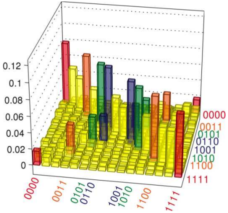
Measures for correlations and complexity based on exponential families

Otfried Gühne, Sönke Niekamp, Tobias Galla



Fachbereich Physik, Universität Siegen

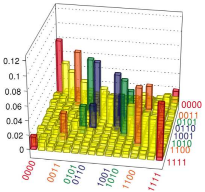
Theoretical Quantum Optics

The problem

The classical problem

Given N particles where each particle is in two possible states. How can we characterize the complexity of a given probability distribution over the state space?

The quantum problem



Given N particles where each particle is a two-level system. How can we characterize the complexity of a given density matrix?

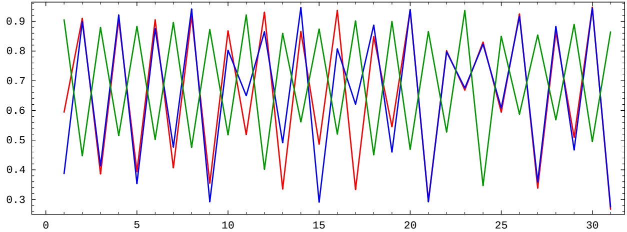
Classical example: coupled iterated maps

Consider N nodes x_i:

$$x_i(t+1) = (1-\varepsilon) f[x_i(t)] + \frac{\varepsilon}{N-1} \sum_{j \neq i} f[x_j(t)]$$

with

$$f(x) = \begin{cases} 2x, & x \leq 1/2 \\ 2(1-x), & x \geq 1/2 \end{cases} \quad \text{or} \quad f(x) = rx(1-x)$$



K. Kaneko, Physica D 41, 137 (1990).

Coarse graining

Take the time series $x_i(t)$ and make a coarse graining:

$$g_i = \begin{cases} 1 & x_i \leq \theta \\ 0 & x_i \geq \theta \end{cases}$$

Time avaraging gives probability distribution:

$$\mathcal{P} : \{0, 1\}^{\times N} \to \mathbb{R}$$

S. Jalan et al. Chaos 12, 033124 (2006), T. Kahle et al., PRE 79, 026201 (2009).

Question

What does this distribution tell us about the underlying complex system?

Information geometry

Question

Given a probability distribution $\mathcal{P} : \{0,1\}^{\times N} \to \mathbb{R}$ is it a thermal state $\mathcal{E}_k$ of an k-particle Hamiltonian $\mathcal{H}_k$?
If not, how far is it in terms of the relative entropy?

$$D(\mathcal{P}||\mathcal{Q}) = \sum_k p_k \log\{\tfrac{p_k}{q_k}\}$$

Complexity measure

Distance to the k-particle Hamiltonians

$$D(\mathcal{P}||\mathcal{E}_k) := \inf_{\mathcal{Q}\in\mathcal{E}_k} D(\mathcal{P}||\mathcal{Q})$$

and then

$$I^k(\mathcal{P}) = D(\mathcal{P}||\mathcal{E}_{k-1}) - D(\mathcal{P}||\mathcal{E}_k)$$

These distances can be computed efficiently.

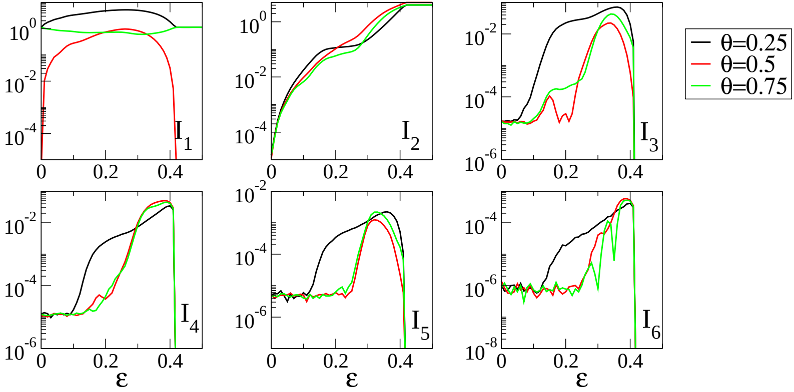
S. Amari, IEEE Trans. Inf. Theor. 47 1701 (2001).

Complexity measures for coupled maps

Observation from Kahle et al.

When the sytem synchonizes, multipartite correlations play a role.

T. Kahle et al., PRE 79, 026201 (2009).



A problem with this approach

Observation

The set of all thermal states $\mathcal{E}_k$ is not invariant under local operations, i.e.

$$\widetilde{P}(\mu) = \sum_{\nu} T^{\text{loc}}_{\mu\nu} P(\nu),$$

where

$$T^{\text{loc}} = \bigotimes_{i=1}^{N} A^{(i)} = \bigotimes_{i=1}^{N} \begin{pmatrix} 1 - a_i & b_i \\ a_i & 1 - b_i \end{pmatrix}.$$

- The distance $D(\mathcal{P}||\mathcal{E}_k)$ can increase under local operations.
- Especially, $D(\mathcal{P}||\mathcal{E}_k)$ can increase from zero to a finite value, if some particle is discarded.

$\Rightarrow$ The quantity $D(\mathcal{P}||\mathcal{E}_k)$ is not equivalent to the notion of correlations in the usual sense.

T. Galla, O. Gühne, arXiv:1107.1180

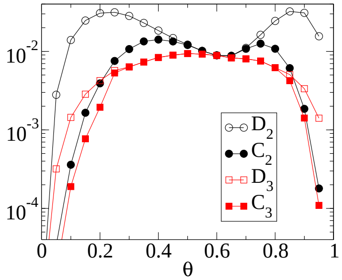
A possible improvement

Idea

Compute the not the distance to $\mathcal{E}_k$, but to the local orbit $\mathcal{L}_k$ of $\mathcal{E}_k$

$$C_k(P) = \inf_{Q \in \mathcal{L}_k} D(P \| Q)$$

Problem: This is numerically difficult to approximate.



T. Galla, O. Gühne, arXiv:1107.1180

The quantum case

Thermal states of two-qubit Hamiltonians are parameterized by

$$\eta_2 = \mathcal{N} \exp\{\sum_{i,a} \lambda_a^{(i)} \sigma_a^{(i)} + \sum_{i,j,a,b} \mu_{ab}^{(ij)} \sigma_a^{(i)} \sigma_b^{(j)}\}$$

Then one can define as before:

$$D(\varrho || \mathcal{Q}_k) := \inf_{\eta \in \mathcal{Q}_k} D(\varrho || \eta)$$

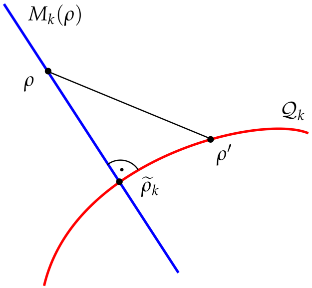
where $D(\varrho||\eta) = Tr(\varrho \log(\varrho) - \varrho \log(\eta))$ is the quantum relative entropy

D.L. Zhou, PRL 101, 180505 (2008), PRA 80, 022113 (2009).

Characterization of the approximation

The following statements are equivalent:

- The state σ_k is the closest state to ϱ in Q_k.
- The state σ_k has the maximal entropy among all states which have the same k-particle marginals as ϱ.
- The state σ_k is in Q_k and has the same k-particle marginals as ϱ.



D.L. Zhou, PRA 80, 022113 (2009), S. Niekamp, Dissertation, 2012

Algorithms to compute the information projection

Zhou's Algorithm

Use the third characterization and try to solve the nonlinear equations.

D.L. Zhou, arXiv:0909.3700

Our Algorithm

- Parameterize an given state in $\mathcal{Q}_2$

$$\eta_2 = \mathcal{N} \exp\{\sum_{i,a} \lambda_a^{(i)} \sigma_a^{(i)} + \sum_{i,j,a,b} \mu_{ab}^{(ij)} \sigma_a^{(i)} \sigma_b^{(j)}\}$$

- An Newton-like optimization for one parameter $\mu_{ab}^{(ij)}$ in order to obtain $\langle \sigma_a^{(i)} \sigma_b^{(j)} \rangle_{\eta_2} = \langle \sigma_a^{(i)} \sigma_b^{(j)} \rangle_{\varrho}$ gives $\mu_{ab}^{(ij)} \mapsto \mu_{ab}^{(ij)} + \varepsilon$ with

$$\varepsilon \approx \frac{\langle \sigma_a^{(i)} \sigma_b^{(j)} \rangle_{\varrho} - \langle \sigma_a^{(i)} \sigma_b^{(j)} \rangle_{\eta_2}}{\Delta^2(\sigma_a^{(i)} \sigma_b^{(j)})_{\eta}}$$

- Start with a maximally mixed η_2 and iterate.

Other algorithms

Other Ideas

- There are iterative algorithms for maximizing the entropy if the mean values of some observables are known.

  Y.S. Teo et al., PRL 107, 020404 (2011)

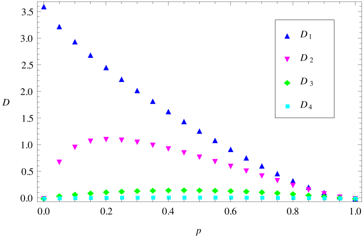
- The entropy is a concave function, which is maximized under linear constraints, so methods from convex programming can be used.

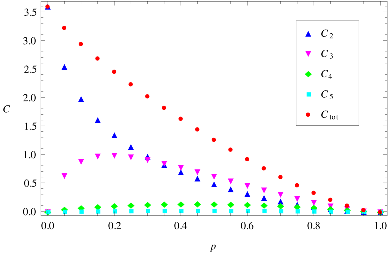
Comparison

- In general, our iteration gives the fastest and best approximation.
- Only if k is large ($=$ many linear contraints), the convex optimization is better.

Consider the five-qubit W state mixed with white noise:

$$|W\rangle = |10000\rangle + |01000\rangle + |00100\rangle + |00010\rangle + |00001\rangle$$

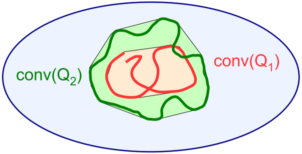




The convex hull of $\mathcal{Q}_k$

Question

- Can we characterize the convex hull of $\mathcal{Q}_k$?



- The convex hull of $\mathcal{Q}_1$ are the fully separable states, so this leads to a generalized notion of entanglement.

Results

- Graph states are generically not in the hull of $\mathcal{Q}_2$
- For some cases, we can obtain fidelity bounds, e.g.

$$F(R_5) = \langle R_5|\varrho|R_5\rangle \geq 1-\varepsilon \Rightarrow \ \varrho \text{ is not in the convex hull of } \mathcal{Q}_2$$

Conclusion

- Exponential families can be used to characterize probability distributions and quantum states.
- For the quantum case, there is an easy algorithm to calculate the distance to $\mathcal{Q}_k$.
- This approach leads to an extended notions of entanglement.

Literature:

- T. Galla, O. Gühne, arXiv:1107.1180
- S. Niekamp, M. Kleinmann, O. Gühne, T. Galla, in preparation.

Funding

FWF Der Wissenschaftsfonds.

Bundesministerium für Bildung und Forschung

THE ROYAL SOCIETY

MARIE CURIE ACTIONS