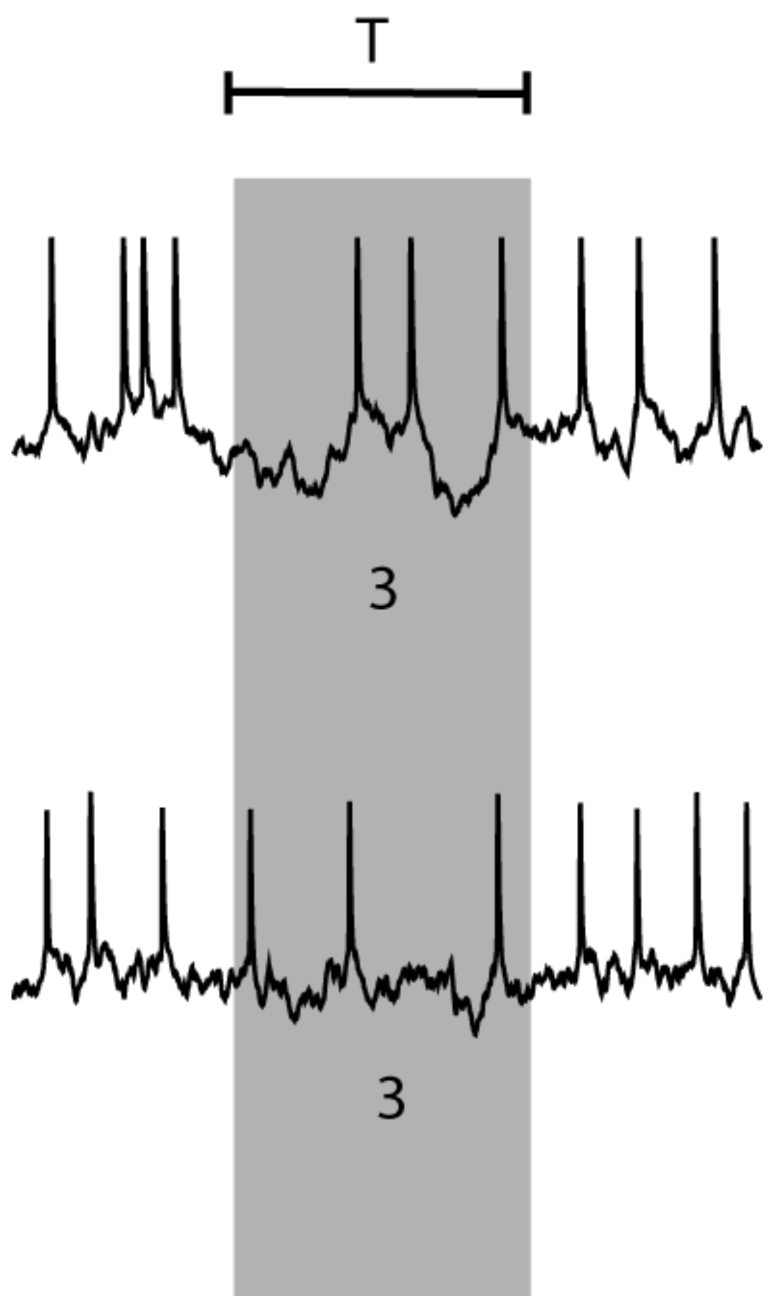
How are structure and dynamics related in neuronal networks?

Synchrony - is probably atypical

Correlation - a measure of dependence

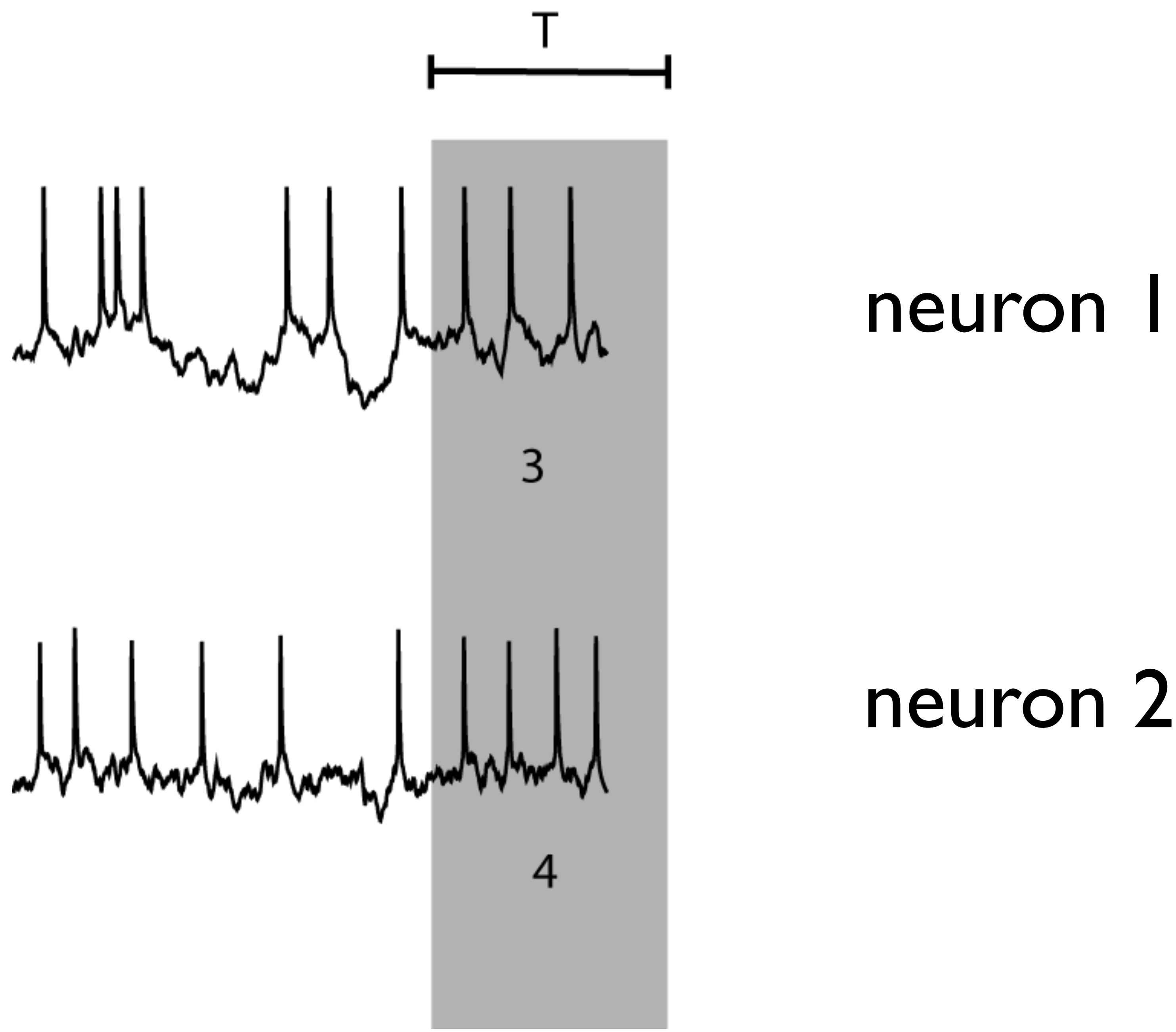


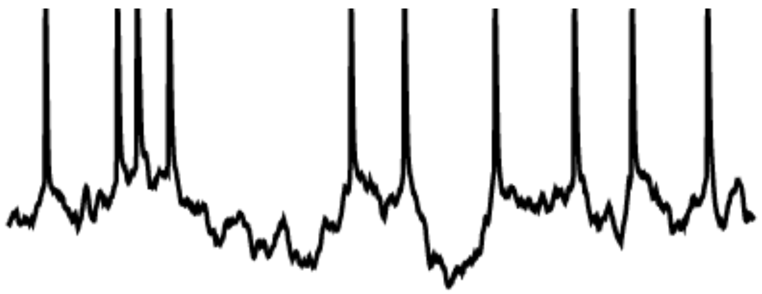

T
4
3
neuron 1
neuron 2



neuron 1

neuron 2


T
3
4
neuron 1
neuron 2



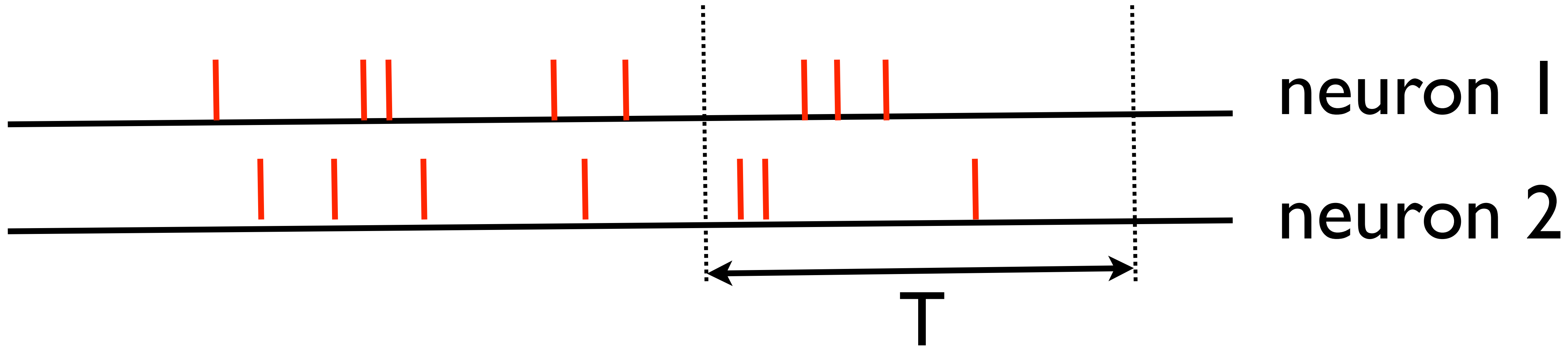
neuron 1

$n_1 =$ 4 3 3 ...

neuron 2

$n_2 =$ 3 3 4 ...

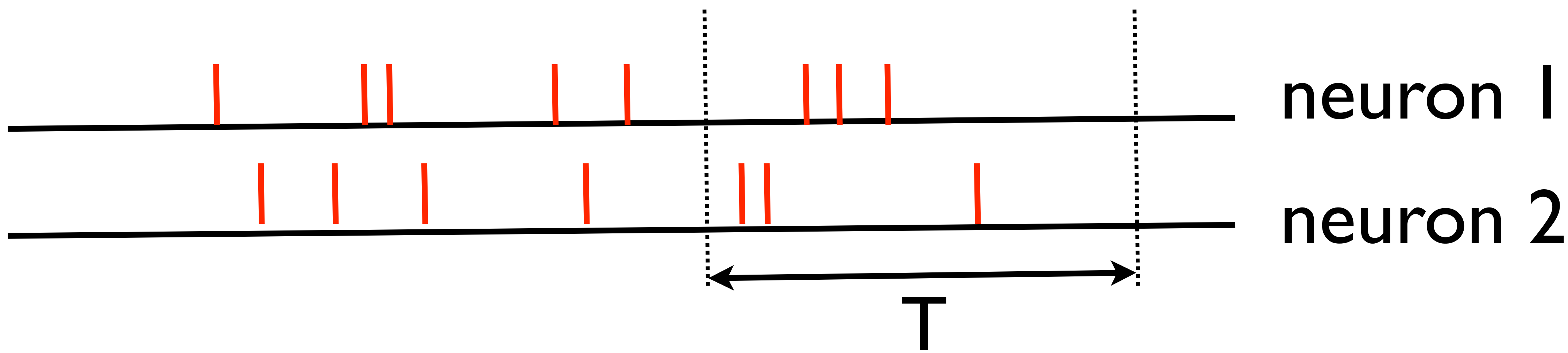
Correlations



n_i - (random) number of spikes of neuron *i* during a time T.
Correlation coefficient of the output is

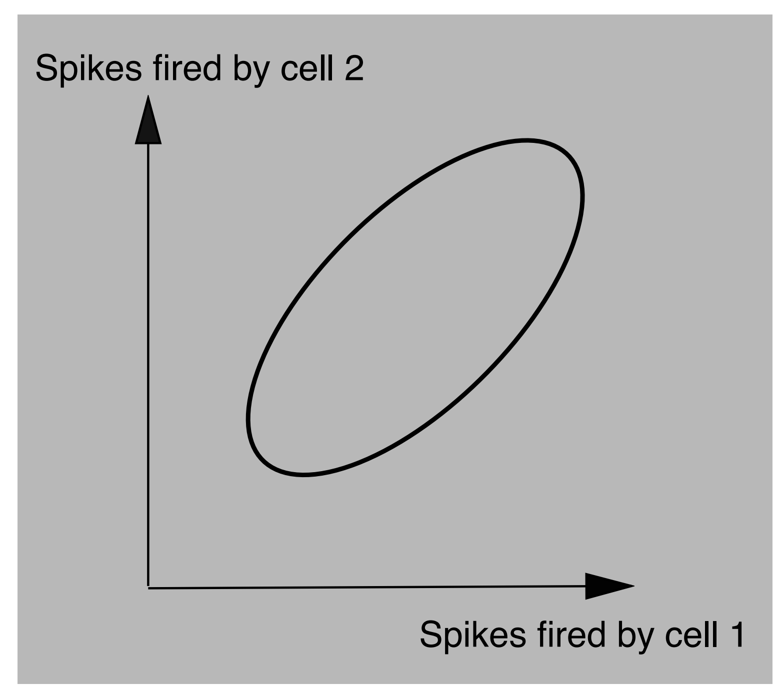
$$\rho_T = \frac{\text{Cov}(n_1, n_2)}{\sqrt{\text{Var}(n_1)\text{Var}(n_2)}}$$

Correlations



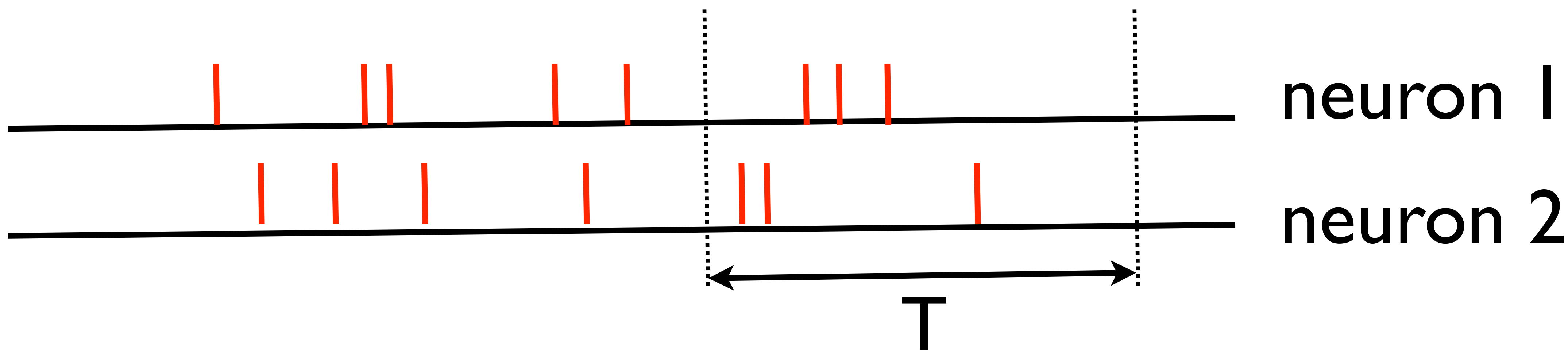
n_i - (random) number of spikes of neuron *i* during a time T.
Correlation coefficient of the output is

$$\rho_T = \frac{\mathrm{Cov}(n_1, n_2)}{\sqrt{\mathrm{Var}(n_1)\mathrm{Var}(n_2)}}$$



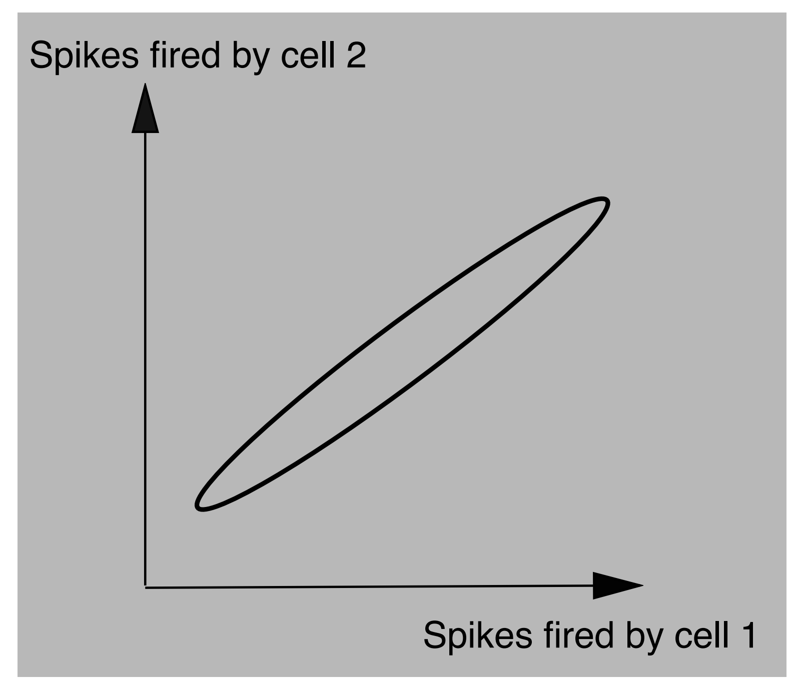
low correlation

Correlations



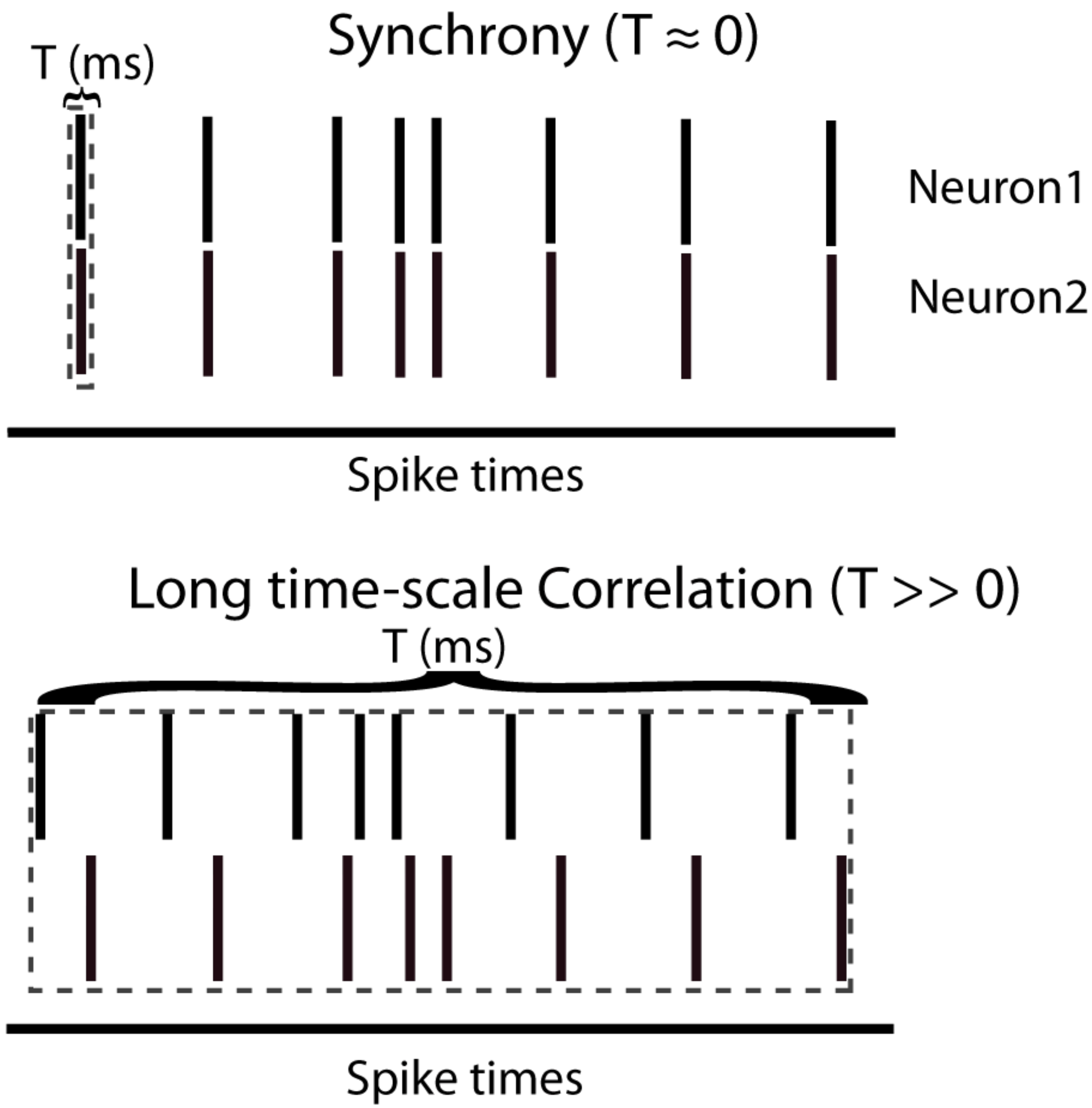
n_i - (random) number of spikes of neuron *i* during a time T.
Correlation coefficient of the output is

$$\rho_T = \frac{\mathrm{Cov}(n_1, n_2)}{\sqrt{\mathrm{Var}(n_1)\mathrm{Var}(n_2)}}$$

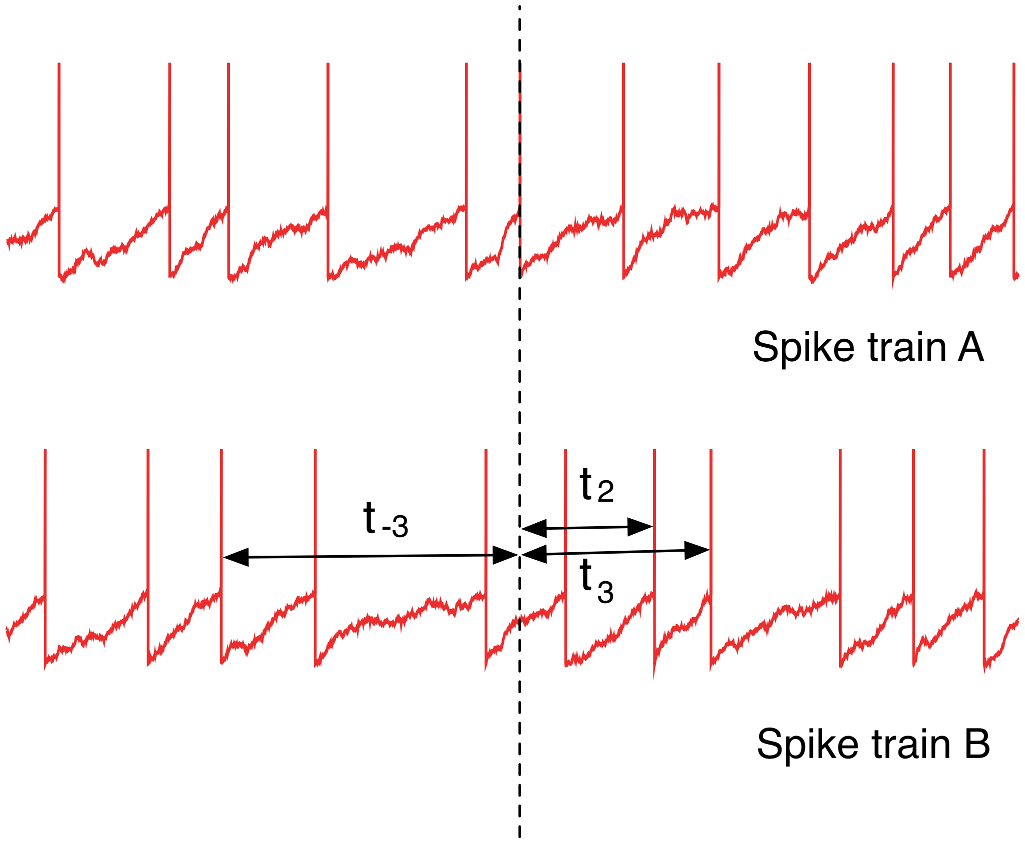


high correlation

Short vs long timescale correlations

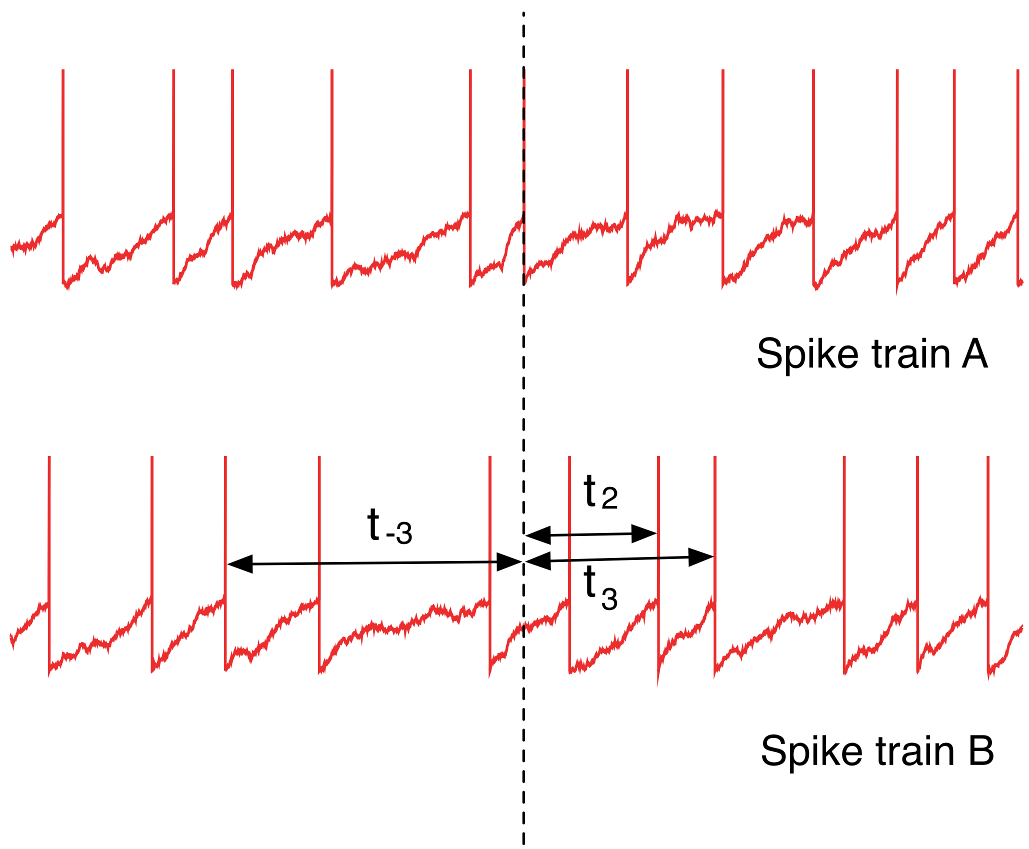


Cross-Correlation Function

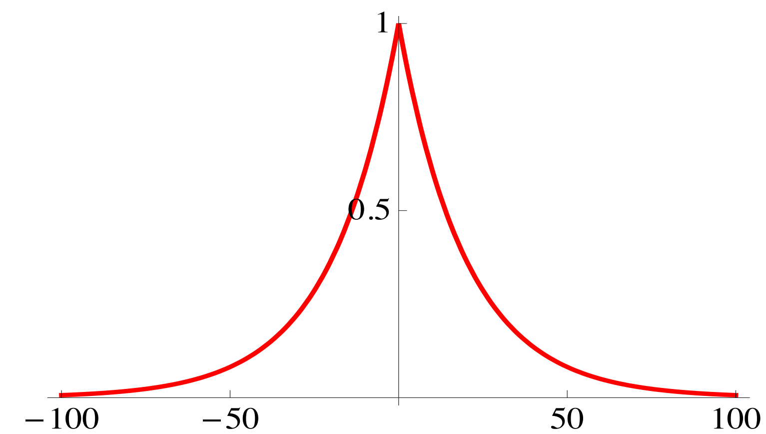


Conditional probability of spike in B, given spike in A.

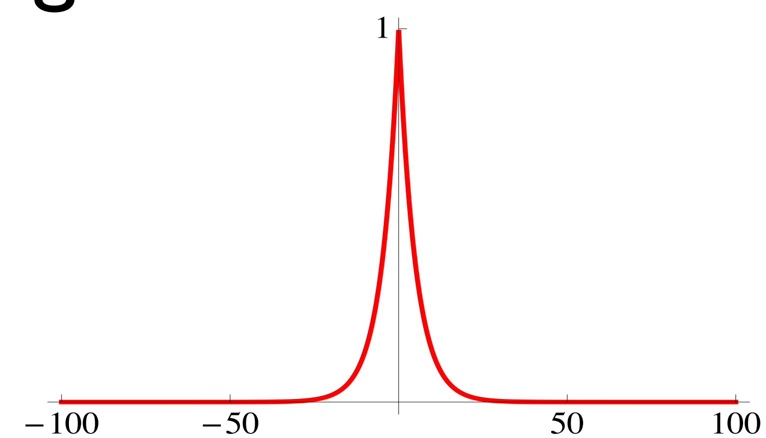
Cross-Correlation Function



Conditional probability of spike in B, given spike in A.

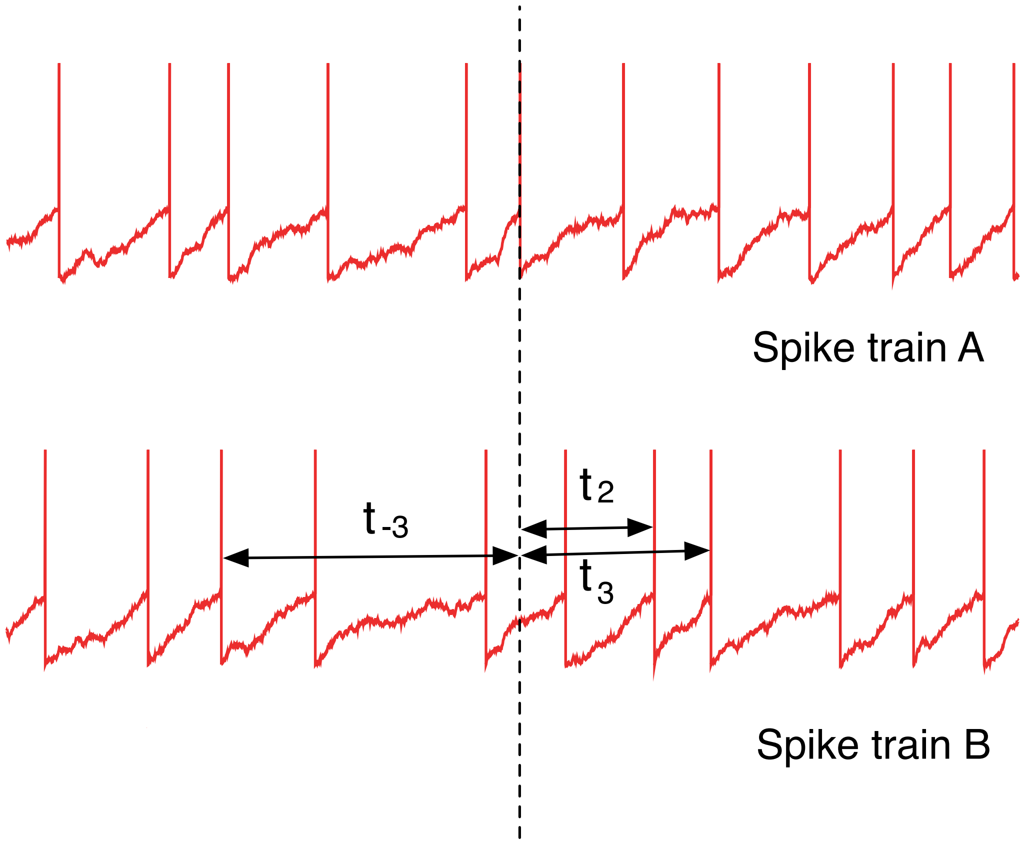


Long-timescale correlation



Short-timescal (synchrony)

Cross-Correlation Function, $C_{i,j}(\tau)$

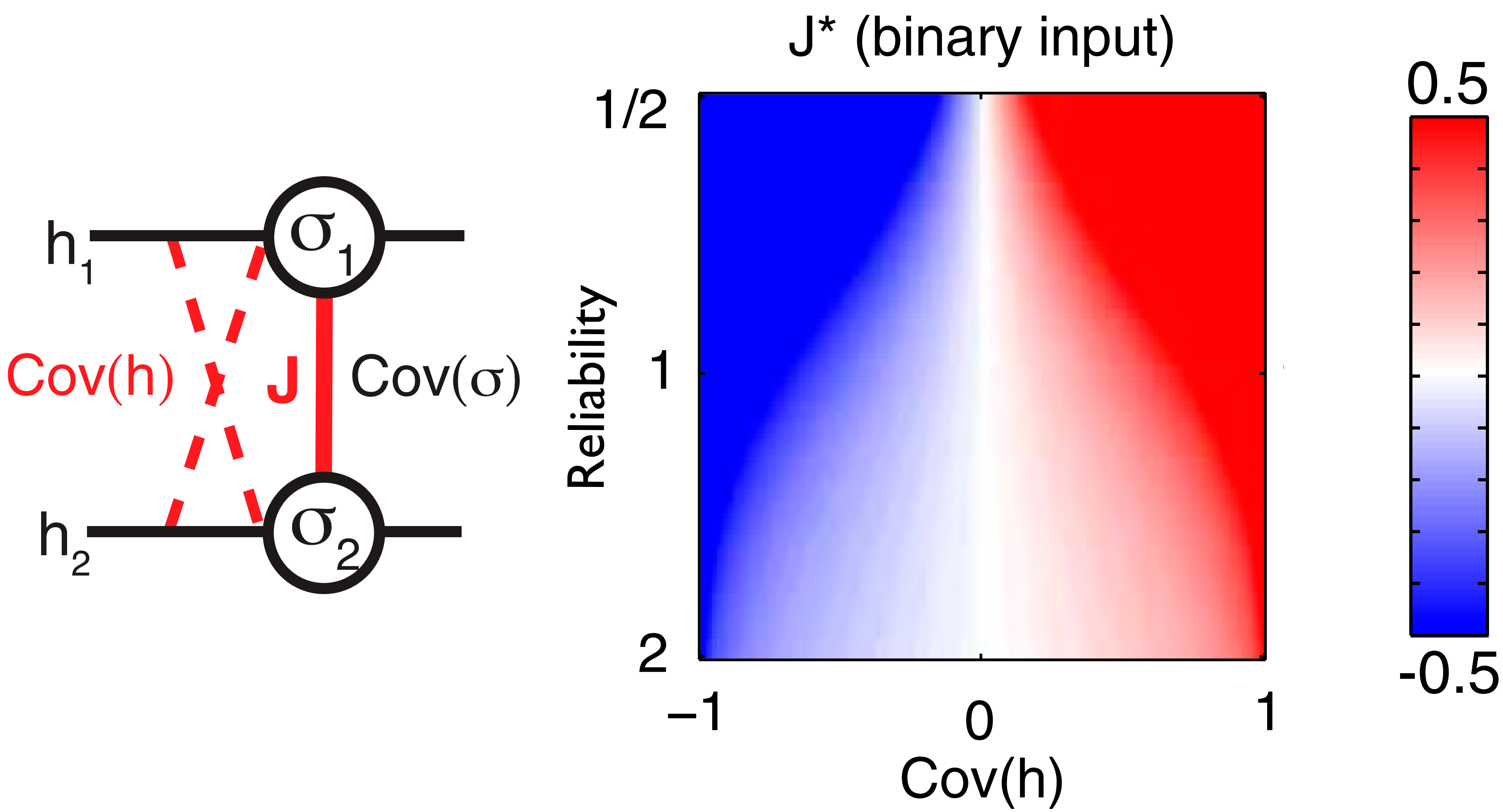


Conditional probability of spike in B, given spike in A.

After normalization

$$\mathbf{C}_{ij}(\tau) = \mathrm{cov}\left(y_i(t+\tau), y_j(t)\right)$$

Correlations Impact Neural Computation



Tkačik, et al. 2010

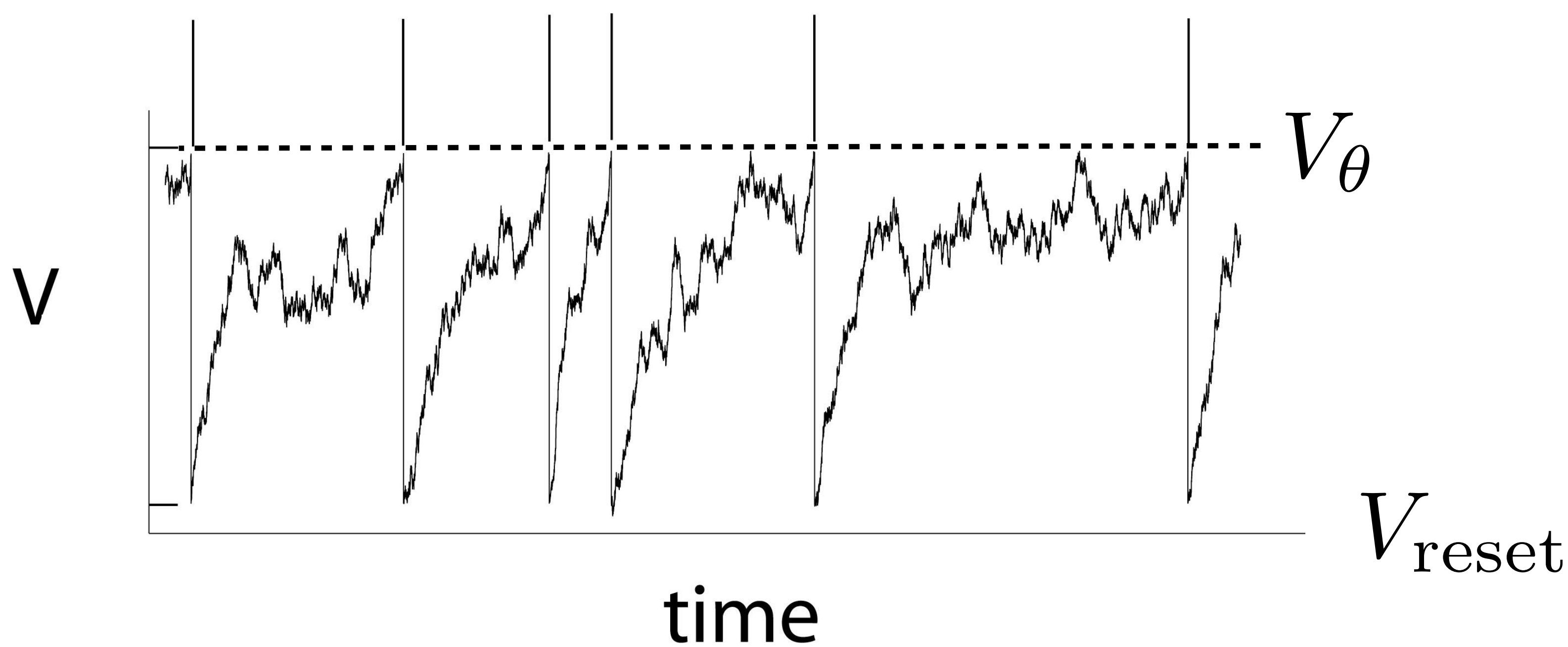
Models of Neurons - Integrate and Fire

$$\frac{dV}{dt} = -\frac{V}{\tau_m} + \Psi(V) + \mu + \sqrt{2D}\eta(t)$$

Subthreshold membrane potential

$$V(t) = V_\theta \Rightarrow V(t^+) = V_{\text{reset}}$$

Fire and Reset



Rate r - number of spikes per second

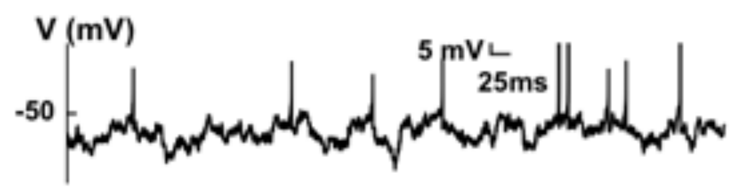
Linear response kernel, A(t)

input to cell, $X(t)$

Linear response kernel, A(t)

input to cell, $X(t)$

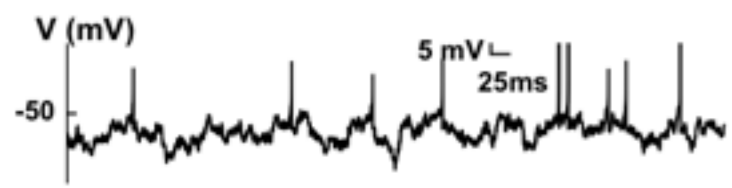
activity of cell



Linear response kernel, A(t)

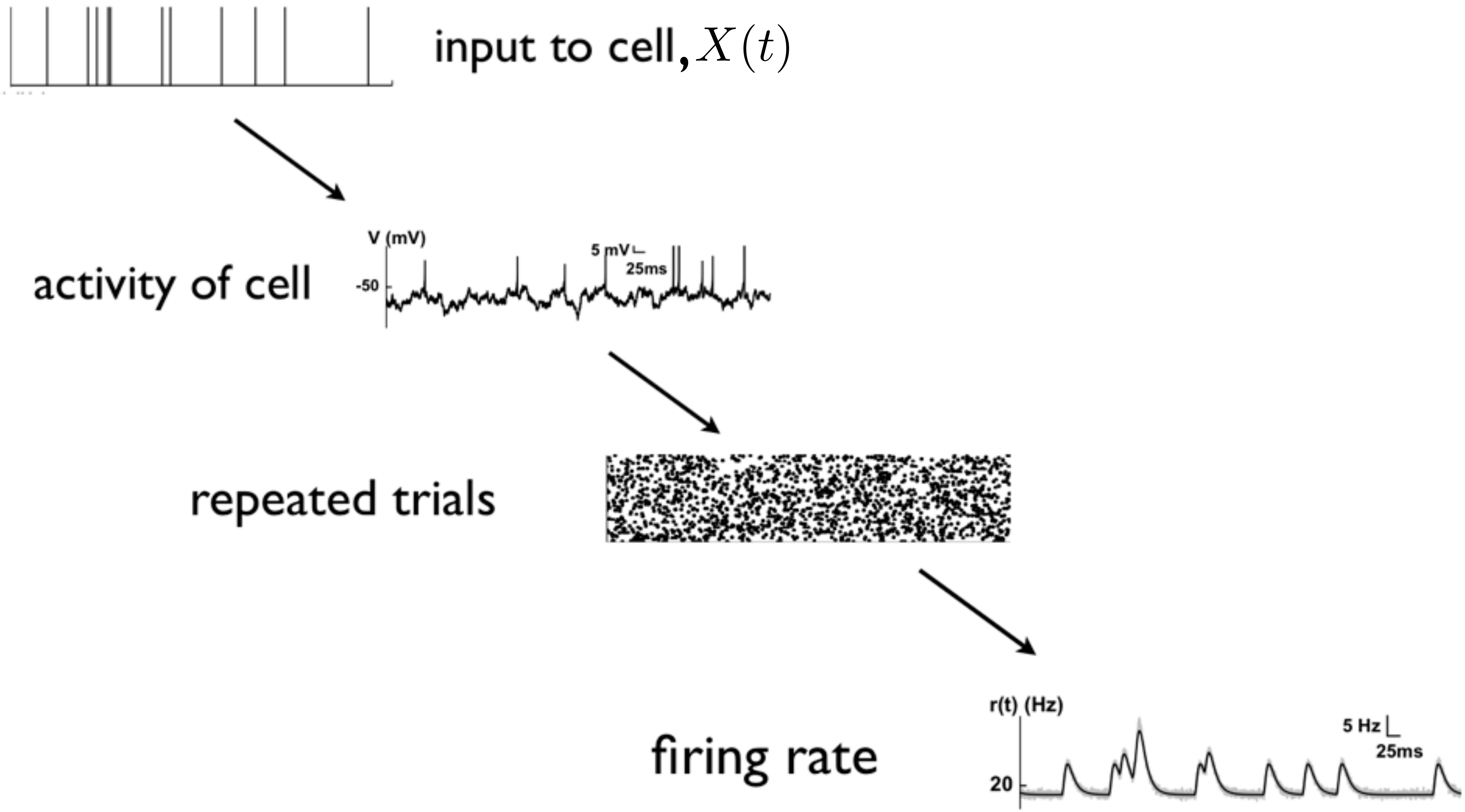
input to cell, $X(t)$

activity of cell



repeated trials

Linear response kernel, A(t)

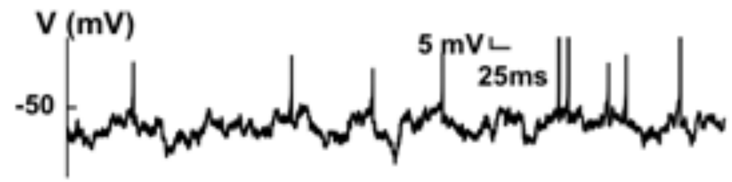


Linear response kernel, A(t)

input to cell, $X(t)$

$$r(t) = r_0 + (A * X)(t)$$

activity of cell



repeated trials

firing rate

Structure or correlations in networks

$$\frac{dV_i}{dt} = -\frac{V_i}{\tau_m} + \Psi(V_i) + \tilde{\mu} + \sqrt{2D}\eta(t) + (f_i - \langle f_i \rangle)$$

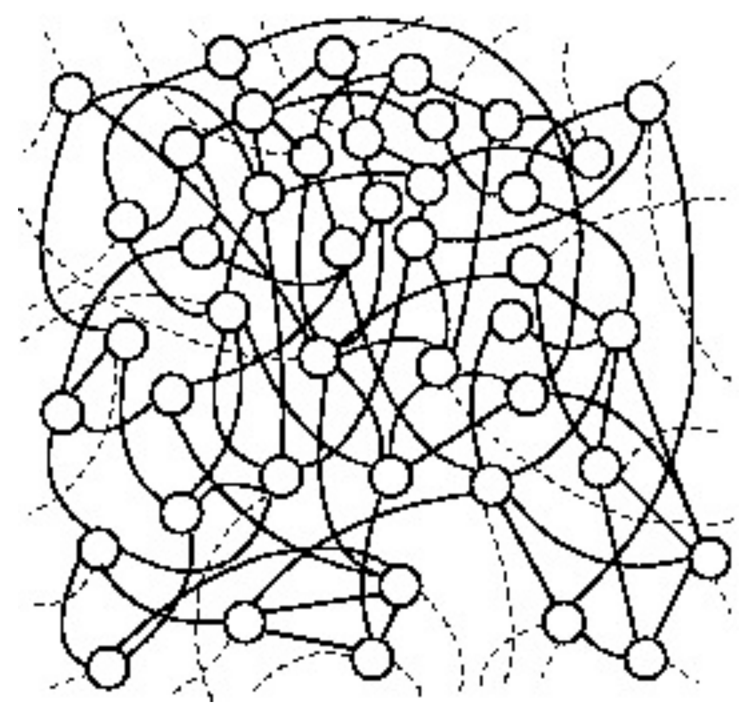
spike generating current → $\Psi(V_i)$

synaptic connections → $(f_i - \langle f_i \rangle)$

$$y_j(t) = \sum_i \delta(t - t_i^j)$$ output spike train of cell *j*

$$f_i(t) = \sum_j (\mathbf{J}_{ij} * y_j)(t)$$ synaptic coupling

$$\mathbf{J}_{ij}(t) = \begin{cases} \mathbf{W}_{ij} \left(\frac{t-\tau_{D,j}}{\tau_{S,j}^2} \right) \exp\left[-\frac{t-\tau_{D,j}}{\tau_{S,j}} \right] & t \geq \tau_{D,j} \\ 0 & t < \tau_{D,j} \end{cases}$$



Nykamp

Can we estimate the correlation structure?

The output of a model neuron is a spike train

$$y_j(t) = \sum_i \delta(t - t_i^j)$$

Linear response gives the output rate as

$$r(t) = r_0 + (A * X)(t)$$

Can we estimate the correlation structure?

The output of a model neuron is a spike train

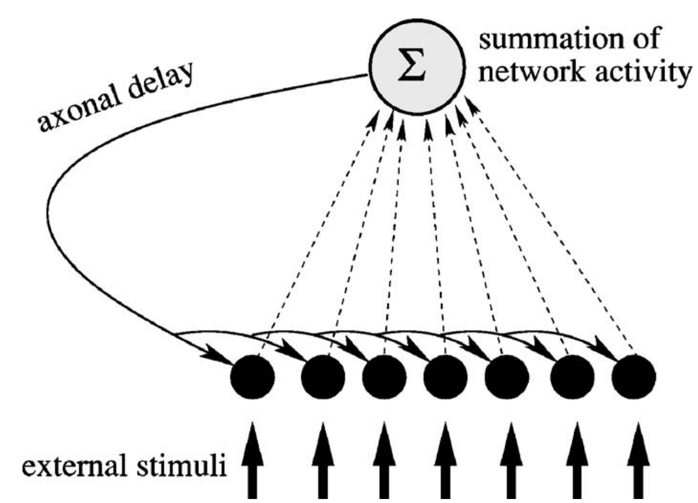
$$y_j(t) = \sum_i \delta(t - t_i^j)$$

Linear response gives the output rate as

$$r(t) = r_0 + (A * X)(t)$$

How do we use this to compute the cross-correlation?

Idea goes back to
Lindner, Doiron, Longtin, 2005



Can we estimate the correlation structure?

$$\frac{dV}{dt} = -\frac{V}{\tau_m} + \Psi(V) + \mu_0 + \sqrt{2D}\eta(t)$$

$$y_0(t) = \sum_i \delta(t - t_i^0)$$

Can we estimate the correlation structure?

$$\frac{dV}{dt} = -\frac{V}{\tau_m} + \Psi(V) + \mu_0 + \sqrt{2D}\eta(t) + X(t)$$

$$y_0(t) = \sum_i \delta(t - t_i^0)$$

Can we estimate the correlation structure?

$$\frac{dV}{dt} = -\frac{V}{\tau_m} + \Psi(V) + \mu_0 + \sqrt{2D}\eta(t) + X(t)$$

Can we estimate the correlation structure?

$$\frac{dV}{dt} = -\frac{V}{\tau_m} + \Psi(V) + \mu_0 + \sqrt{2D}\eta(t) + X(t)$$

$$y(t) = \sum_i \delta(t - t_i)$$

Can we estimate the correlation structure?

$$\frac{dV}{dt} = -\frac{V}{\tau_m} + \Psi(V) + \mu_0 + \sqrt{2D}\eta(t) + X(t)$$

$$y(t) = \sum_i \delta(t - t_i)$$

Use linear response to obtain a mixed point/continuous process

$$y(t) \approx y^1(t) = y^0(t) + (A * X)(t)$$

Can we estimate the correlation structure?

$$\frac{dV}{dt} = -\frac{V}{\tau_m} + \Psi(V) + \mu_0 + \sqrt{2D}\eta(t) + X(t)$$

$$y(t) = \sum_i \delta(t - t_i)$$

Use linear response to obtain a mixed point/continuous process

$$y(t) \approx y^1(t) = y^0(t) + (A * X)(t)$$

Which averages out to the right thing

$$r(t) \approx r_0 + (A * X)(t)$$

Approximate network correlations

The linear response approximation now takes the form

$$y_i^1(t) = y_i^0(t) + \sum_{\text{all inputs}} \left(\mathbf{K}_{i,j} * [y_j^0 - r_j]\right)(t)$$

$$\mathbf{K}_{i,j} = (A_i * \mathbf{J}_{i,j})(t)$$

We can use this to approximate the cross-covariances

$$\begin{aligned}\mathbf{C}_{ij}(\tau) \approx \mathbf{C}_{ij}^1(\tau) &= \mathbf{E}\left\{(y_i^1(t+\tau) - r_i)(y_j^1(t) - r_j)\right\} \\ &= \delta_{ij}\mathbf{C}_{ii}^0(\tau) + (\mathbf{K}_{ij} * \mathbf{C}_{jj}^0)(\tau) + (\mathbf{K}_{ji}^- * \mathbf{C}_{ii}^0)(\tau) + \sum_k (\mathbf{K}_{ik} * \mathbf{K}_{jk}^- * \mathbf{C}_{kk}^0)(\tau)\end{aligned}$$

Ostojic, Brunel, Hakim, 2009,
Trousdale, Yu, Shea-Brown, Josić, 2011

Impact of non-immediate neighbors

We use an iterative construction

$$\mathbf{y}^{n+1}(t) = \mathbf{y}^0(t) + (\mathbf{K} * [\mathbf{y}^n - \mathbf{r}])\,(t)$$
$$= \mathbf{y}^0(t) + \sum_{k=1}^{n+1} \left(\mathbf{K}^{(k)} * [\mathbf{y}^0 - \mathbf{r}]\right)(t)$$

Which gives the *n*-th approximation to the cross-correlation

After taking the Fourier transform, and the limit $n \to \infty$

$$\tilde{\mathbf{C}}^{\infty}(\omega) = \lim_{n\to\infty} \tilde{\mathbf{C}}^n(\omega) = (\mathbf{I} - \tilde{\mathbf{K}}(\omega))^{-1}\tilde{\mathbf{C}}^0(\omega)(\mathbf{I} - \tilde{\mathbf{K}}^*(\omega))^{-1}$$

Impact of non-immediate neighbors

We use an iterative construction

$$\mathbf{y}^{n+1}(t) = \mathbf{y}^0(t) + (\mathbf{K} * [\mathbf{y}^n - \mathbf{r}])(t)$$
$$= \mathbf{y}^0(t) + \sum_{k=1}^{n+1} \left(\mathbf{K}^{(k)} * [\mathbf{y}^0 - \mathbf{r}]\right)(t)$$

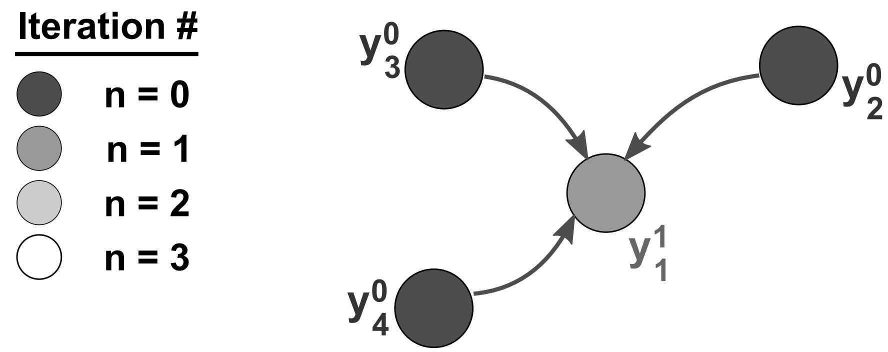
Which gives the *n*-th approximation to the cross-correlation

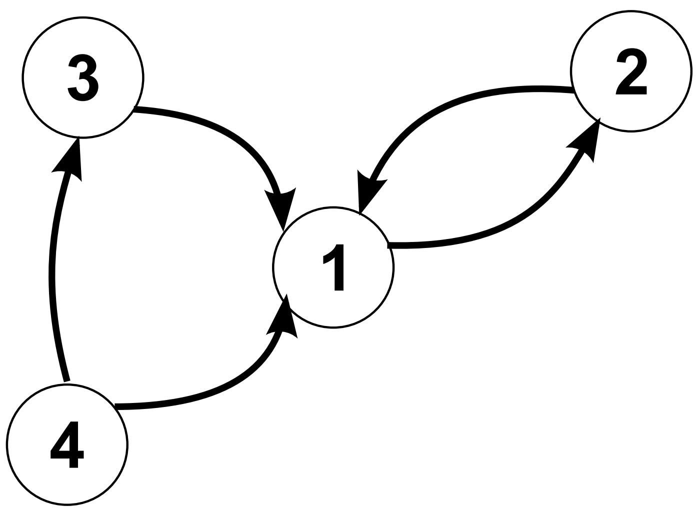
After taking the Fourier transform, and the limit $n \to \infty$

$$\tilde{\mathbf{C}}^\infty(\omega) = \lim_{n\to\infty} \tilde{\mathbf{C}}^n(\omega) = (\mathbf{I} - \tilde{\mathbf{K}}(\omega))^{-1}\tilde{\mathbf{C}}^0(\omega)(\mathbf{I} - \tilde{\mathbf{K}}^*(\omega))^{-1}$$

$$\mathbf{K}_{i,j} = (A_i * \mathbf{J}_{i,j})(t)$$

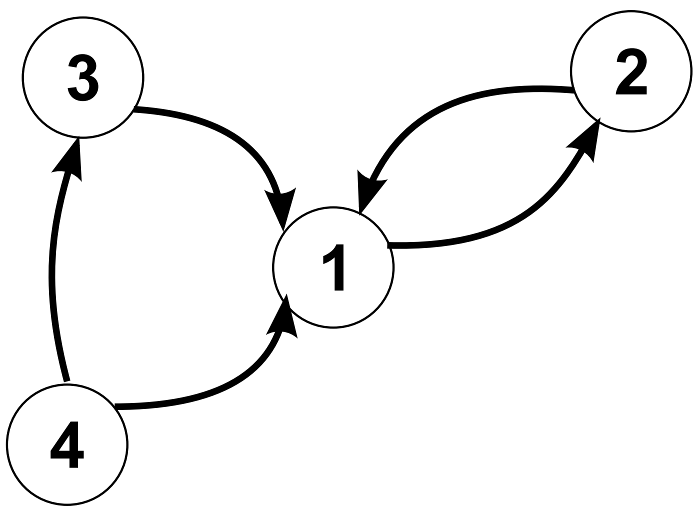
The iterative construction

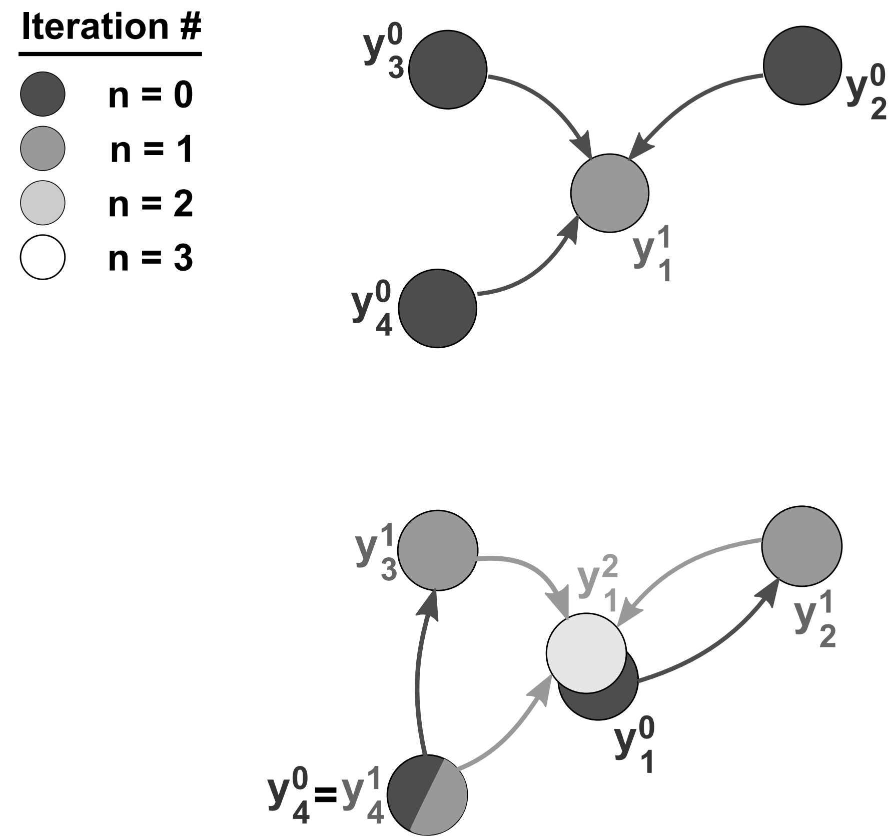




Rangan 2009
Pernice, Staube, Cardanobile, Rotter 2011
Trousdale, Yu, Shea-Brown, Josić, 2011

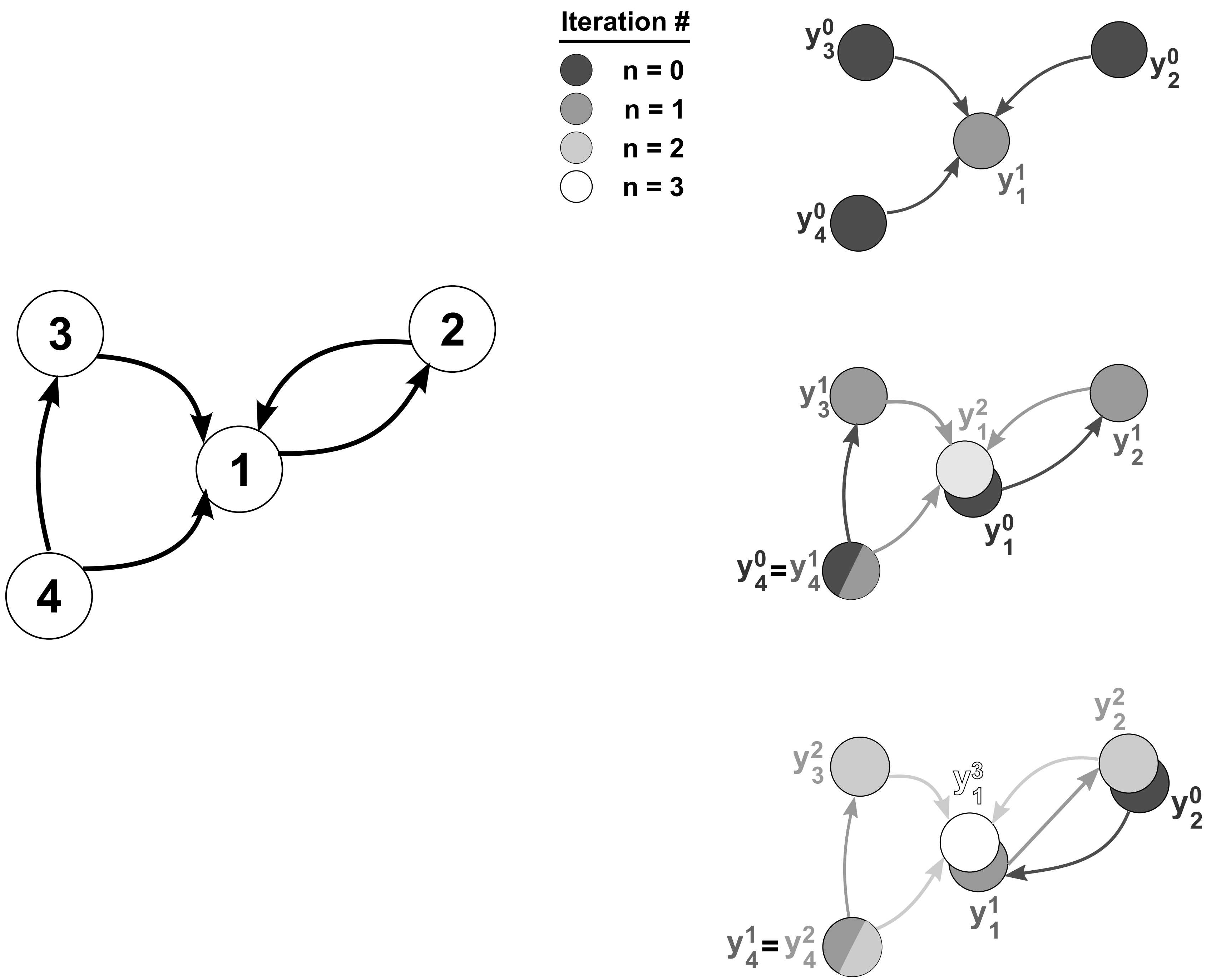
The iterative construction





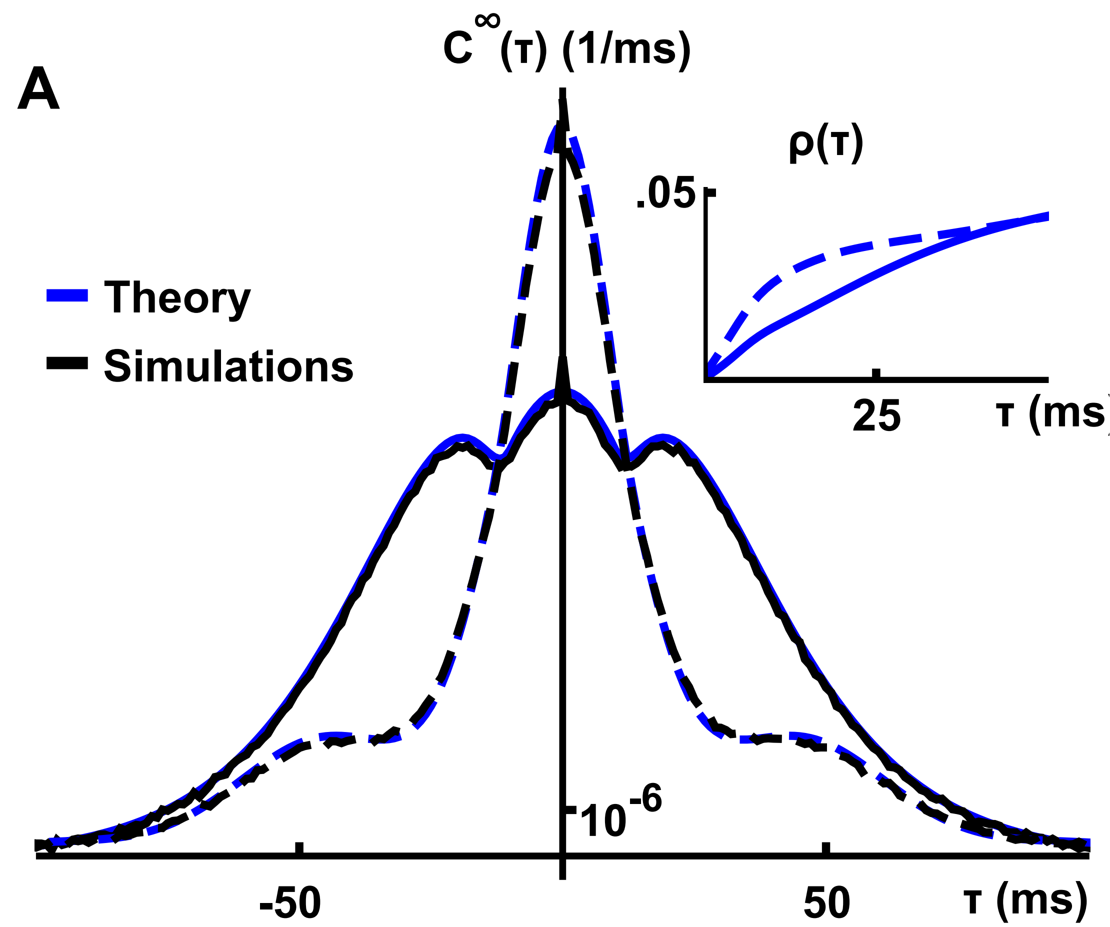
Rangan 2009
Pernice, Staube, Cardanobile, Rotter 2011
Trousdale, Yu, Shea-Brown, Josić, 2011

The iterative construction



Rangan 2009
Pernice, Staube, Cardanobile, Rotter 2011
Trousdale, Yu, Shea-Brown, Josić, 2011

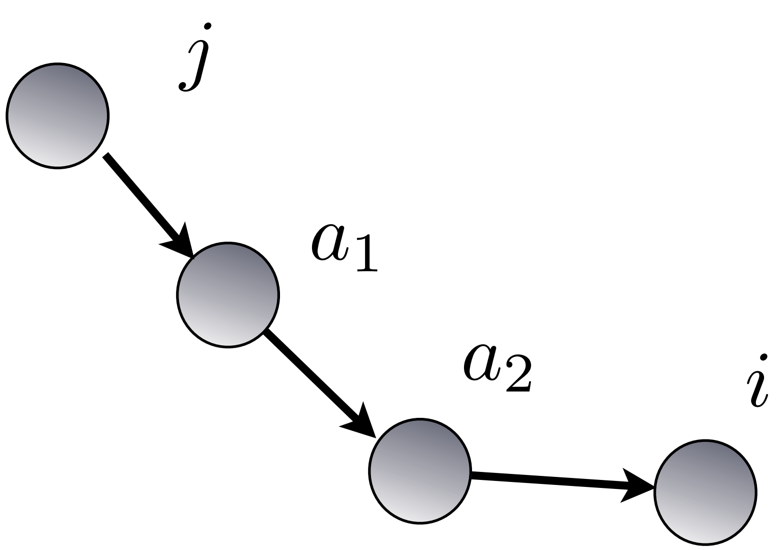
The approximation works well



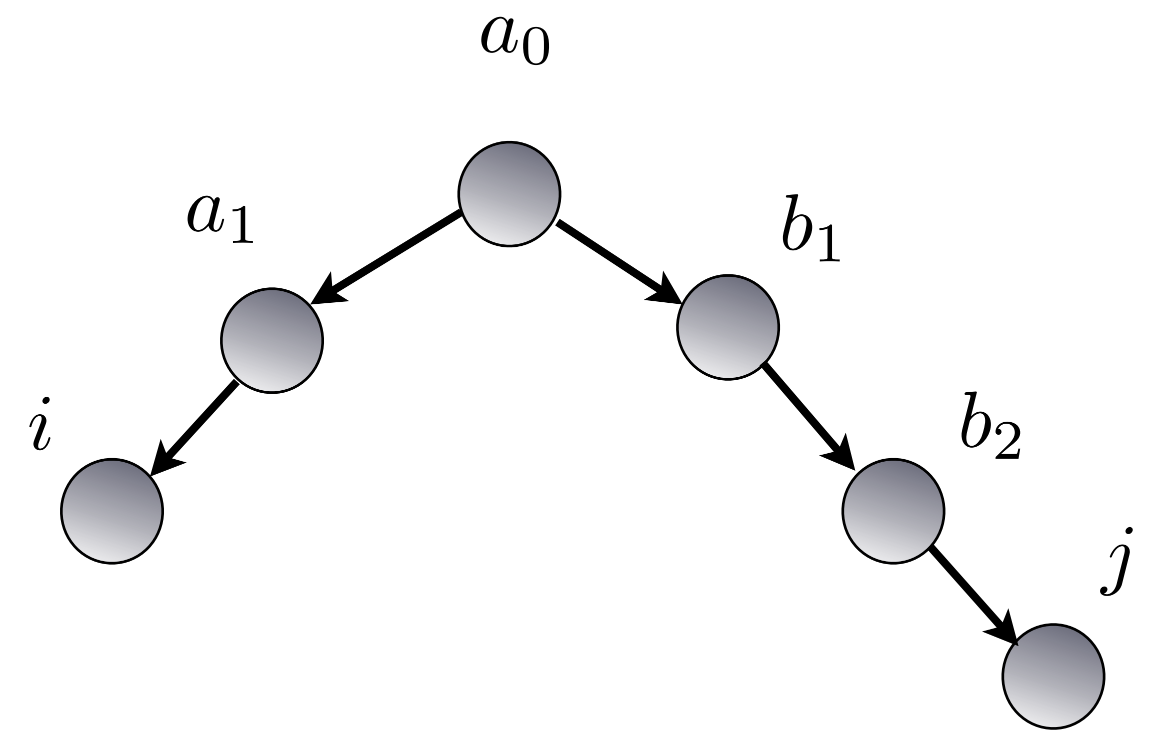
Cross-correlation between two excitatory cells as we shift from balance to excess inhibition

Expansion in terms of paths through the graph

$$\mathbf{C}^{\infty}(\tau) = \lim_{n\to\infty} \sum_{k,l}^{n} \left(\mathbf{K}^{(k)} * \mathbf{C}^{0} * \mathbf{K}^{(l)T} \right)(\tau)$$

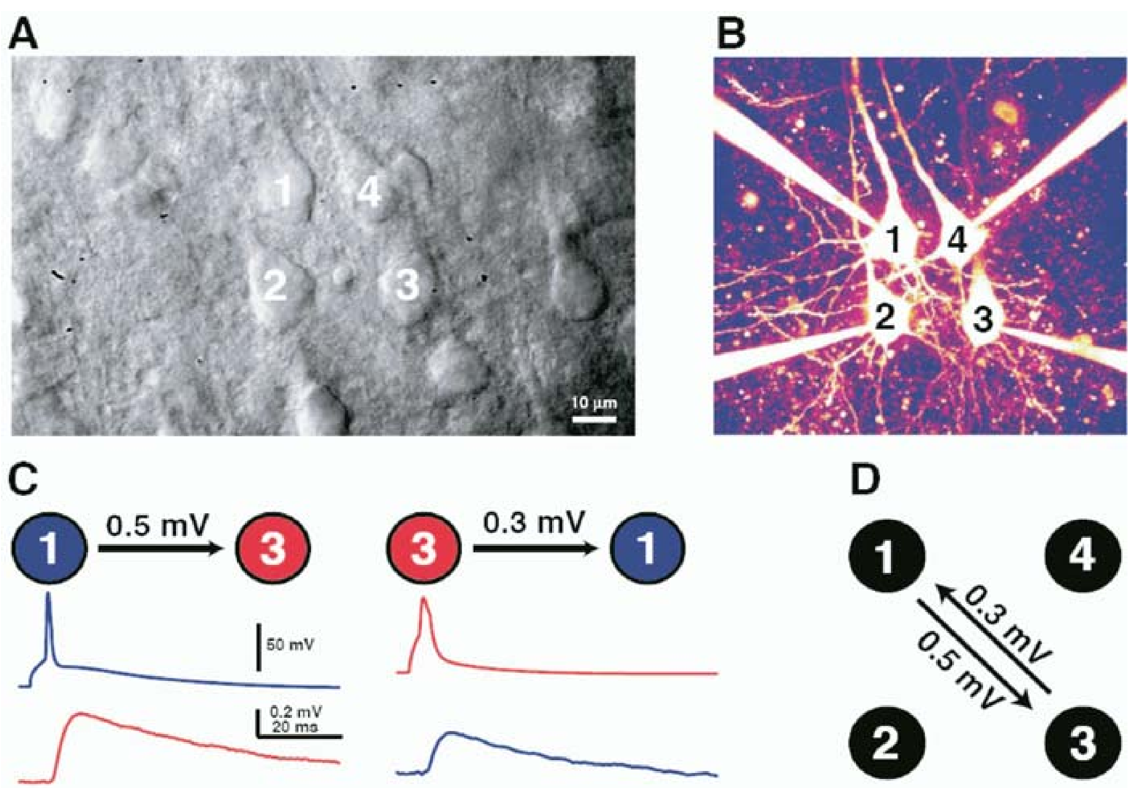


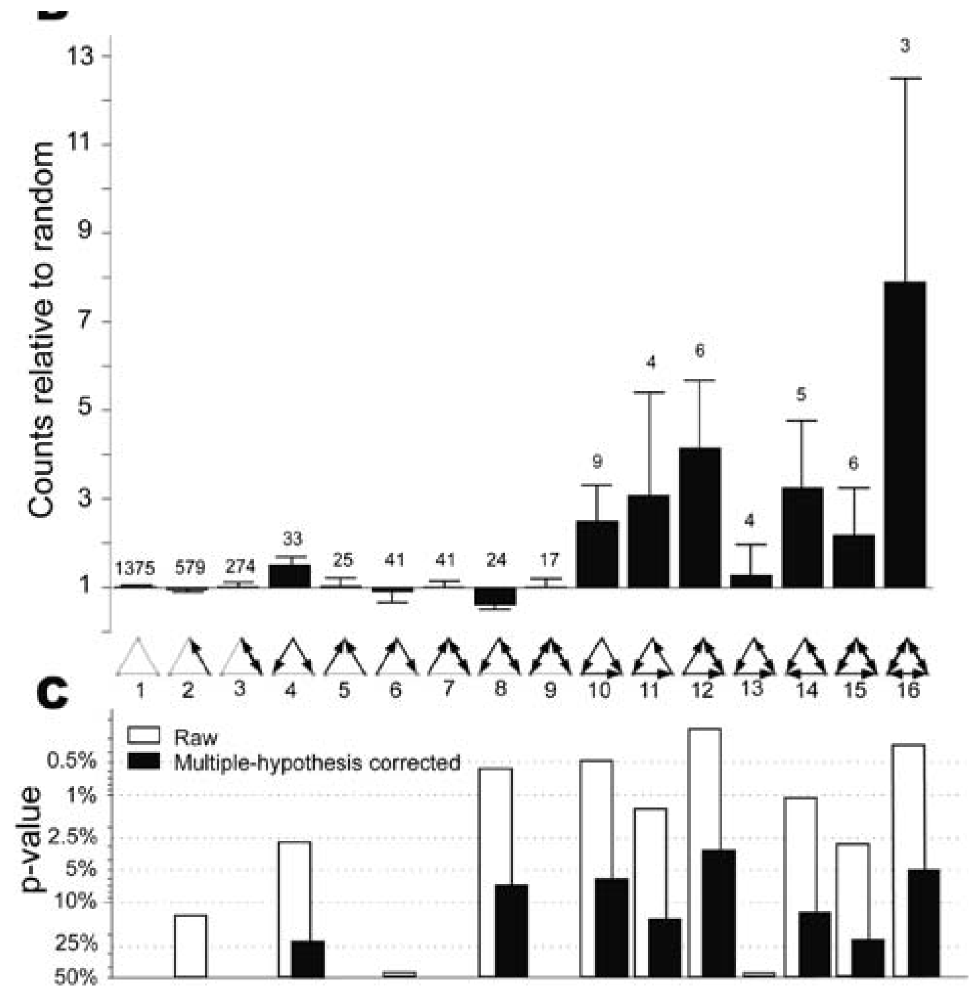
$$\tilde{\mathbf{K}}_{ia_{n-1}} \tilde{\mathbf{K}}_{a_{n-1}a_{n-2}} \cdots \tilde{\mathbf{K}}_{a_1 j} \tilde{\mathbf{C}}^{0}_{jj}$$



$$\tilde{\mathbf{K}}_{ia_{n-1}} \tilde{\mathbf{K}}_{a_{n-1}a_{n-2}} \cdots \tilde{\mathbf{K}}_{a_1 a_0} \tilde{\mathbf{C}}^{0}_{a_0 a_0} \tilde{\mathbf{K}}^{*}_{a_0 b_1} \cdots \tilde{\mathbf{K}}^{*}_{b_{m-2} b_{m-1}} \tilde{\mathbf{K}}^{*}_{b_{m-1} j}$$

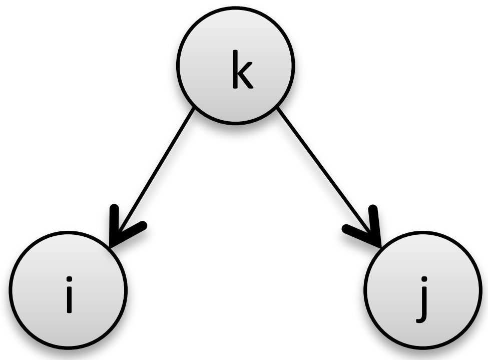
How does local structure determine correlations?





Song, et al. 2005

How do small motifs impact the correlation structure?

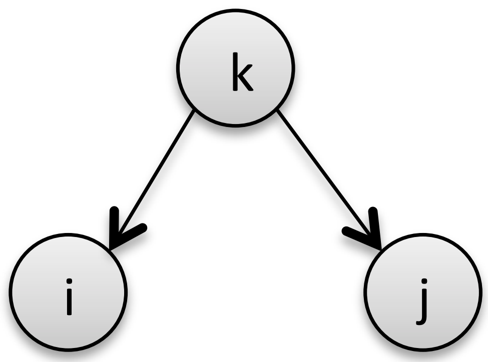


diverging

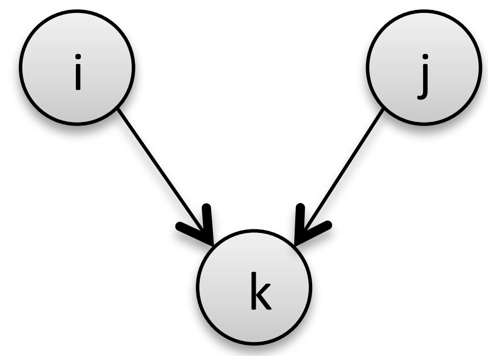
$$q_{\text{div}} = \sum_{i,j,k} (\mathbf{W}^0_{i,k}\mathbf{W}^0_{j,k})/N^3 - p^2$$

Sporns and Kötter, 2004

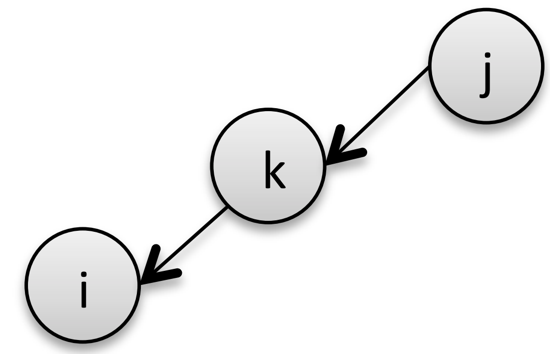
How do small motifs impact the correlation structure?



diverging



converging



chain

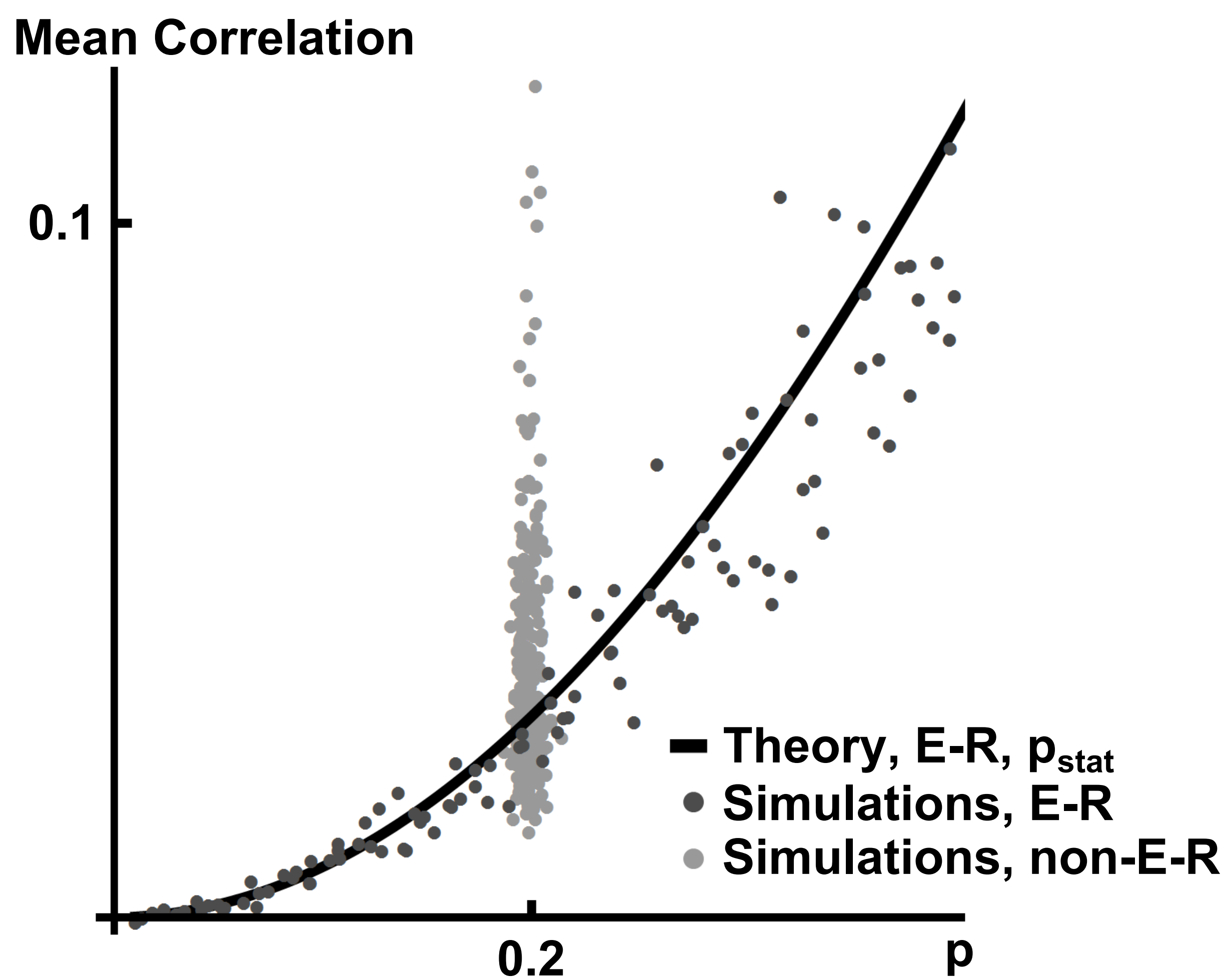
$$q_{\text{div}} = \sum_{i,j,k} (\mathbf{W}^0_{i,k}\mathbf{W}^0_{j,k})/N^3 - p^2$$

$$q_{\text{con}}$$

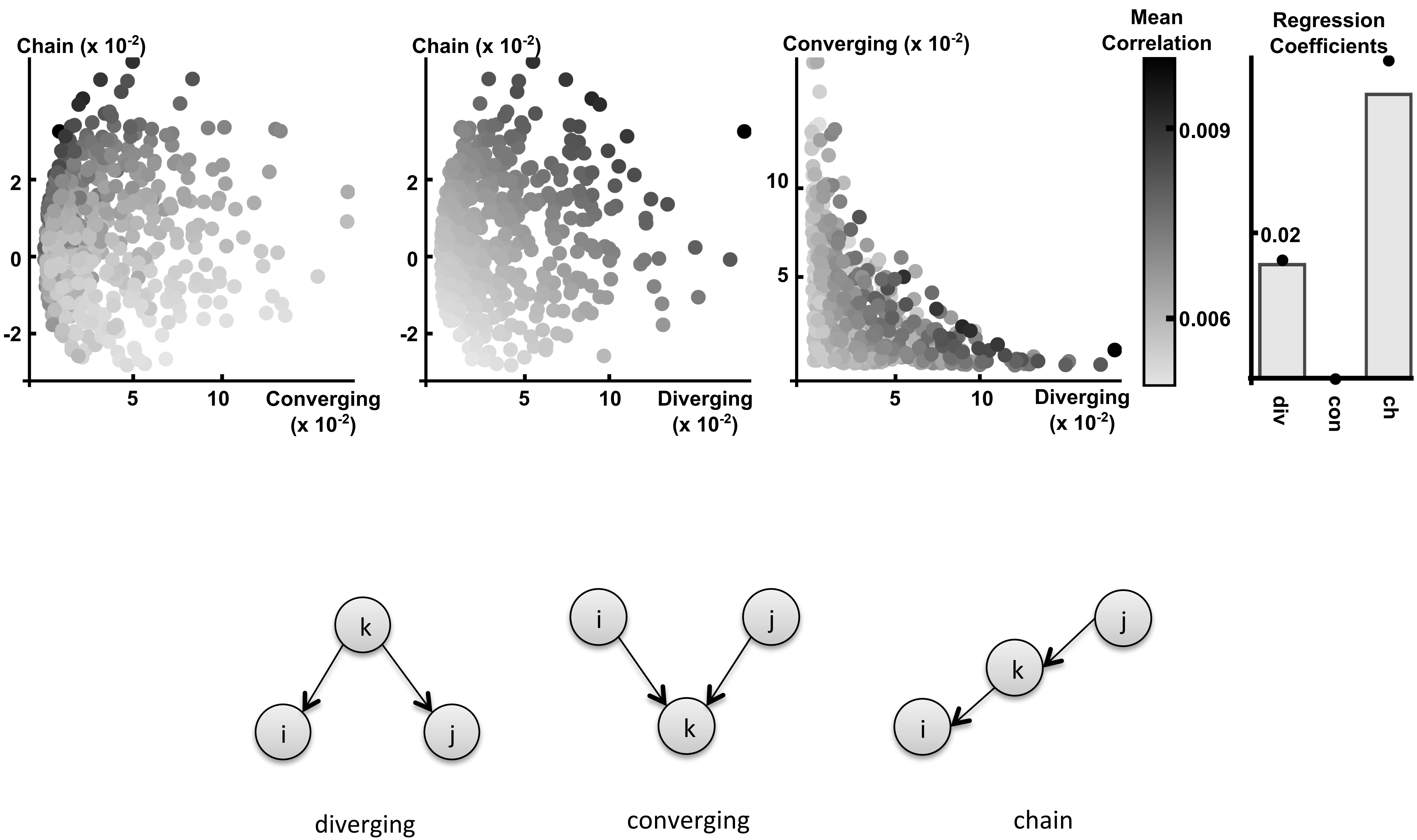
$$q_{\text{ch}}$$

Sporns and Kötter, 2004

Mean correlations in structured networks



How do small motifs impact the correlation structure?



Correlations with homogeneity

$$\tilde{\mathbf{C}}^{\infty}(\omega) = (\mathbf{I} - \tilde{\mathbf{K}}(\omega))^{-1} \langle \tilde{y}^{0}(\omega) \tilde{y}^{0*}(\omega) \rangle (\mathbf{I} - \tilde{\mathbf{K}}^{*}(\omega))^{-1}$$

Assuming homogeneity in uncoupled cells, and evaluating at $\omega=0$

$$\tilde{\mathbf{C}}^{\infty}(0) = \tilde{C}^{0}(0)(\mathbf{I} - \tilde{A}\mathbf{W})^{-1}(\mathbf{I} - \tilde{A}\mathbf{W}^{T})^{-1}$$

After expanding and truncating at second order in connection strength, writing $wW^{0} = W$

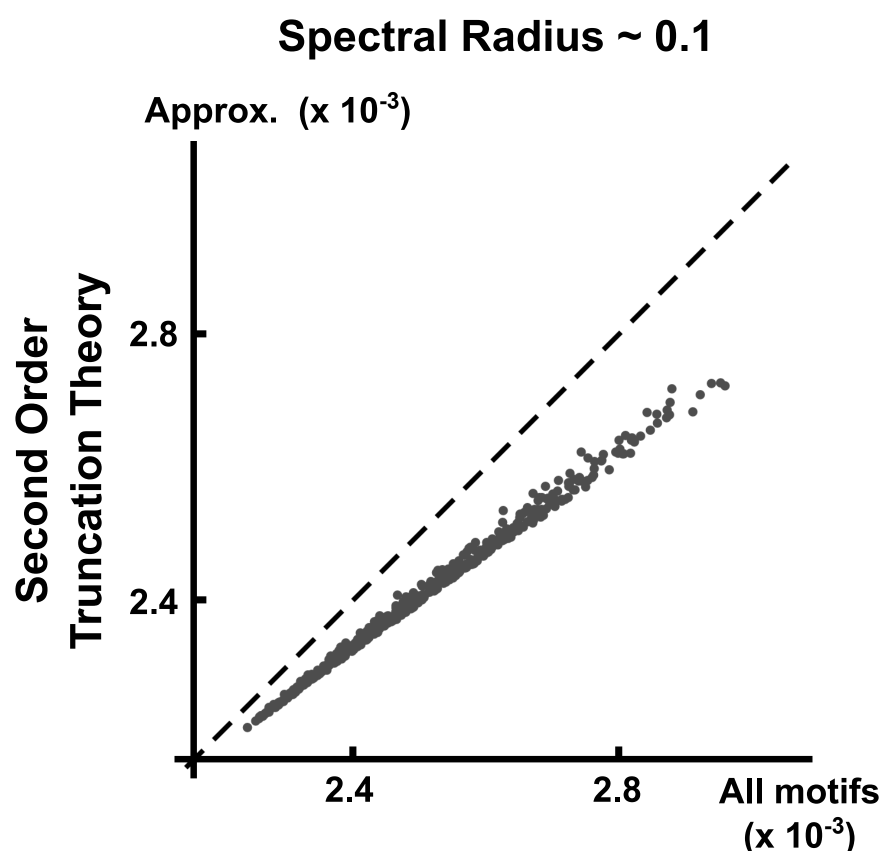
$$\frac{\tilde{\mathbf{C}}^{\infty}}{\tilde{C}^{0}} \approx I + \tilde{A}w\mathbf{W}^{0} + \tilde{A}w\mathbf{W}^{0T} + \left(\tilde{A}w\right)^{2} \mathbf{W}^{0}\mathbf{W}^{0T} + \left(\tilde{A}w\right)^{2} (\mathbf{W}^{0})^{2} + \left(\tilde{A}w\right)^{2} (\mathbf{W}^{0T})^{2}$$

Averagd network correlations

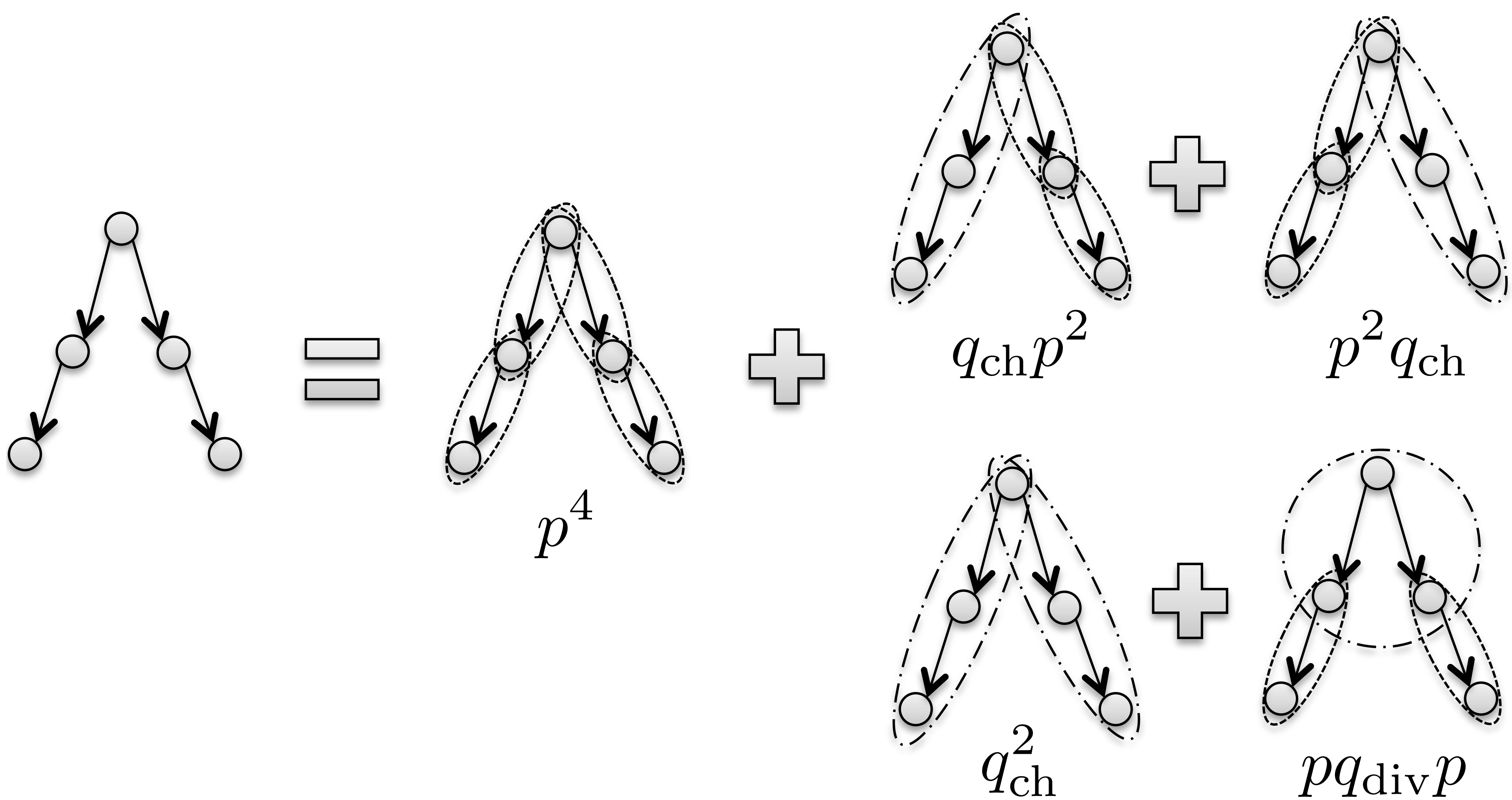
$$\frac{\tilde{\mathbf{C}}^{\infty}}{\tilde{C}^0} \approx I + \tilde{A}w\mathbf{W}^0 + \tilde{A}w\mathbf{W}^{0T} + \left(\tilde{A}w\right)^2 \mathbf{W}^0\mathbf{W}^{0T} + \left(\tilde{A}w\right)^2 (\mathbf{W}^0)^2 + \left(\tilde{A}w\right)^2 (\mathbf{W}^{0T})^2$$

Averaging over the network

$$\frac{\langle \tilde{\mathbf{C}}^{\infty} \rangle}{\tilde{C}^0} \approx \frac{1}{N} + 2\tilde{A}wp + 3N\left(\tilde{A}w\right)^2 p^2 + N\left(\tilde{A}w\right)^2 q_{\mathrm{div}} + 2N\left(\tilde{A}w\right)^2 q_{\mathrm{ch}}$$



Resumming



Resumming

$$\frac{\langle \tilde{\mathbf{C}}^{\infty} \rangle}{\tilde{C}^0} = \frac{1}{N}\left(1 - \sum_{n=1}^{\infty} (N\tilde{A}w)^n \mathbf{L}^T \mathbf{W}_n^0 \mathbf{L}\right)^{-1} \left(1 + \sum_{n,m=1}^{\infty} (N\tilde{A}w)^{n+m} \mathbf{L}^T \mathbf{W}_{n,m}^0 \mathbf{L}\right) \cdot \left(1 - \sum_{m=1}^{\infty} (N\tilde{A}w)^m \mathbf{L}^T \mathbf{W}_m^{0T} \mathbf{L}\right)^{-1},$$

Keeping contribution of second order motifs

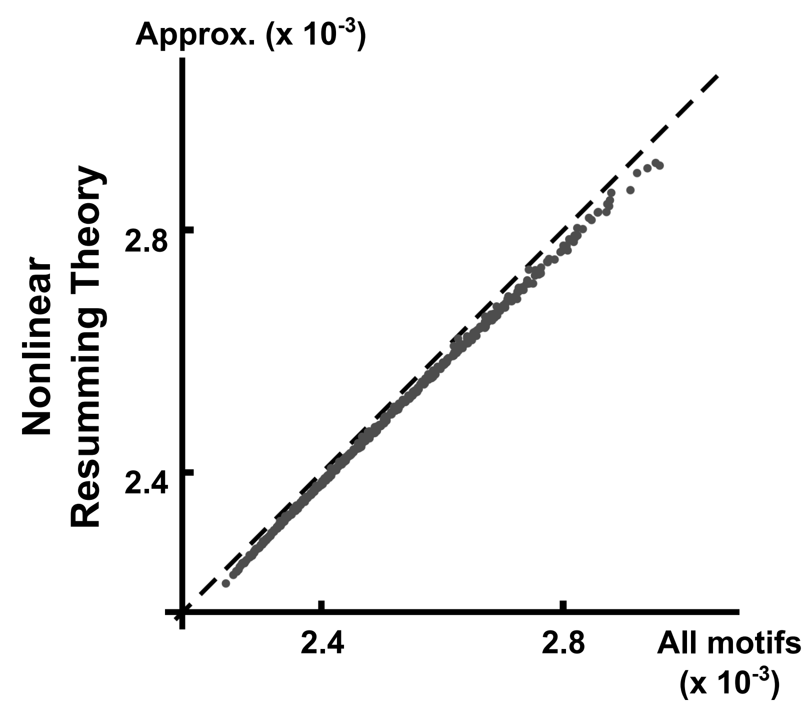
$$\frac{\langle \tilde{\mathbf{C}}^{\infty} \rangle}{\tilde{C}^0} = \frac{1}{N} \frac{1 + \left(N\tilde{A}w\right)^2 q_{\mathrm{div}}}{\left[1 - \left(N\tilde{A}w\right) p - \left(N\tilde{A}w\right)^2 q_{\mathrm{ch}}\right]^2}.$$

Resumming

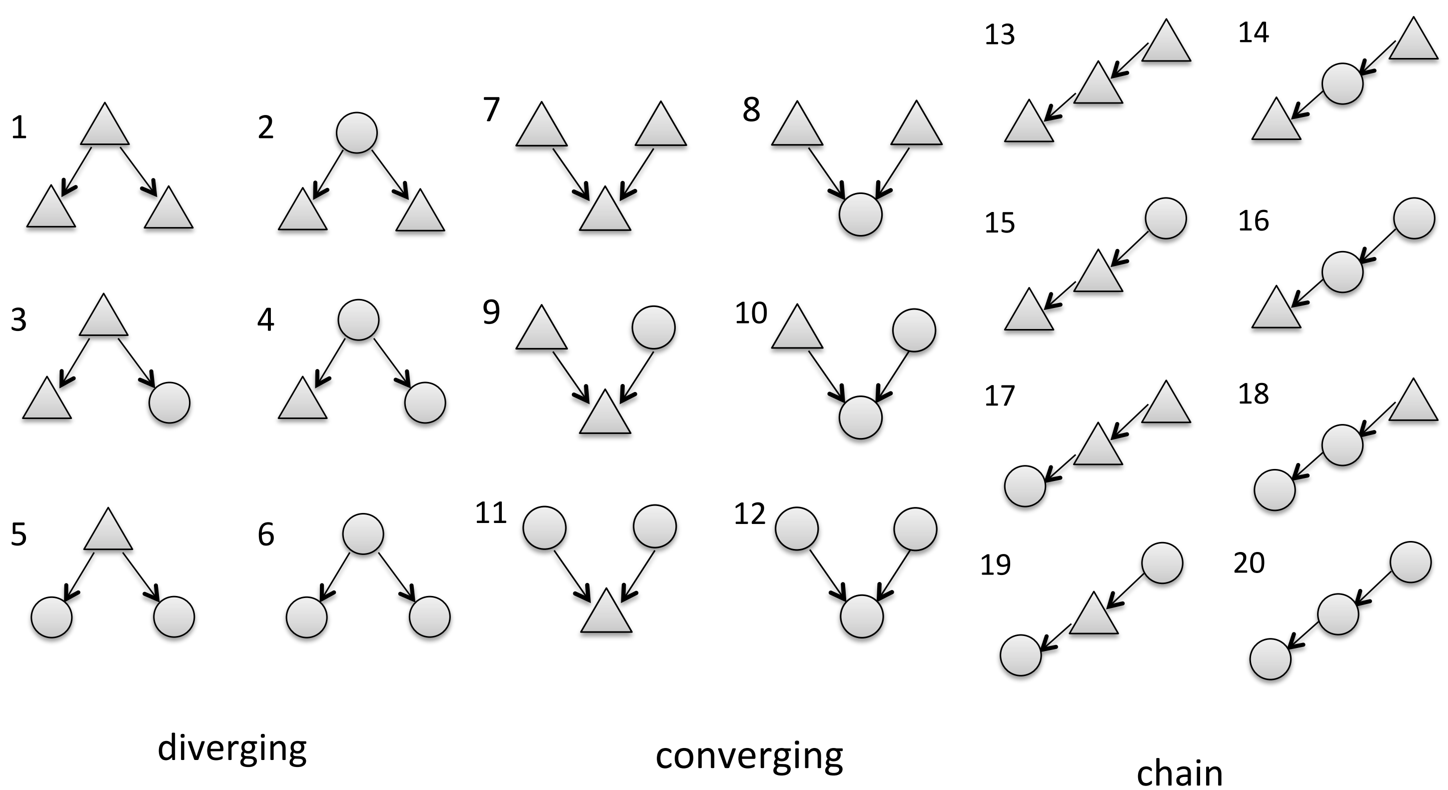
$$\frac{\langle \tilde{\mathbf{C}}^{\infty} \rangle}{\tilde{C}^0} = \frac{1}{N}\left(1-\sum_{n=1}^{\infty}(N\tilde{A}w)^n \mathbf{L}^T \mathbf{W}_n^0 \mathbf{L}\right)^{-1}\left(1+\sum_{n,m=1}^{\infty}(N\tilde{A}w)^{n+m}\mathbf{L}^T\mathbf{W}_{n,m}^0\mathbf{L}\right) \cdot \left(1-\sum_{m=1}^{\infty}(N\tilde{A}w)^m \mathbf{L}^T \mathbf{W}_m^{0T}\mathbf{L}\right)^{-1},$$

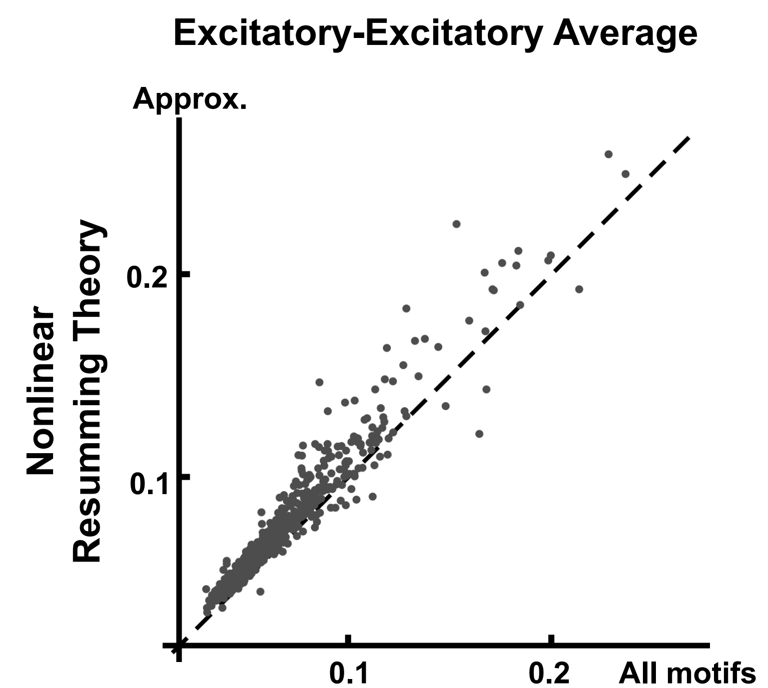
Keeping contribution of second order motifs

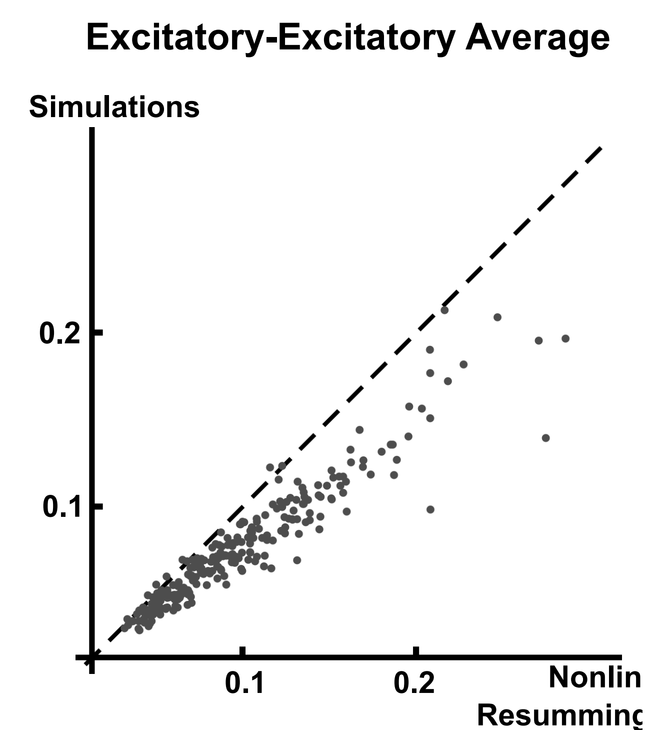
$$\frac{\langle \tilde{\mathbf{C}}^{\infty} \rangle}{\tilde{C}^0} = \frac{1}{N}\frac{1+\left(N\tilde{A}w\right)^2 q_{\mathrm{div}}}{\left[1-\left(N\tilde{A}w\right)p-\left(N\tilde{A}w\right)^2 q_{\mathrm{ch}}\right]^2}.$$



Theory extends to EI







Conclusion

- Linear response theory can be used to understand the statistical structure of population activity.

- Cross-correlation functions can be understood in terms of contributions from paths through the network. Thus architecture and population activity can be related.

- This local theory applies to any network where interactions can be linearized

- There is a lot more to do - see Bullmore and Sporns, *Nat Neurosci*, 2009

University of Houston

Manisha Bhardwaj
Becky Chen
Manuel Lopez
Robert Rosenbaum (Pitt)
James Trousdale

University of Washington, Seattle

Eric Shea-Brown
Yu Hu

University of Pittsburgh

Brent Doiron