# Traveling waves on small-world networks

**Thomas M. Isele**

*Institut für Theoretische Physik, TU Berlin, Germany*

## Introduction

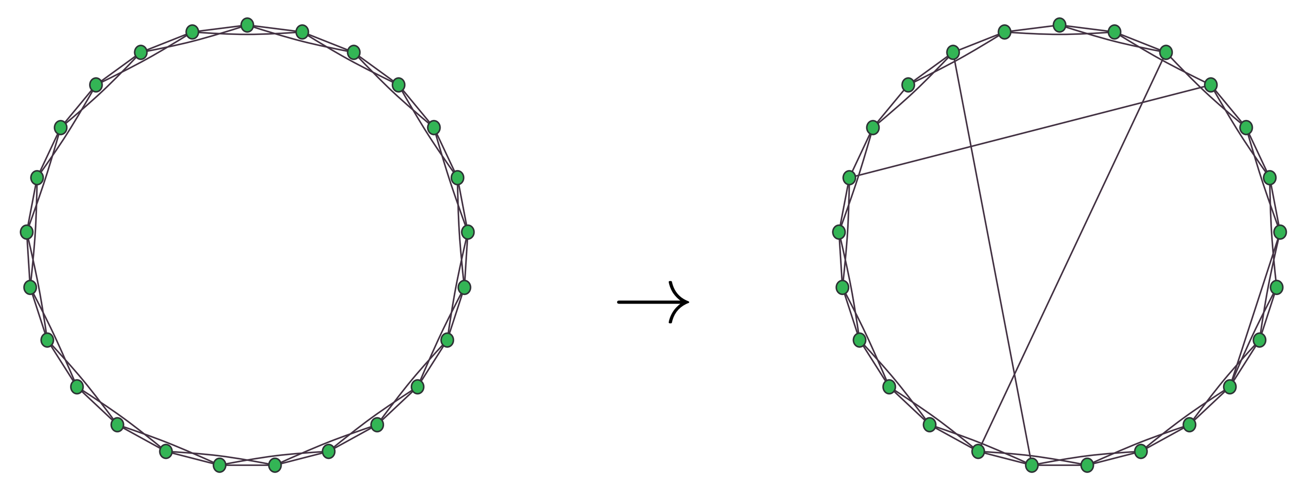
We investigate the propagation and the propagation death of excitation waves on small-world networks. We observe a sharp boundary in the rewiring probability beyond which propagation of excitation waves is not observed anymore. This is reminiscent of an upper boundary in rewiring probability for finding sustained activity in small-world networks [SIN07b]. We are able to approximate this boundary in a mean field model. We also observe another effect we do not yet fully understand, which makes propagation death occur at unusually low rewiring probabilities for distinct values of nearest-neighbor number and coupling strength.

## Model

**Dynamics:** The nodes in our network obey the FitzHugh-Nagumo differential equations [IZH06]. They are coupled by the difference in the activator concentration (diffusive coupling of the activator).

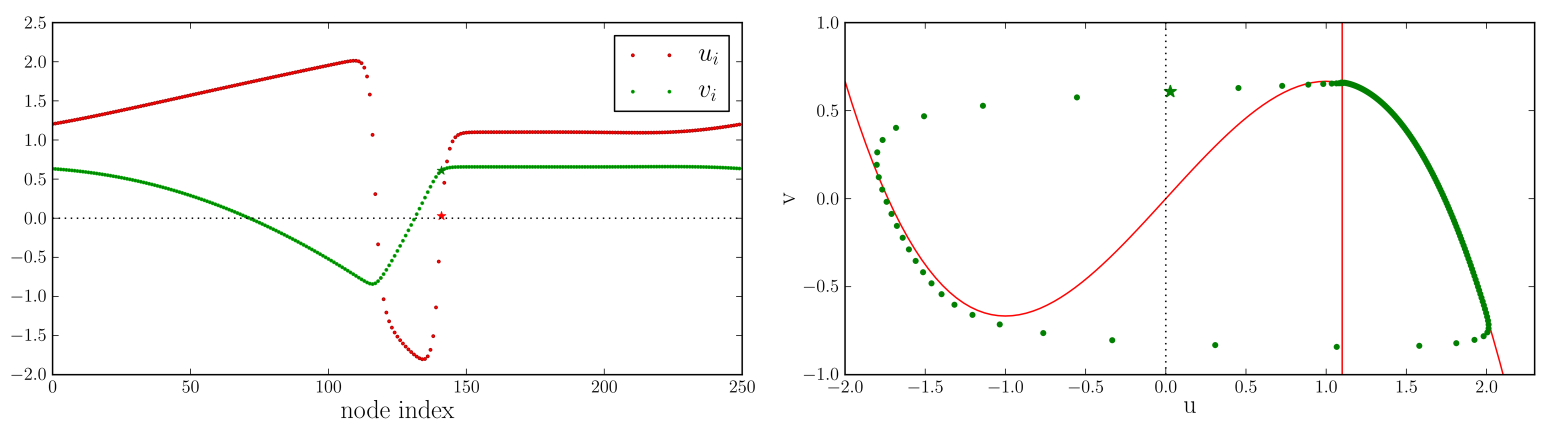
$$\begin{aligned}\dot{u}_i &= u_i - \frac{u_i^3}{3} - v_i + \kappa \sum_j A_{ij}\,(u_j - u_i)\,,\\ \dot{v}_i &= \epsilon\,(u_i - \beta)\,,\end{aligned} \tag{1}$$

with $A_{ij}$ being the adjacency matrix of the network.
The parameters of the FitzHugh-Nagumo model are chosen as $\beta = 1.1$, $\epsilon = 0.04$, such that the model is in the excitable regime. The coupling strength $\kappa$ will be varied.
**Topology:** We examine the dynamics on a Watts-Strogatz small-world model [WAT98]. In constructing the model we start out with a ring network with $N$ nodes and nearest-neighbor number $2k$.
For the construction of this model, every link is chosen for rewiring with a probability $p$. One of the ends of such a chosen link is connected to a new node which is chosen at random from the entire network.



Ring network with $N = 25$ and $k = 2$, as is and rewired with $p = 0.1$.

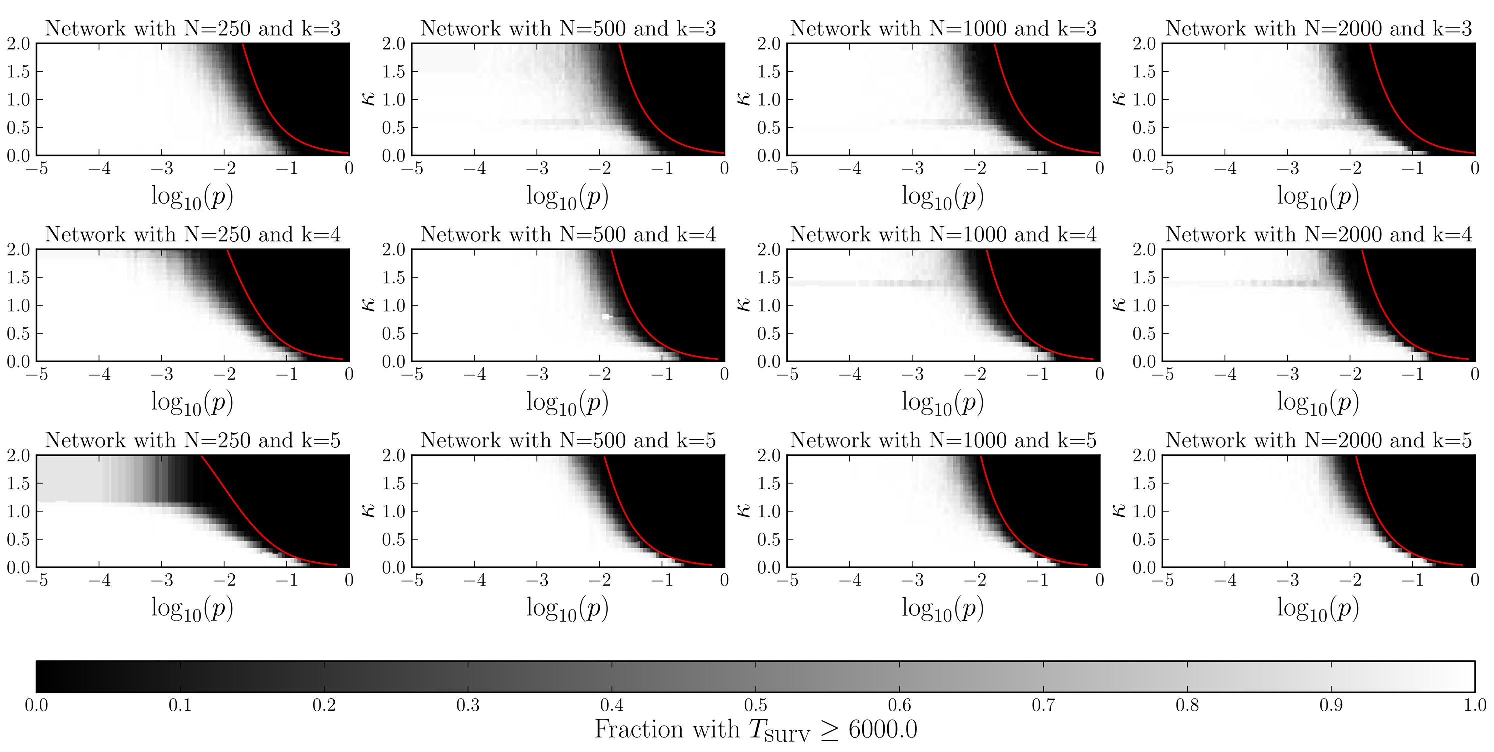
**Initial Conditions:** In order to examine the fate of traveling waves on small world networks, we use as initial condition for the simulation a traveling wave on the ring network that the small-world model is constructed from.



Initial conditions for $N = 250$, $k = 3$ and $\kappa = 0.2$ by node index and in a nullcline diagram.

## Numerical Results

We simulated the dynamics on small-world networks generated from rings with $N = 250,\ 500,\ 1000,\ 2000$ and $k = 3,\ 4,\ 5$. The small-world networks were generated with 81 rewiring probabilities from $p = 10^{-5}$ to $p = 10^{0}$ and for each probability (and network) 200 realizations were generated. Each simulation was done with coupling strengths from $\kappa = 0.05$ to $\kappa = 2.0$, with the proper traveling waves as initial conditions. For every simulation, the time until the collapse of the solution was recorded. (Cutoff at 6000 time units as maximal simulation time).




The fraction of solutions with survival times $T_{\text{surv}} <= 6000$ as a function of $\log_{10}(p)$ and $\kappa$. The red line is the approximation by the mean-field model.

## Meanfield Model

The first approximation we do is to say that the small-world shortcuts are added to the network (instead of replacing). This way, the adjacency matrix can be viewed as a sum

$$A_{ij} = D_{ij} + S_{ij} \tag{2}$$

of the adjacency matrix of a regular ring network $D_{ij}$ and that of a random network $S_{ij}$. We can understand the nodes of a regular ring network as discretization points (with distance between the nodes $\mathrm{d}x = 1$) of an excitable medium. For large enough probabilities $p$, and broad enough waves, we can approximate the effect of the random links in $S_{ij}$ by a mean-field term. The expression for this excitable medium is

$$\begin{aligned}\dot{u} &= u - \tfrac{u^3}{3} - v + D\Delta u + \delta(u - \bar{u})\\ \dot{v} &= \epsilon\,(u - \beta)\end{aligned} \tag{3}$$

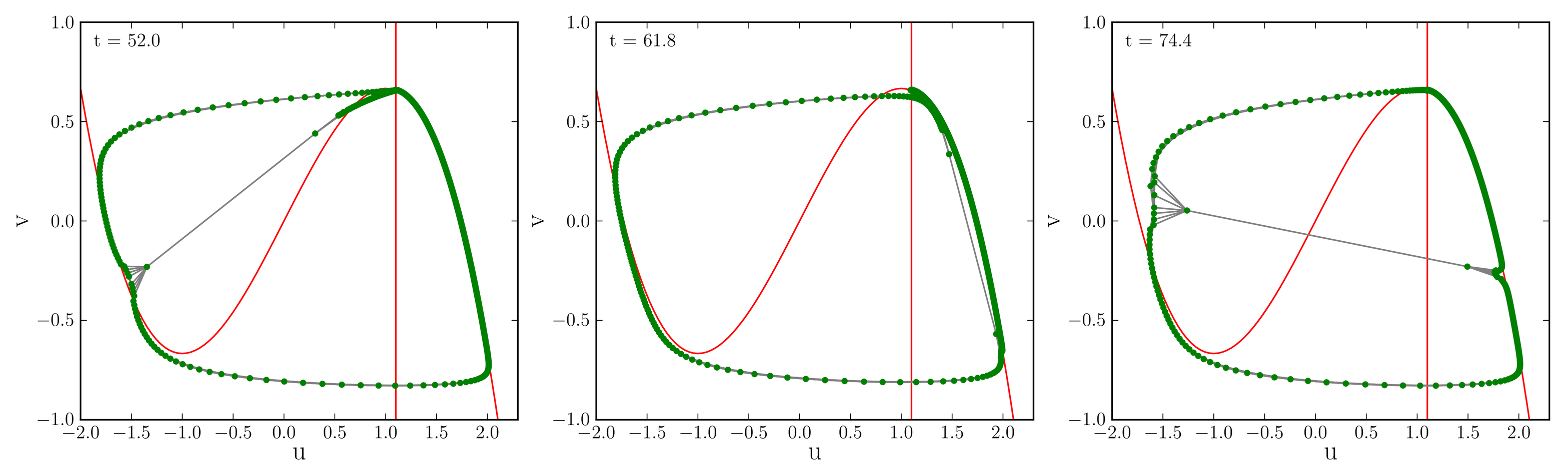
with an effective diffusion coefficient and mean-field strength

$$D = \kappa \textstyle\sum_{j=1}^{k} j^2, \qquad \delta = \kappa k p. \tag{4}$$

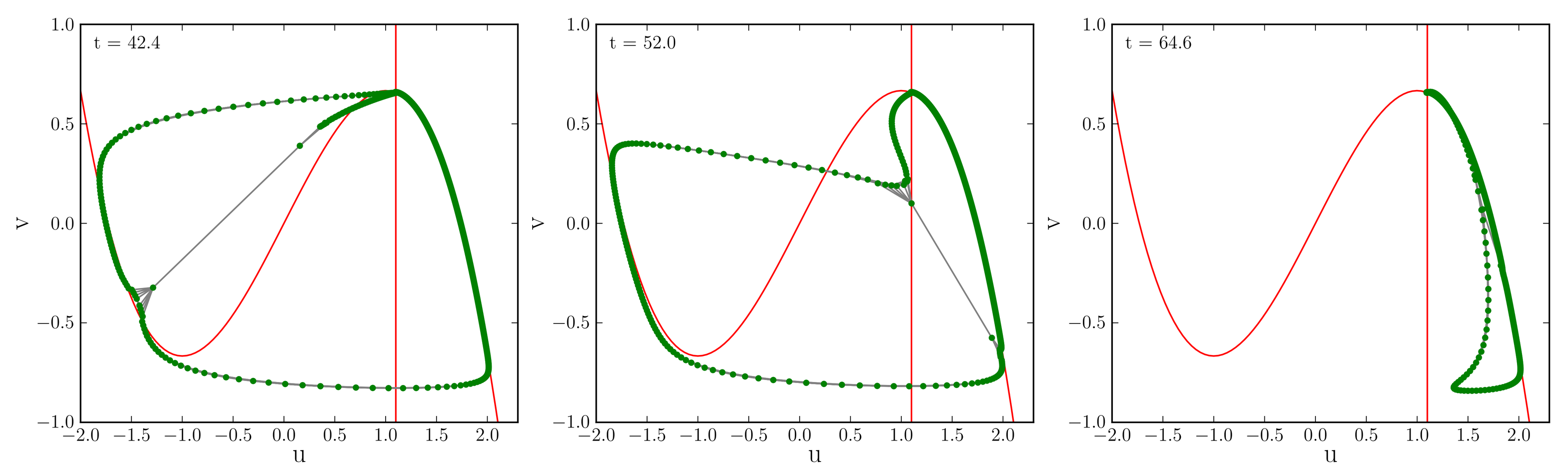
When raising $\delta$, we come across a fold bifurcation at $\delta_0(D)$. Using AUTO to track $\delta_0(D)$ and calculating $p_0(\kappa)$ using (4), we can approximate the point of propagation death.

## Resonance effects

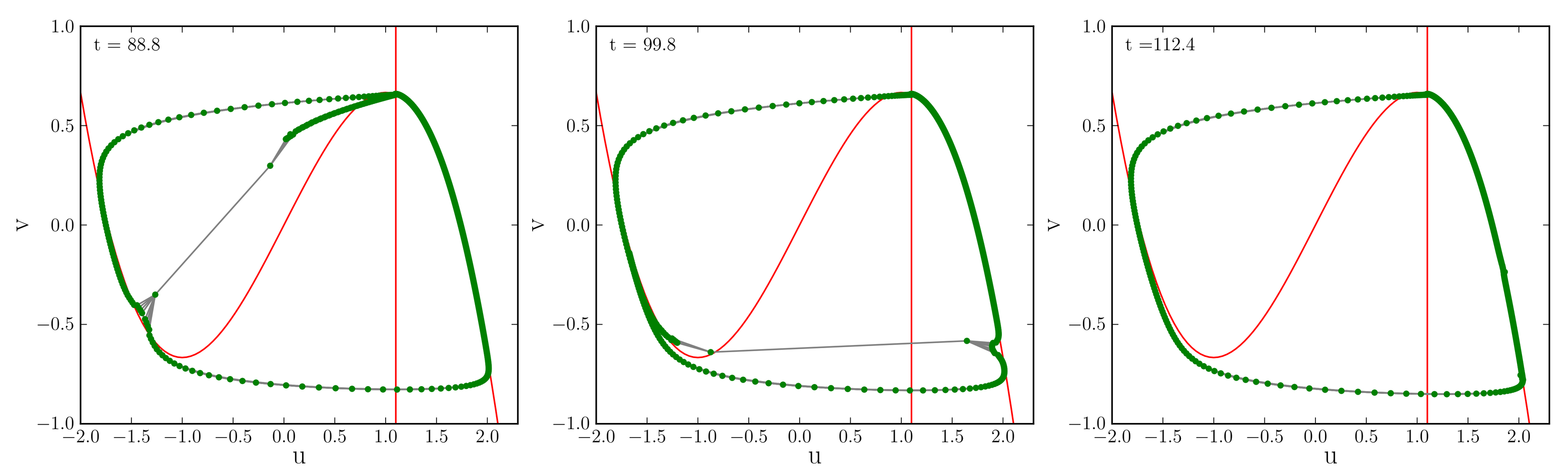
When having a close look at the survival rates for $N = 1000$ and $k = 4$, for instance, one notices that the survival rate at a coupling strength of $\kappa = 1.4$ begins to deteriorate very early. The coupling strength at which this happens is the same for $N = 2000$ and $k = 4$ but it changes when the nearest-neighbor number changes. We have an exemplary look at the time series for one realization of a small-world network with $N = 1000$, $k = 4$ and $p = 10^{-4}$. In this particular realization there is only one small-world shortcut present.




Three timeslices for coupling strength $\kappa = 1.2$




Three timeslices for coupling strength $\kappa = 1.4$




Three timeslices for coupling strength $\kappa = 1.6$

The solution with $\kappa = 1.4$ suffers propagation death shortly after the last displayed timeslice, whereas the other two depicted solutions propagate until maximum simulation time is reached.

## Acknowledgments

This work was done in the SFB910, project A1. The author would like to thank Eckehard Schöll, Judith Lehnert and Thomas Dahms for support and helpful discussions.

## References

[IZH06] E. M. Izhikevich and R. A. FitzHugh: FitzHugh-Nagumo model, Scholarpedia **1**, 1349 (2006).

[SIN07b] S. Sinha, J. Saramäki, and K. Kaski: Emergence of self-sustained patterns in small-world excitable media, Phys. Rev. E **76** (2007).

[WAT98] D. J. Watts and S. H. Strogatz: Collective dynamics of 'small-world' networks, Nature **393**, 440–442 (1998).

International workshop: MAPCON12, Mathematical Physics of Complex Networks: From Graph Theory to Biological Physics, Dresden, Germany May 14-18, 2012 , contact: physikant@gmail.com