Beyond mean-field theory: High-accuracy approximation of binary-state dynamics on networks

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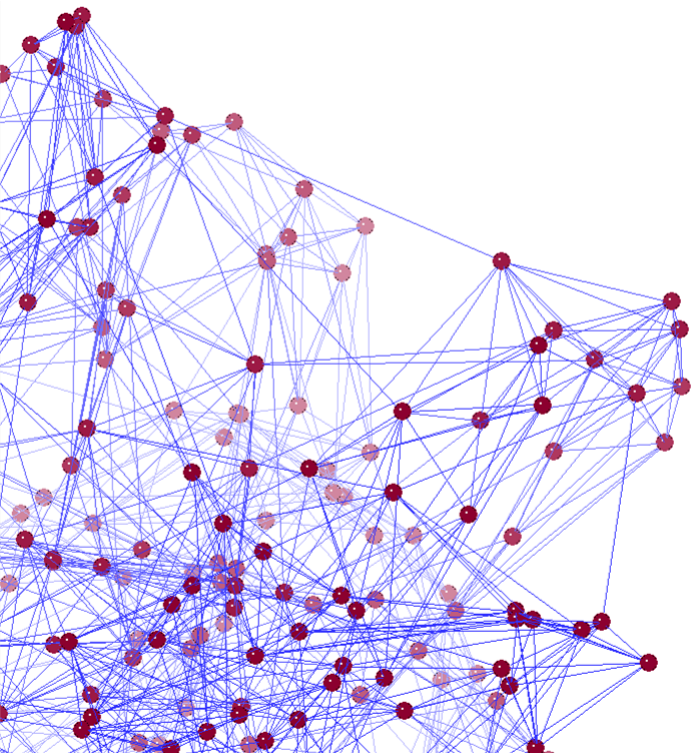
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On the ensemble of (static, undirected, $N \to \infty$) random networks with degree distribution P_k: is it possible to accurately predict macroscopic outcomes for given stochastic (binary-state) dynamical processes?

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Each node is either infected or susceptible.
Infected nodes become susceptible at rate μ;
an infected node infects each of its susceptible neighbours at rate λ.

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Mean-field (MF) theory:
Pastor-Satorras and Vespignani (2001)

Pair approximation (PA):
Levin and Durrett (1996); Eames and Keeling (2002)

Approx. Master Equations (AME):
Marceau et al, PRE (2010), Lindquist et al, J. Math. Biol. (2011)

On the ensemble of (static, undirected, $N \to \infty$) random networks with degree distribution P_k: is it possible to accurately predict macroscopic outcomes for given stochastic (binary-state) dynamical processes?

Voter model
Each node has an opinion (let's call these "infected" or "susceptible"). At each time step ($dt = 1/N$), a randomly-chosen node is updated.
The chosen node updates its opinion by picking a neighbour at random and copying the opinion of that neighbour.

MF: Sood and Redner (2005)

PA: Vazquez and Eguíluz (2008)

General binary-state stochastic dynamics:

- Each node (of N) is in one of two states at any time – call these states "susceptible" and "infected".

- A randomly-chosen fraction $\rho(0)$ of nodes are initially infected.

- In a small time step dt, a fraction dt of nodes are updated (often $dt = 1/N$).

- A updating node that is susceptible becomes infected with probability $F_{k,m}\, dt$, where k is the node's degree and m is the number of its neighbours that are infected:

- Notation: $F_{k,m}\, dt =$ infection probability for a k-degree susceptible node with m infected neighbours.
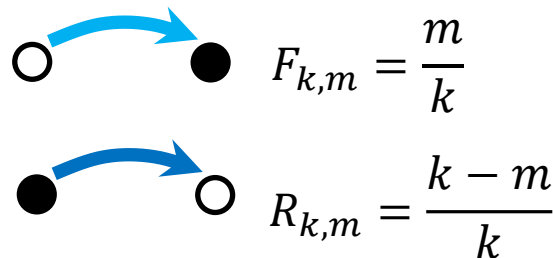

- Similarly: $R_{k,m}\, dt =$ recovery probability for a k-degree infected node with m infected neighbours.

Examples

Voter model
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At each time step ($dt = 1/N$), a randomly-chosen node is updated.
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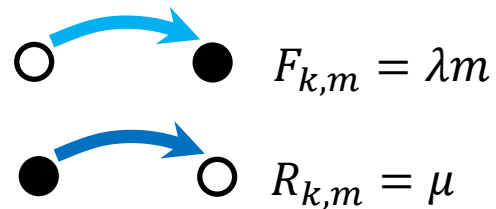and copying the opinion of that neighbour.

$$F_{k,m} = \frac{m}{k}$$

$$R_{k,m} = \frac{k - m}{k}$$

Examples

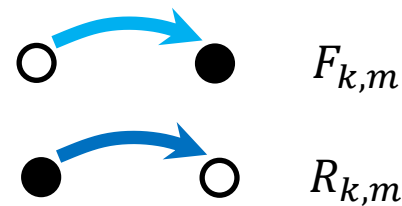**SIS (susceptible-infected-susceptible) model for disease spread**
Each node is either infected or susceptible.
Infected nodes become susceptible at rate μ;
an infected node infects each of its susceptible neighbours at rate λ.

$$F_{k,m} = \lambda m$$

$$R_{k,m} = \mu$$

Further examples

$F_{k,m}$

$R_{k,m}$

Synergy in Spreading Processes: From Exploitative to Explorative Foraging Strategies

Francisco J. Pérez-Reche,[1,2,*] Jonathan J. Ludlam,[3] Sergei N. Taraskin,[4] and Christopher A. Gilligan[5]

[1]*Department of Chemistry, University of Cambridge, Cambridge, United Kingdom*
[2]*SIMBIOS Centre, University of Abertay, Dundee, United Kingdom*
[3]*Churchill College, University of Cambridge, Cambridge, United Kingdom*
[4]*St. Catharine's College and Department of Chemistry, University of Cambridge, Cambridge, United Kingdom*
[5]*Department of Plant Sciences, University of Cambridge, Cambridge, United Kingdom*
(Received 27 October 2010; published 24 May 2011)

An epidemiological model which incorporates synergistic effects that allow the infectivity and/or susceptibility of hosts to be dependent on the number of infected neighbors is proposed. Constructive synergy induces an exploitative behavior which results in a rapid invasion that infects a large number of hosts. Interfering synergy leads to a slower and sparser explorative foraging strategy that traverses larger distances by infecting fewer hosts. The model can be mapped to a dynamical bond percolation with spatial correlations that affect the mechanism of spread but do not influence the critical behavior of epidemics.

Further examples

$F_{k,m}$

$R_{k,m}$
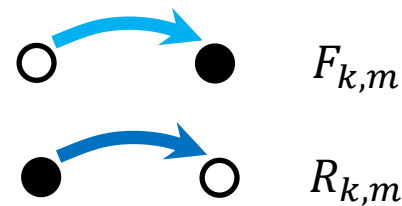
Infectious Disease Modeling of Social Contagion in Networks

Alison L. Hill[1,2]*, **David G. Rand**[1,3], **Martin A. Nowak**[1,4,5], **Nicholas A. Christakis**[6,7,8]

1 Program for Evolutionary Dynamics, Harvard University, Cambridge, Massachusetts, United States of America, 2 Biophysics Program and Harvard-MIT Division of Health Sciences and Technology, Harvard University, Cambridge, Massachusetts, United States of America, 3 Department of Psychology and Berkman Center for Internet and Society, Harvard University, Cambridge, Massachusetts, United States of America, 4 Department of Mathematics, Harvard University, Cambridge, Massachusetts, United States of America, 5 Department of Organismic and Evolutionary Biology, Harvard University, Cambridge, Massachusetts, United States of America, 6 Department of Medicine, Harvard Medical School, Boston, Massachusetts, United States of America, 7 Department of Health Care Policy, Harvard Medical School, Boston, Massachusetts, United States of America, 8 Department of Sociology, Harvard University, Cambridge, Massachusetts, United States of America

Abstract

Many behavioral phenomena have been found to spread interpersonally through social networks, in a manner similar to infectious diseases. An important difference between social contagion and traditional infectious diseases, however, is that behavioral phenomena can be acquired by non-social mechanisms as well as through social transmission. We introduce a novel theoretical framework for studying these phenomena (the SISa model) by adapting a classic disease model to include the possibility for 'automatic' (or 'spontaneous') non-social infection. We provide an example of the use of this framework

Further examples

$F_{k,m}$

$R_{k,m}$
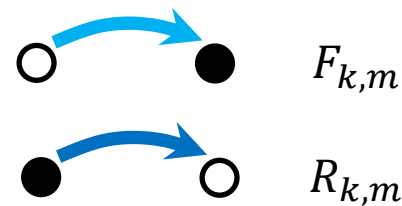
Dynamic Opinion Model and Invasion Percolation

Jia Shao,[1] Shlomo Havlin,[2] and H. Eugene Stanley[1]

[1]*Center for Polymer Studies and Department of Physics, Boston University, Boston, Massachusetts 02215, USA*
[2]*Minerva Center and Department of Physics, Bar-Ilan University, 52900 Ramat-Gan, Israel*
(Received 25 March 2009; published 1 July 2009)

We propose a "nonconsensus" opinion model that allows for stable coexistence of two opinions by forming clusters of agents holding the same opinion. We study this nonconsensus model on lattices, several model complex networks, and a real-life social network. We find that the model displays a phase transition behavior characterized by a large spanning cluster of nodes holding the same opinion appearing when the concentration of nodes holding the same opinion (even minority) is above a certain threshold. Because of the clustering (community support) of agents holding the same opinion, these clusters cannot be invaded by the other opinion (similar to incompressible fluids). Our extensive simulations show that the nonconsensus opinion model appears to belong to the same universality class as invasion percolation.

(Monotone) threshold models of "complex contagion"
[Granovetter (1978), Watts (2002), Centola & Macy (2007)]

- Each node i has a (frozen) threshold r_i, and a binary state ("susceptible"/"infected").

- A randomly-chosen fraction $\rho(0)$ of nodes are initially infected.

- Asynchronous updating: A fraction dt of nodes update in time step dt.

- Update rule: compare the fraction of infected neighbours m_i/k_i to r_i. Node i is infected if $m_i/k_i \geq r_i$, but unchanged otherwise

- $F_{k,m}\, dt =$ infection probability for a k-degree susceptible node with m infected neighbours.
- For example, if all thresholds are identical ($r_i = r \; \forall\, i$):

$$F_{k,m} = \begin{cases} 0 & \text{for } m < kr \\ 1 & \text{for } m \geq kr \end{cases}$$

- Monotone case: no recovery, so $\quad R_{k,m} \equiv 0$

Monotone threshold model

$$F_{k,m} = \begin{cases} 0 & \text{for } m < kr \\ 1 & \text{for } m \geq kr \end{cases}$$

random 3-regular graph, $r = 2/3$

Mean-field (MF) theory

Numerical simulations
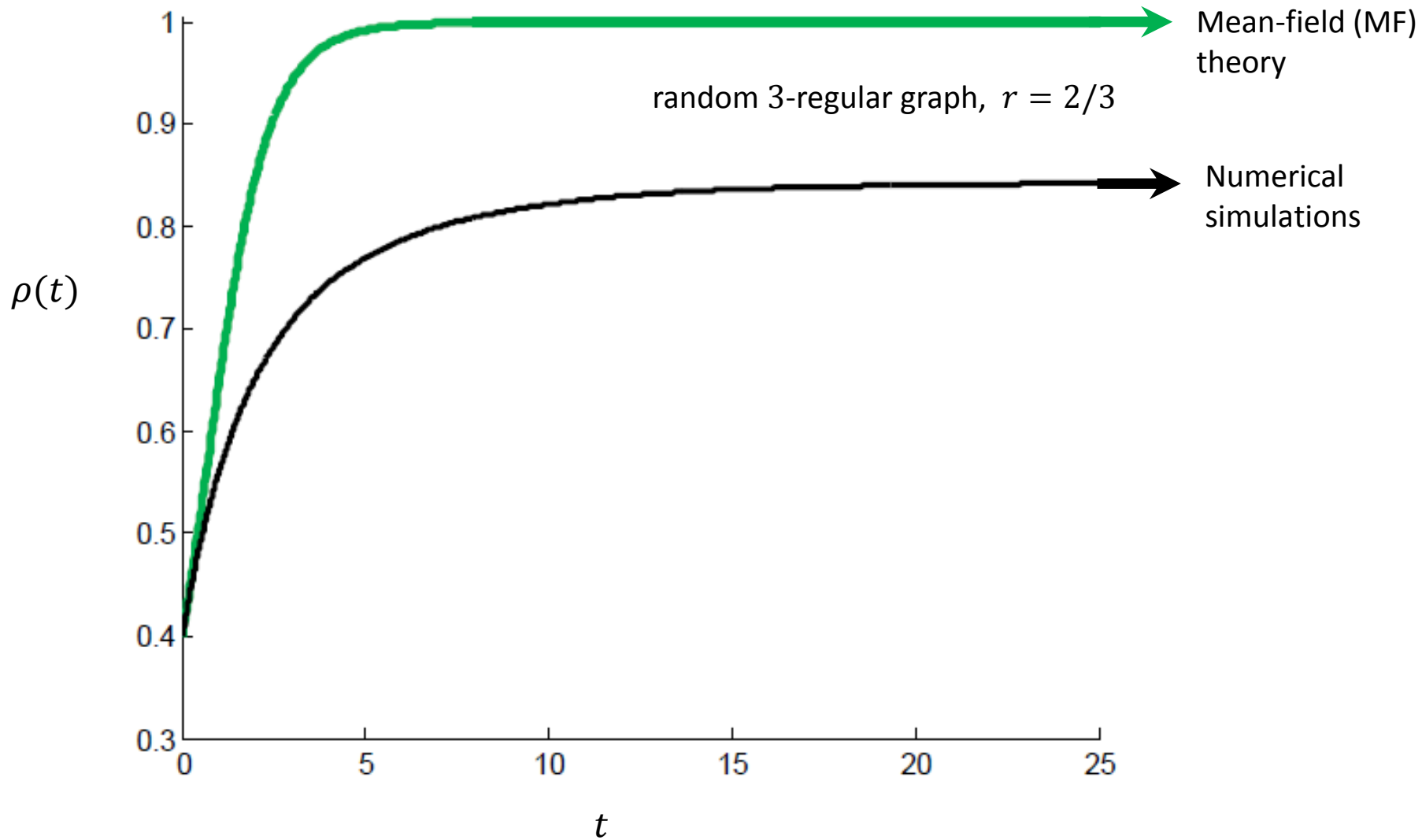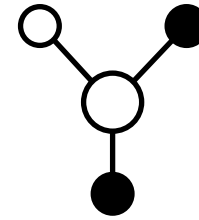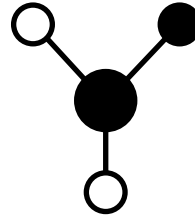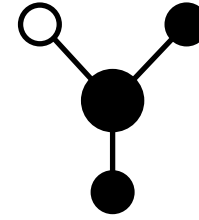
Random z-regular graphs

S_{m-1} class S_m class S_{m+1} class
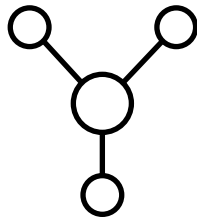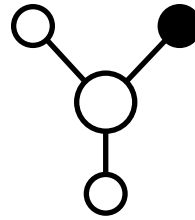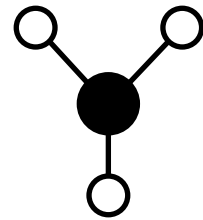
I_{m-1} class I_m class I_{m+1} class

$s_m(t) =$ size of S_m class at time t (for $m = 0, 1, \ldots, z$)

$\quad\quad = $ fraction of nodes which are susceptible and have m infected neighbours at time t

$i_m(t) = $ fraction of nodes which are infected and have m infected neighbours at time t
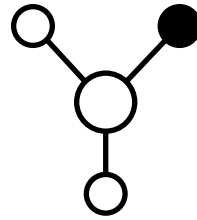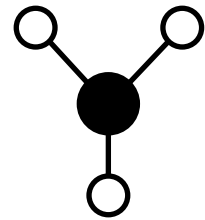
$s_m(0) = (1 - \rho(0))B_{z,m}(\rho(0))$

$i_m(0) = \rho(0)B_{z,m}(\rho(0))$

[cf. Marceau et al, PRE (2010),
       Lindquist et al, J. Math. Biol. (2011)]
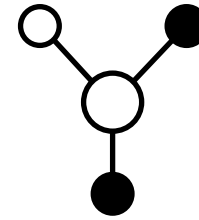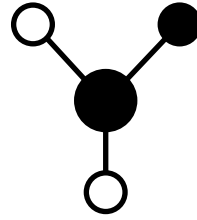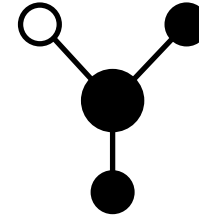
S_{m-1} class S_m class S_{m+1} class
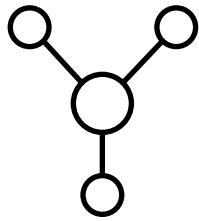
I_{m-1} class I_m class I_{m+1} class

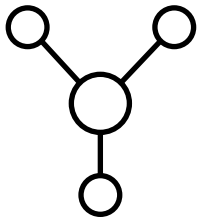$s_m(t) =$ fraction of nodes which are susceptible and have m infected neighbours at time t

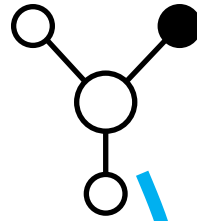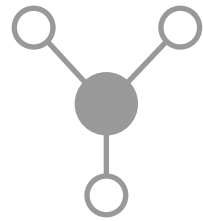$i_m(t) =$ fraction of nodes which are infected and have m infected neighbours at time t
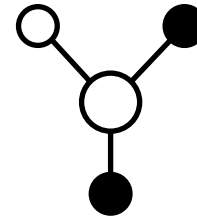
= number of S-I edges
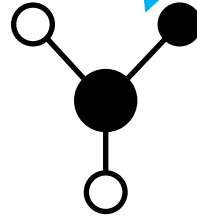
$$= N \sum_{m=0}^{z} m s_m$$

S_{m-1} class S_m class S_{m+1} class

F_m
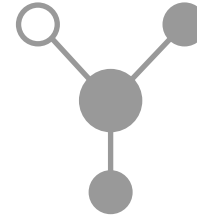
I_{m-1} class I_m class I_{m+1} class

$$\frac{d}{dt} s_m = -F_m s_m + \cdots$$

for $m = 0, 1, \ldots, z$

$s_m(t) =$ fraction of nodes which are susceptible and have m infected neighbours at time t

$F_m \, dt =$ infection probability for a susceptible node with m infected neighbours

e.g., threshold model on random z-regular graph:

$$F_m \equiv F_{z,m} = \begin{cases} 0 & \text{for } m < zr \\ 1 & \text{for } m \geq zr \end{cases}$$

S_{m-1} class S_m class S_{m+1} class

β^s

F_m
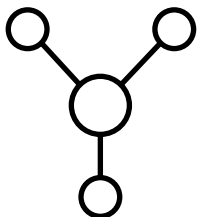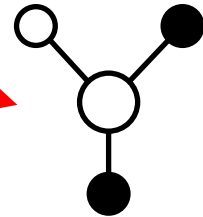
I_{m-1} class I_m class I_{m+1} class

$$\frac{d}{dt}s_m = -F_m s_m - \beta^s (z - m)s_m + \cdots \qquad \text{for } m = 0, 1, \ldots, z$$

S_{m-1} class $\quad S_m$ class $\quad S_{m+1}$ class

β^s

β^s

F_m

I_{m-1} class $\quad I_m$ class $\quad I_{m+1}$ class

$$\frac{d}{dt} s_m = -F_m s_m - \beta^s(z-m)s_m + \beta^s(z-m+1)s_{m-1} \quad \text{for } m = 0,1,\dots,z$$

S_{m-1} class $\quad S_m$ class $\quad S_{m+1}$ class

$\beta^s \qquad \beta^s$

F_m

I_{m-1} class $\quad I_m$ class $\quad I_{m+1}$ class

$$\frac{d}{dt}s_m = -{\color{cyan}F_m}s_m - {\color{red}\beta^s}(\,z-m)s_m + {\color{red}\beta^s}(z-m+1)s_{m-1} \quad \text{for } m = 0,1,\dots,z$$

$\beta^s dt = \cdots$

$$\frac{d}{dt} s_m = -F_m s_m - \beta^s (z-m) s_m + \beta^s (z-m+1) s_{m-1} \quad \text{for } m = 0, 1, \ldots, z$$

$$\beta^s dt = \cdots$$

S_{m-1} class $\xrightarrow{\beta^s}$ S_m class $\xrightarrow{\beta^s}$ S_{m+1} class

F_m

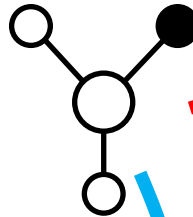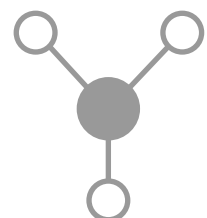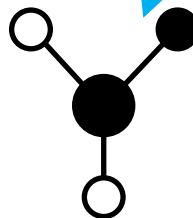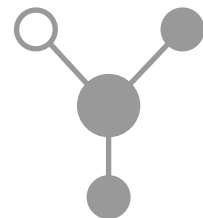I_{m-1} class I_m class I_{m+1} class

$$\frac{d}{dt}s_m = -{\color{blue}F_m}s_m - {\color{red}\beta^s}(z-m)s_m + {\color{red}\beta^s}(z-m+1)s_{m-1} \quad \text{for } m = 0,1,\dots,z$$

$$\beta^s = \frac{\sum_{m=0}^{z}(z-m)F_m s_m}{\sum_{m=0}^{z}(z-m)s_m} \quad =$$

S_{m-1} class $\quad$ S_m class $\quad$ S_{m+1} class

β^s $\quad$ β^s

F_m

I_{m-1} class $\quad$ I_m class $\quad$ I_{m+1} class
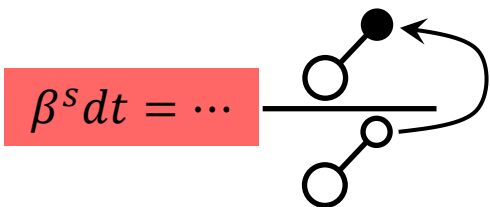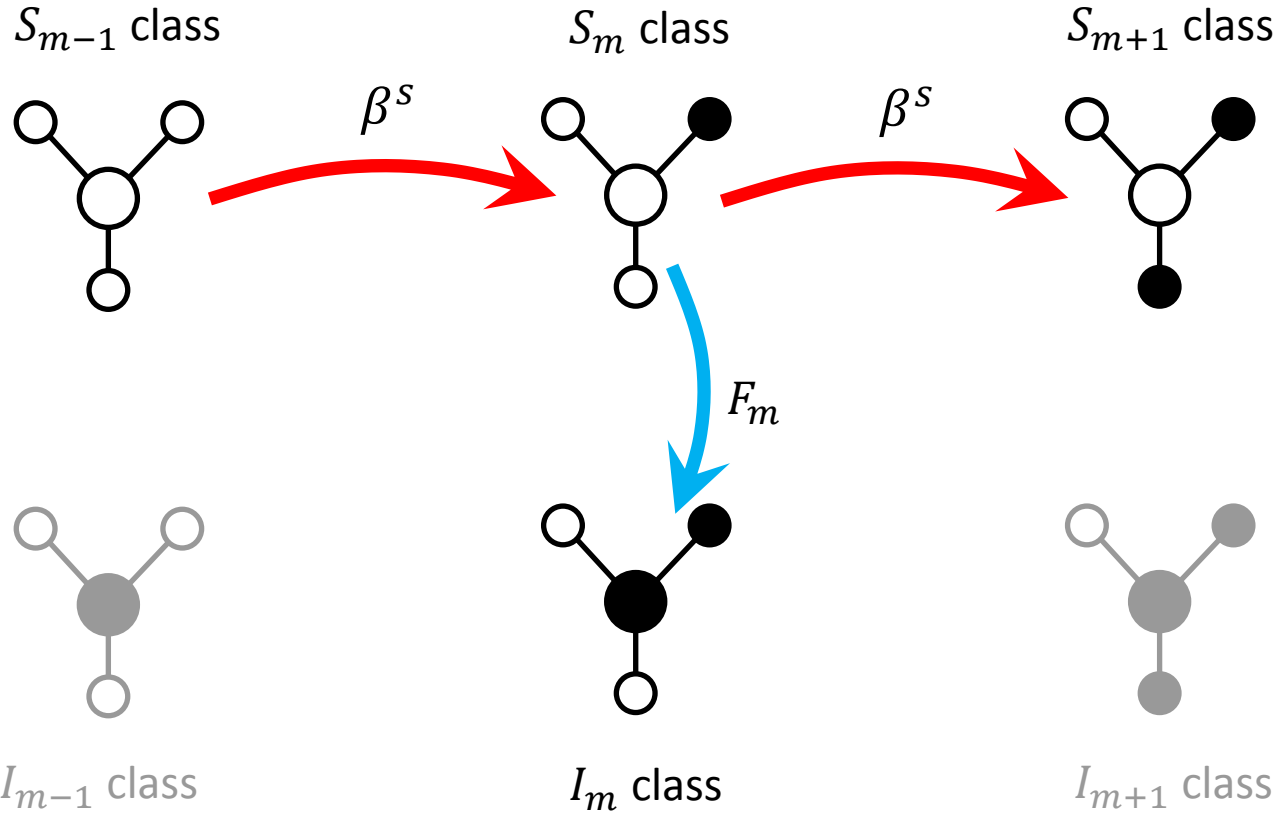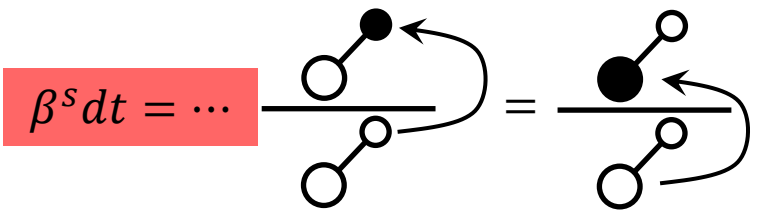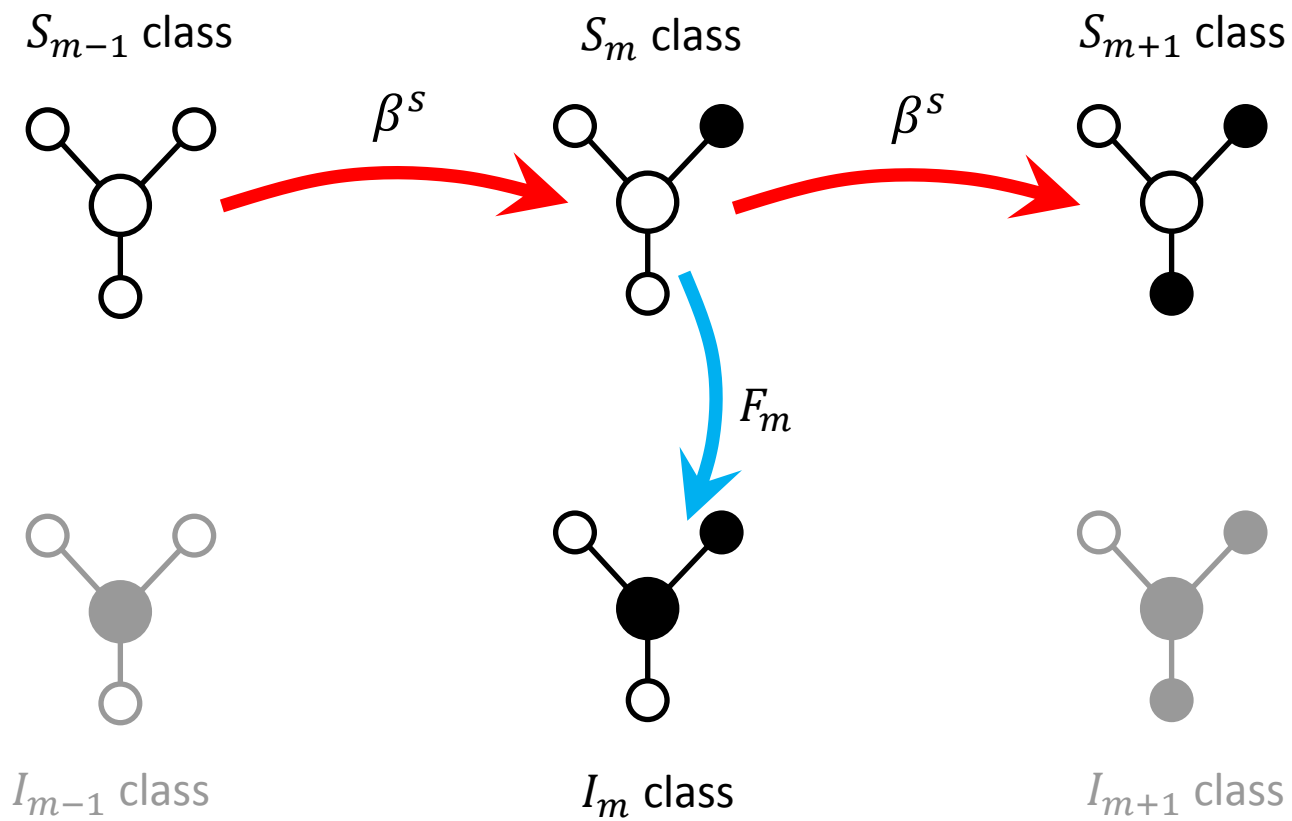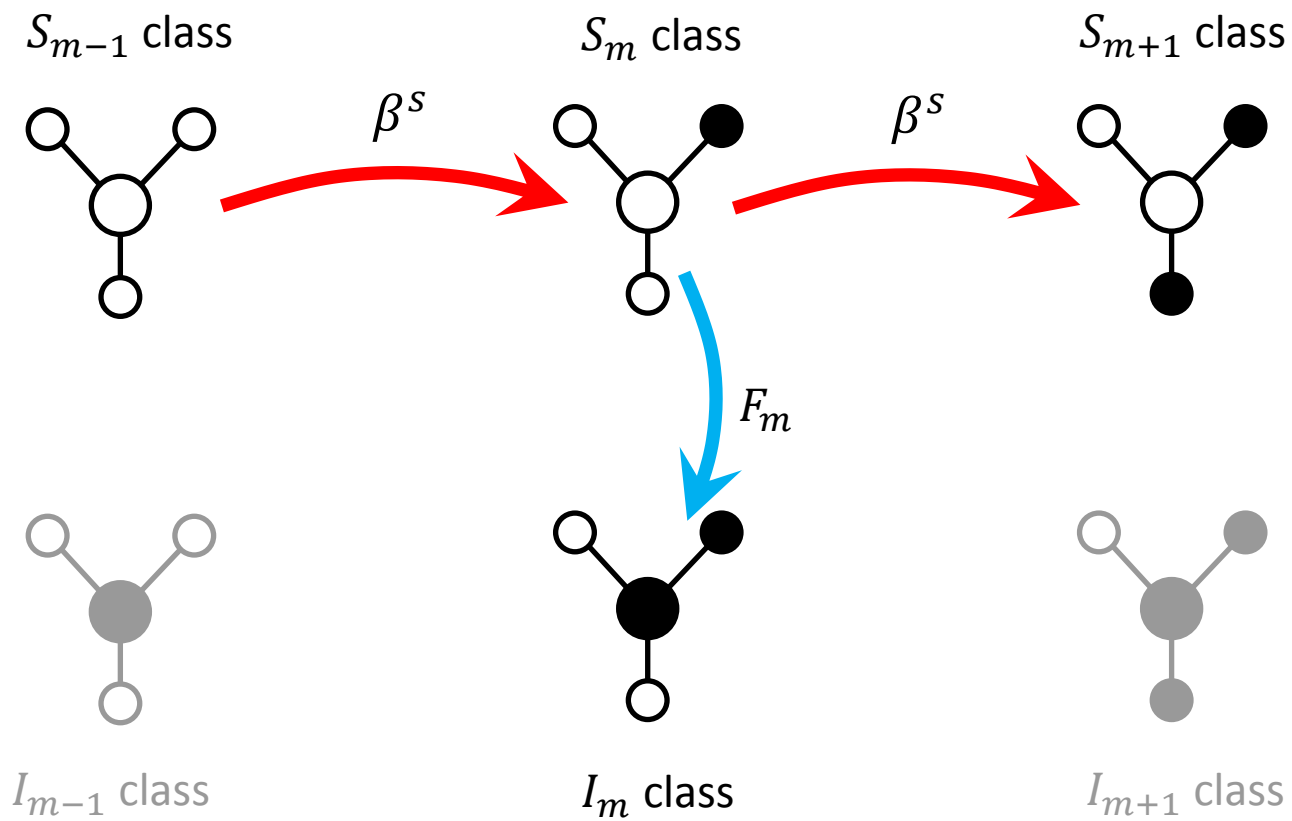
$$\frac{d}{dt}s_m = -F_m s_m - \beta^s(z-m)s_m + \beta^s(z-m+1)s_{m-1} \quad \text{for } m = 0,1,\dots,z$$

$$\beta^s = \frac{\sum_{m=0}^{z}(z-m)F_m s_m}{\sum_{m=0}^{z}(z-m)s_m}$$

$$s_m(0) = (1-\rho(0))B_{z,m}(\rho(0))$$

$$\rho(t) = 1 - \sum_{m=0}^{z} s_m(t)$$

Monotone threshold model

$$F_{k,m} = \begin{cases} 0 & \text{for } m < kr \\ 1 & \text{for } m \geq kr \end{cases}$$

random 3-regular graph, $r = 2/3$

Mean-field (MF) theory

Numerical simulations

$\rho(t)$

t

Monotone threshold model

$$F_{k,m} = \begin{cases} 0 & \text{for } m < kr \\ 1 & \text{for } m \geq kr \end{cases}$$

Mean-field (MF) theory

random 3-regular graph, $r = 2/3$

Approximate master equation (AME)

$\rho(t)$

t

S_{m-1} class $\quad \beta^s \quad$ S_m class $\quad \beta^s \quad$ S_{m+1} class

F_m

I_{m-1} class $\qquad I_m$ class $\qquad I_{m+1}$ class

$$\frac{d}{dt}s_m = -F_m s_m - \beta^s(z-m)s_m + \beta^s(z-m+1)s_{m-1} \quad \text{for } m = 0,1,\dots,z$$

$$\beta^s = \frac{\sum_{m=0}^{z}(z-m)F_m s_m}{\sum_{m=0}^{z}(z-m)s_m}$$

$$s_m(0) = (1-\rho(0))B_{z,m}(\rho(0))$$

$$\rho = 1 - \sum_{m=0}^{z} s_m$$

S_{m-1} class β^s S_m class β^s S_{m+1} class

F_m

I_{m-1} class I_m class I_{m+1} class

$$\frac{d}{dt}s_m = -F_m s_m \qquad -\beta^s(z-m)s_m + \beta^s(z-m+1)s_{m-1} \qquad \text{for } m = 0,1,\dots,z$$

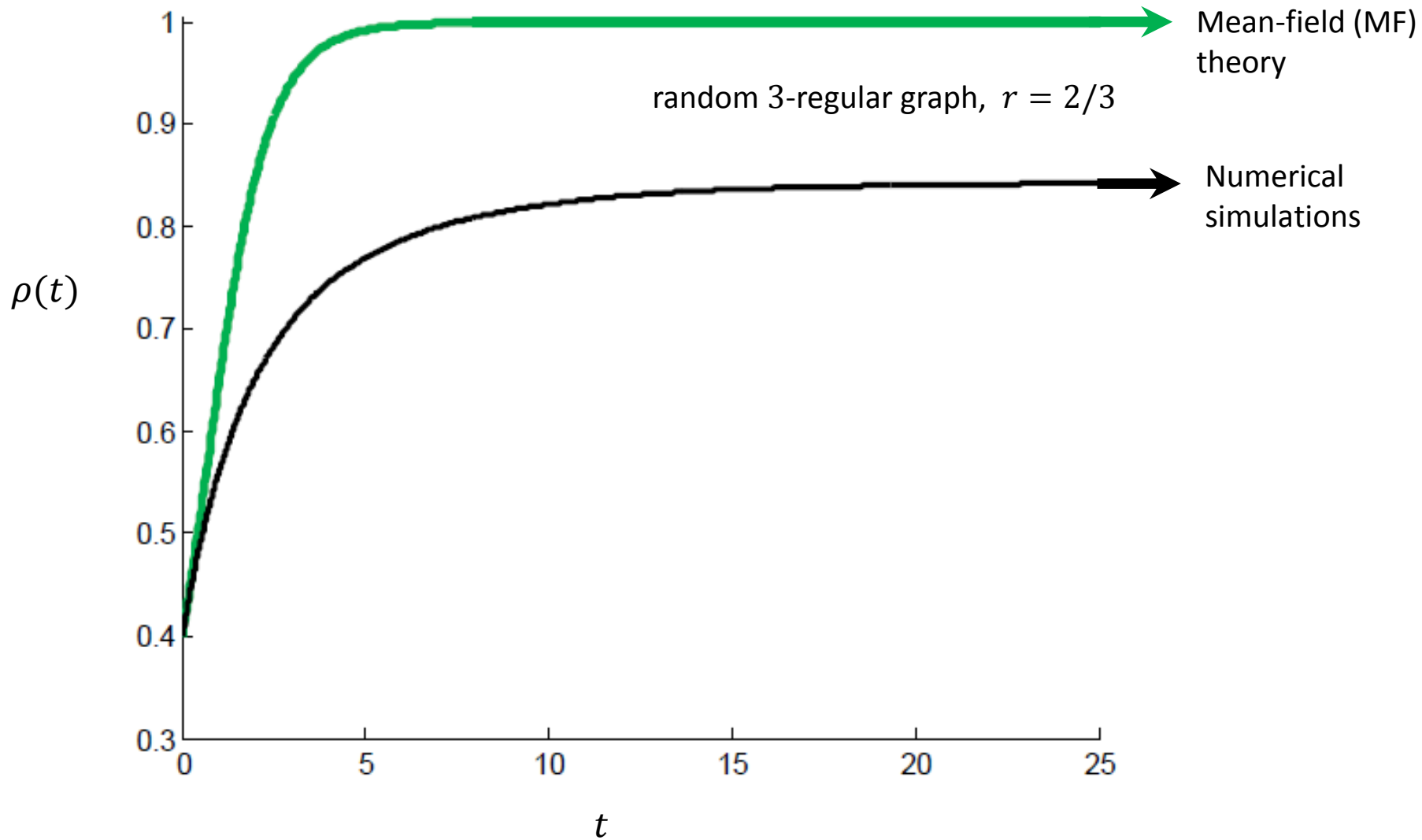$$\beta^s = \frac{\sum_{m=0}^{z}(z-m)F_m s_m}{\sum_{m=0}^{z}(z-m)s_m}$$

$$s_m(0) = (1-\rho(0))B_{z,m}(\rho(0))$$

$$\rho = 1 - \sum_{m=0}^{z} s_m$$

S_{m-1} class S_m class S_{m+1} class

β^s β^s

γ^s γ^s

R_m F_m

I_{m-1} class I_m class I_{m+1} class

$$\frac{d}{dt}s_m = -F_m s_m + R_m i_m - (\gamma^s m + \beta^s(z-m))s_m + \beta^s(z-m+1)s_{m-1} + \gamma^s(m+1)s_{m+1}$$

$R_m\, dt =$ recovery probability for an
infected node with m infected
neighbours

$$\beta^s = \frac{\sum_{m=0}^{z}(z-m)F_m s_m}{\sum_{m=0}^{z}(z-m)s_m}$$

e.g., non-monotone threshold model:

$$R_{k,m} = \begin{cases} 1 \text{ for } m < kr \\ 0 \text{ for } m \geq kr \end{cases}$$
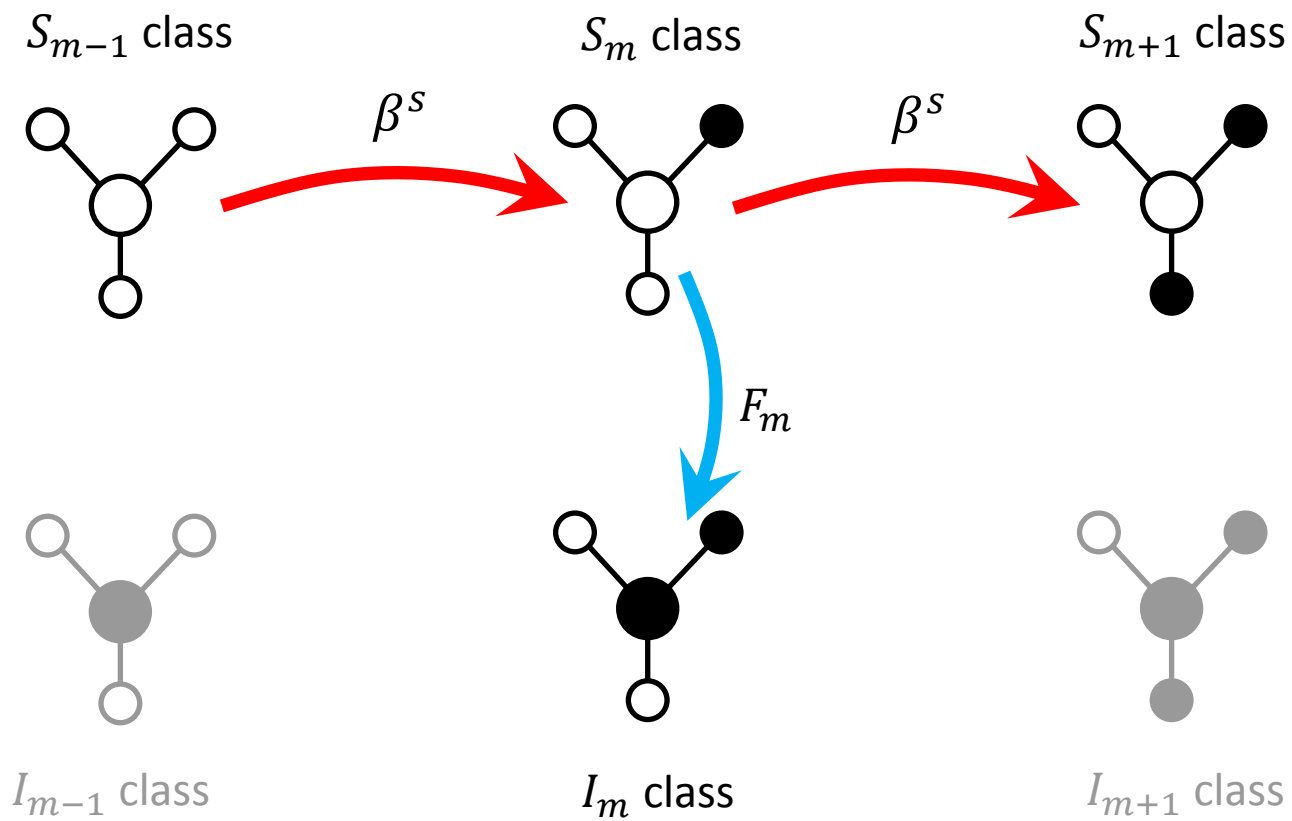
S_{m-1} class S_m class S_{m+1} class

β^s β^s

γ^s γ^s

R_m F_m

I_{m-1} class I_m class I_{m+1} class

$$\frac{d}{dt}s_m = -F_m s_m + R_m i_m - (\gamma^s m + \beta^s(z-m))s_m + \beta^s(z-m+1)s_{m-1} + \gamma^s(m+1)s_{m+1}$$

$$\beta^s = \frac{\sum_{m=0}^{z}(z-m)F_m s_m}{\sum_{m=0}^{z}(z-m)s_m}$$

$$\gamma^s = \frac{\sum_{m=0}^{z}(z-m)R_m i_m}{\sum_{m=0}^{z}(z-m)i_m}$$
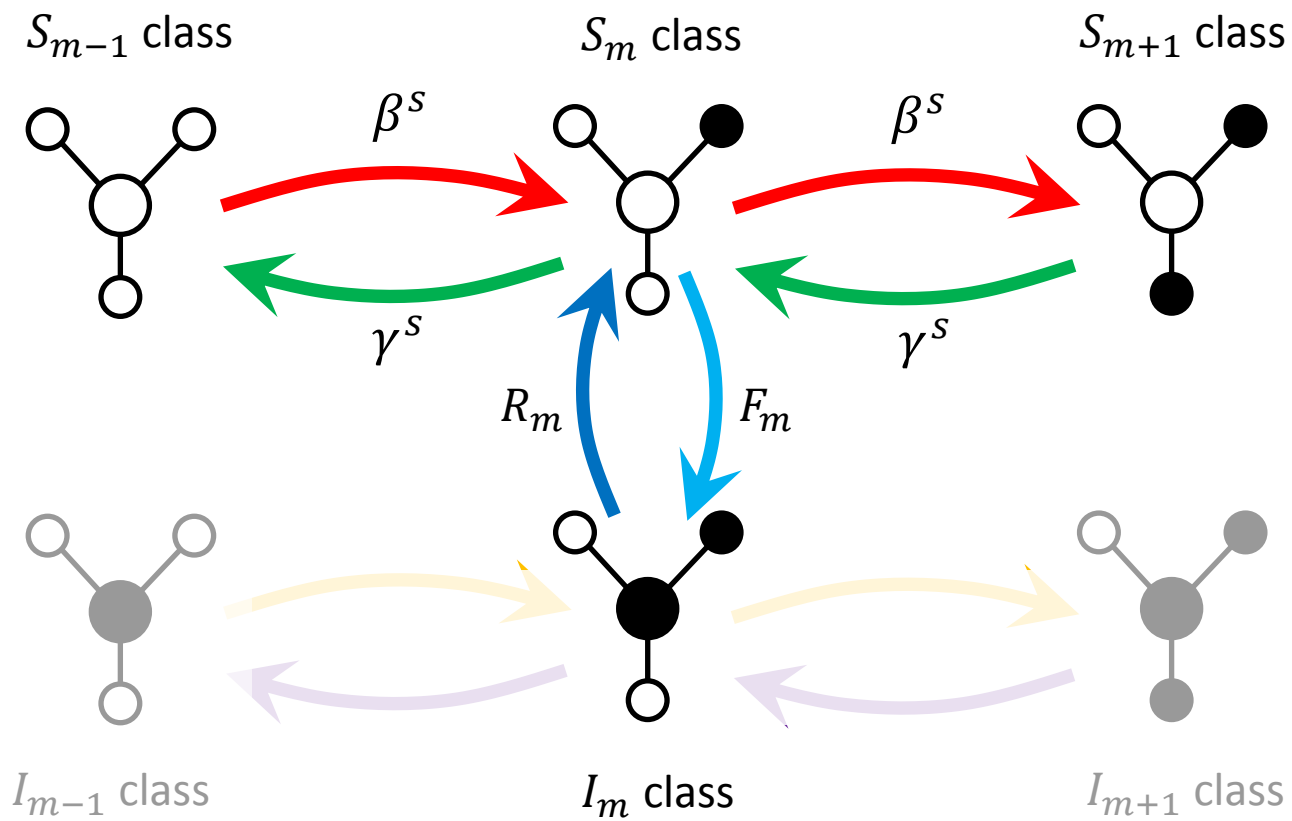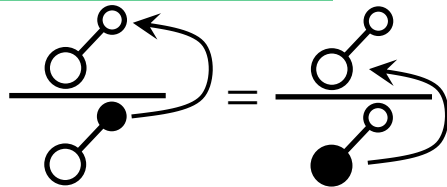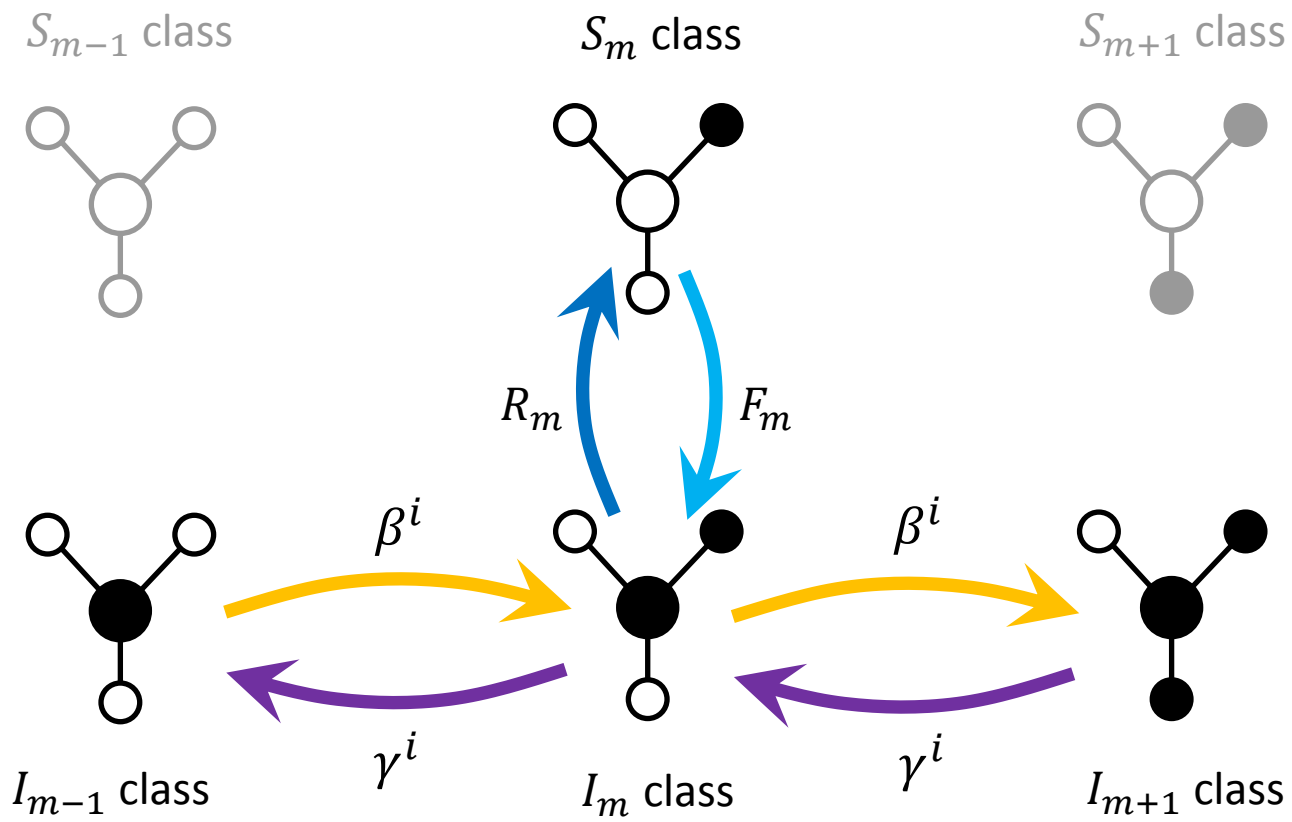
$$s_m(0) = (1-\rho(0))B_{z,m}(\rho(0))$$

S_{m-1} class S_m class S_{m+1} class

R_m F_m

β^i β^i

γ^i γ^i

I_{m-1} class I_m class I_{m+1} class

$$\frac{d}{dt}s_m = -F_m s_m + R_m i_m - \left(\gamma^s m + \beta^s (z-m)\right)s_m + \beta^s (z-m+1)s_{m-1} + \gamma^s (m+1)s_{m+1}$$

$$\frac{d}{dt}i_m = -R_m i_m + F_m s_m - \left(\gamma^i m + \beta^i (z-m)\right)i_m + \beta^i (z-m+1)i_{m-1} + \gamma^i (m+1)i_{m+1}$$

$$\beta^s = \frac{\sum_{m=0}^{z}(z-m)F_m s_m}{\sum_{m=0}^{z}(z-m)s_m}$$

$$\gamma^s = \frac{\sum_{m=0}^{z}(z-m)R_m i_m}{\sum_{m=0}^{z}(z-m)i_m}$$

$$s_m(0) = (1-\rho(0))B_{z,m}(\rho(0))$$

$$i_m(0) = \rho(0)B_{z,m}(\rho(0))$$

$$\beta^i = \frac{\sum_{m=0}^{z}mF_m s_m}{\sum_{m=0}^{z}ms_m}$$

$$\gamma^i = \frac{\sum_{m=0}^{z}mR_m i_m}{\sum_{m=0}^{z}mi_m}$$

$$\frac{d}{dt}s_m = -F_m s_m + R_m i_m - (\gamma^s m + \beta^s(z-m))s_m + \beta^s(z-m+1)s_{m-1} + \gamma^s(m+1)s_{m+1}$$

$$\frac{d}{dt}i_m = -R_m i_m + F_m s_m - (\gamma^i m + \beta^i(z-m))i_m + \beta^i(z-m+1)i_{m-1} + \gamma^i(m+1)i_{m+1}$$

$$\beta^s = \frac{\sum_{m=0}^{z}(z-m)F_m s_m}{\sum_{m=0}^{z}(z-m)s_m}$$
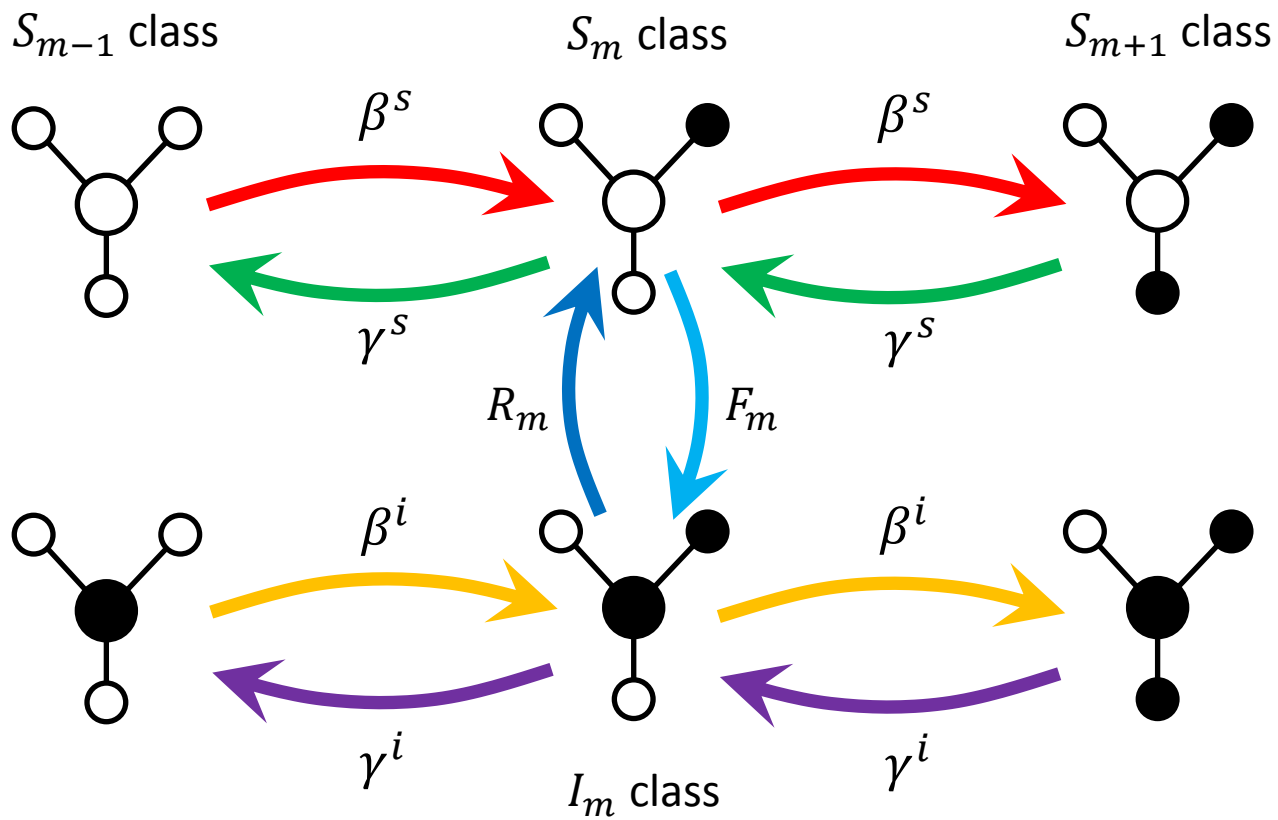
$$\gamma^s = \frac{\sum_{m=0}^{z}(z-m)R_m i_m}{\sum_{m=0}^{z}(z-m)i_m}$$

$$s_m(0) = (1-\rho(0))B_{z,m}(\rho(0))$$

$$i_m(0) = \rho(0)B_{z,m}(\rho(0))$$

$$\beta^i = \frac{\sum_{m=0}^{z} m F_m s_m}{\sum_{m=0}^{z} m s_m}$$

$$\gamma^i = \frac{\sum_{m=0}^{z} m R_m i_m}{\sum_{m=0}^{z} m i_m}$$

$$\rho = \sum_{m=0}^{z} i_m = 1 - \sum_{m=0}^{z} s_m$$

Non-monotone threshold model

$$F_{k,m} = \begin{cases} 0 & \text{for } m < kr \\ 1 & \text{for } m \geq kr \end{cases}$$

$$R_{k,m} = \begin{cases} 1 & \text{for } m < kr \\ 0 & \text{for } m \geq kr \end{cases}$$

RRG, $z = 3$, $r = 2/3$

$\rho(t)$

AME

MF theory

t

$$\frac{d}{dt}s_m = -F_m s_m + R_m i_m - (\gamma^s m + \beta^s(z-m))s_m + \beta^s(z-m+1)s_{m-1} + \gamma^s(m+1)s_{m+1}$$

$$\frac{d}{dt}i_m = -R_m i_m + F_m s_m - (\gamma^i m + \beta^i(z-m))i_m + \beta^i(z-m+1)i_{m-1} + \gamma^i(m+1)i_{m+1}$$

$$\beta^s = \frac{\sum_{m=0}^{z}(z-m)F_m s_m}{\sum_{m=0}^{z}(z-m)s_m}$$

$$\gamma^s = \frac{\sum_{m=0}^{z}(z-m)R_m i_m}{\sum_{m=0}^{z}(z-m)i_m}$$
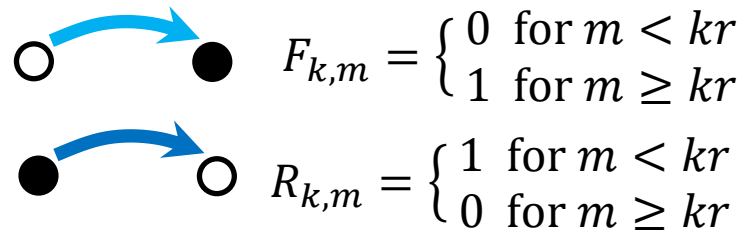
$$s_m(0) = (1-\rho(0))B_{z,m}(\rho(0))$$
$$i_m(0) = \rho(0)B_{z,m}(\rho(0))$$

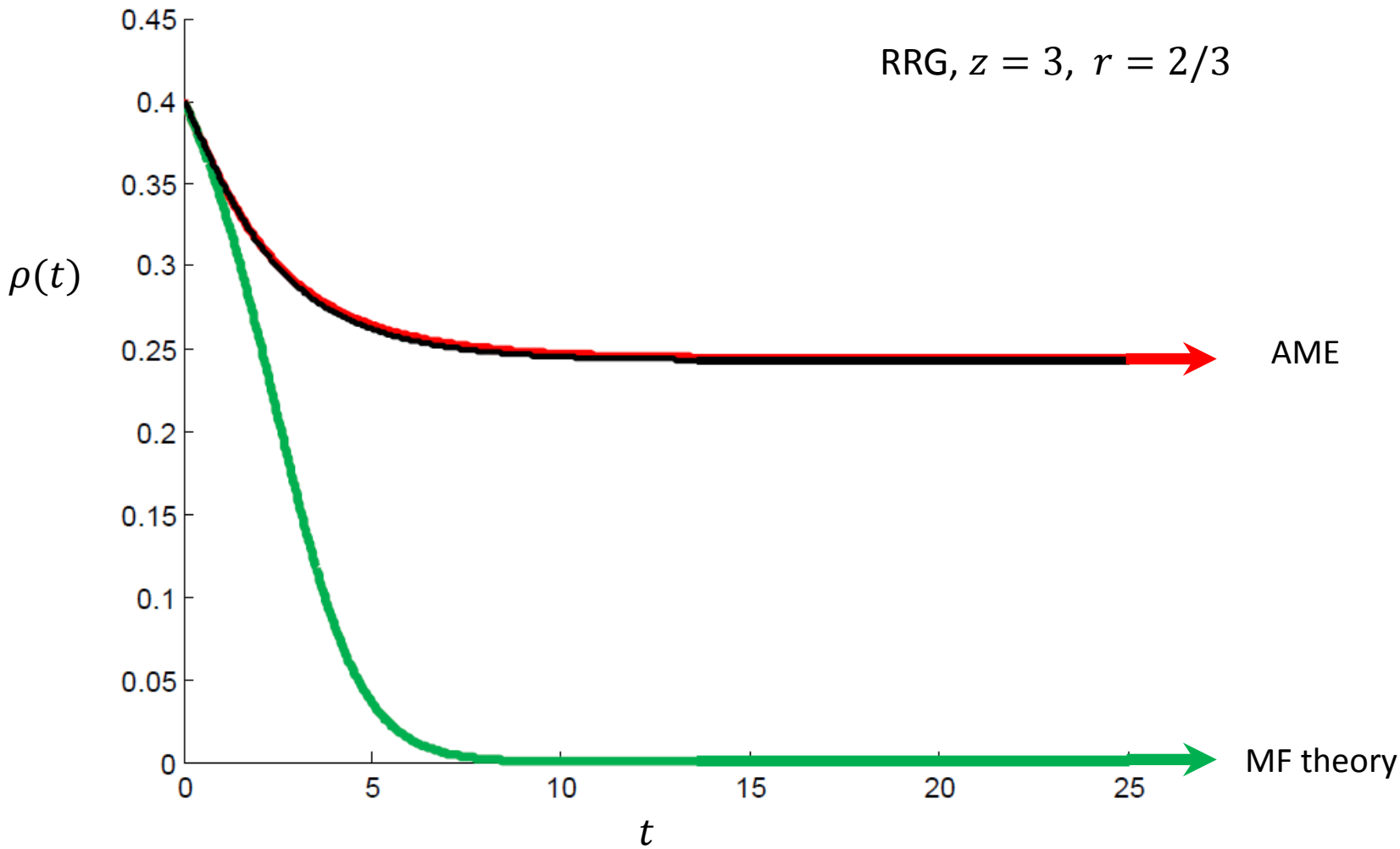$$\beta^i = \frac{\sum_{m=0}^{z} m F_m s_m}{\sum_{m=0}^{z} m s_m}$$

$$\gamma^i = \frac{\sum_{m=0}^{z} m R_m i_m}{\sum_{m=0}^{z} m i_m}$$

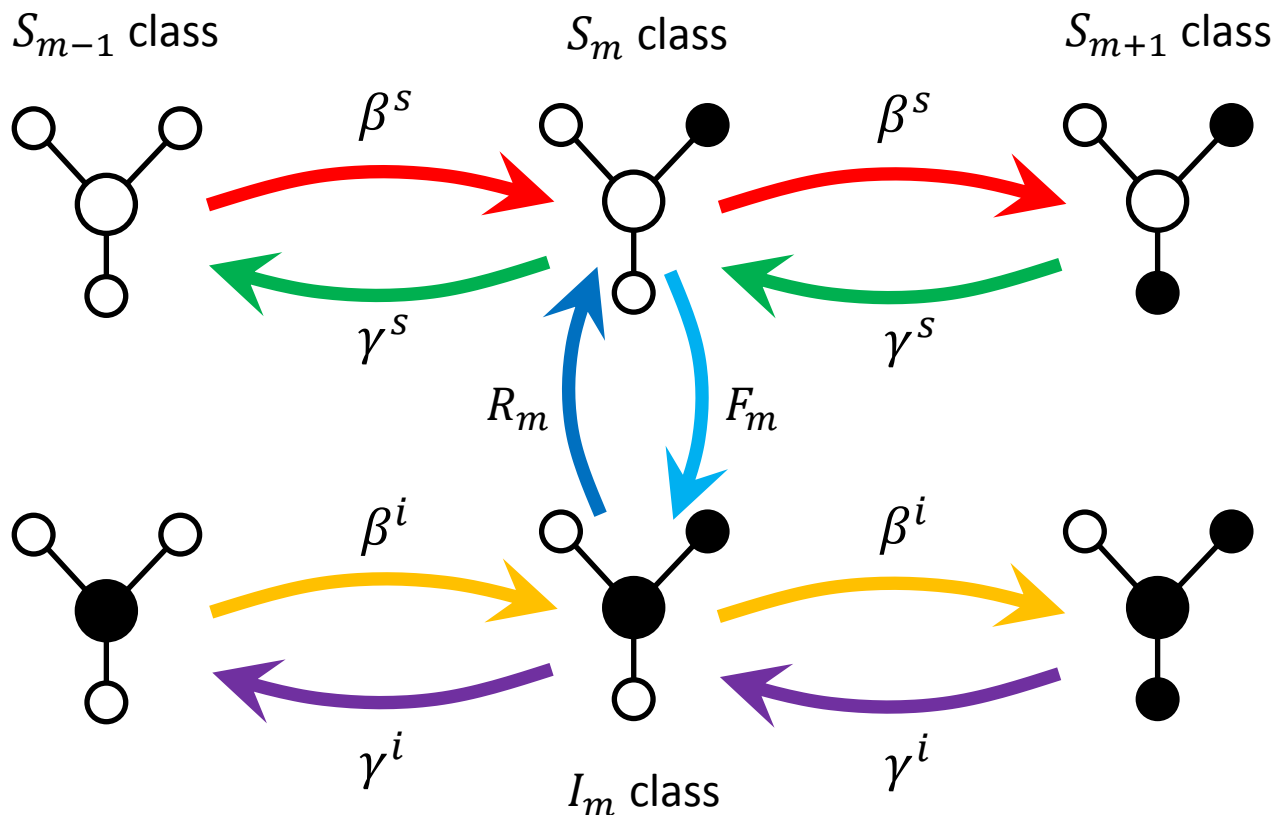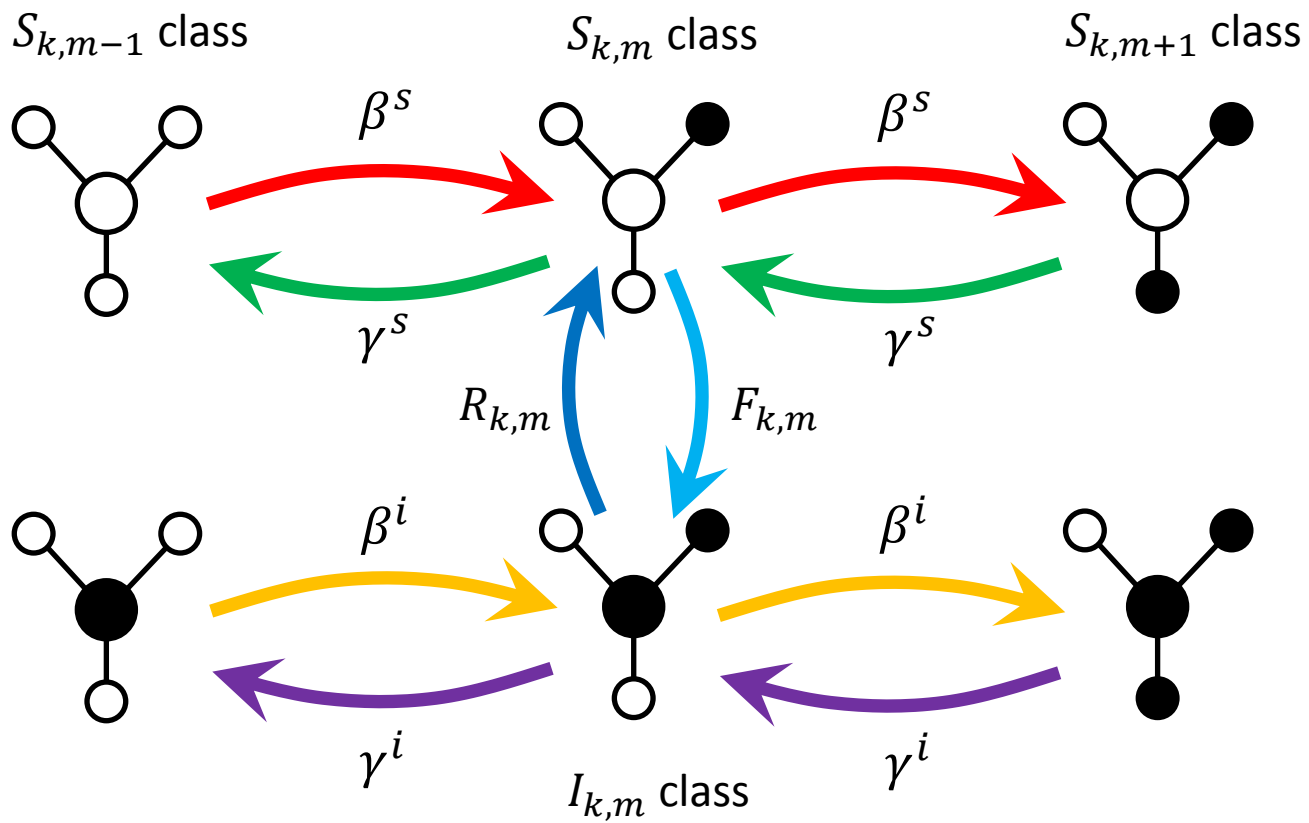$$\rho = \sum_{m=0}^{z} i_m = 1 - \sum_{m=0}^{z} s_m$$

General degree distribution P_k

$S_{k,m-1}$ class $S_{k,m}$ class $S_{k,m+1}$ class

β^s β^s

γ^s γ^s

$R_{k,m}$ $F_{k,m}$

β^i β^i

γ^i γ^i

$I_{k,m}$ class

$$\frac{d}{dt}s_{k,m} = -F_{k,m}s_{k,m} + R_{k,m}i_{k,m} - (\gamma^s m + \beta^s(k-m))s_{k,m} + \beta^s(k-m+1)s_{k,m-1} + \gamma^s(m+1)s_{k,m+1}$$

$$\frac{d}{dt}i_{k,m} = -R_{k,m}i_{k,m} + F_{k,m}s_{k,m} - (\gamma^i m + \beta^i(k-m))i_{k,m} + \beta^i(k-m+1)i_{k,m-1} + \gamma^i(m+1)i_{k,m+1}$$

$$\beta^s = \frac{\sum P_k \sum_{m=0}^{k}(k-m)F_{k,m}s_{k,m}}{\sum P_k \sum_{m=0}^{k}(k-m)s_{k,m}}$$

$$\gamma^s = \frac{\sum P_k \sum_{m=0}^{k}(k-m)R_{k,m}i_{k,m}}{\sum P_k \sum_{m=0}^{k}(k-m)i_{k,m}}$$

$$s_{k,m}(0) = (1-\rho_k(0))B_{k,m}(\rho_k(0))$$

$$i_{k,m}(0) = \rho_k(0)B_{k,m}(\rho_k(0))$$

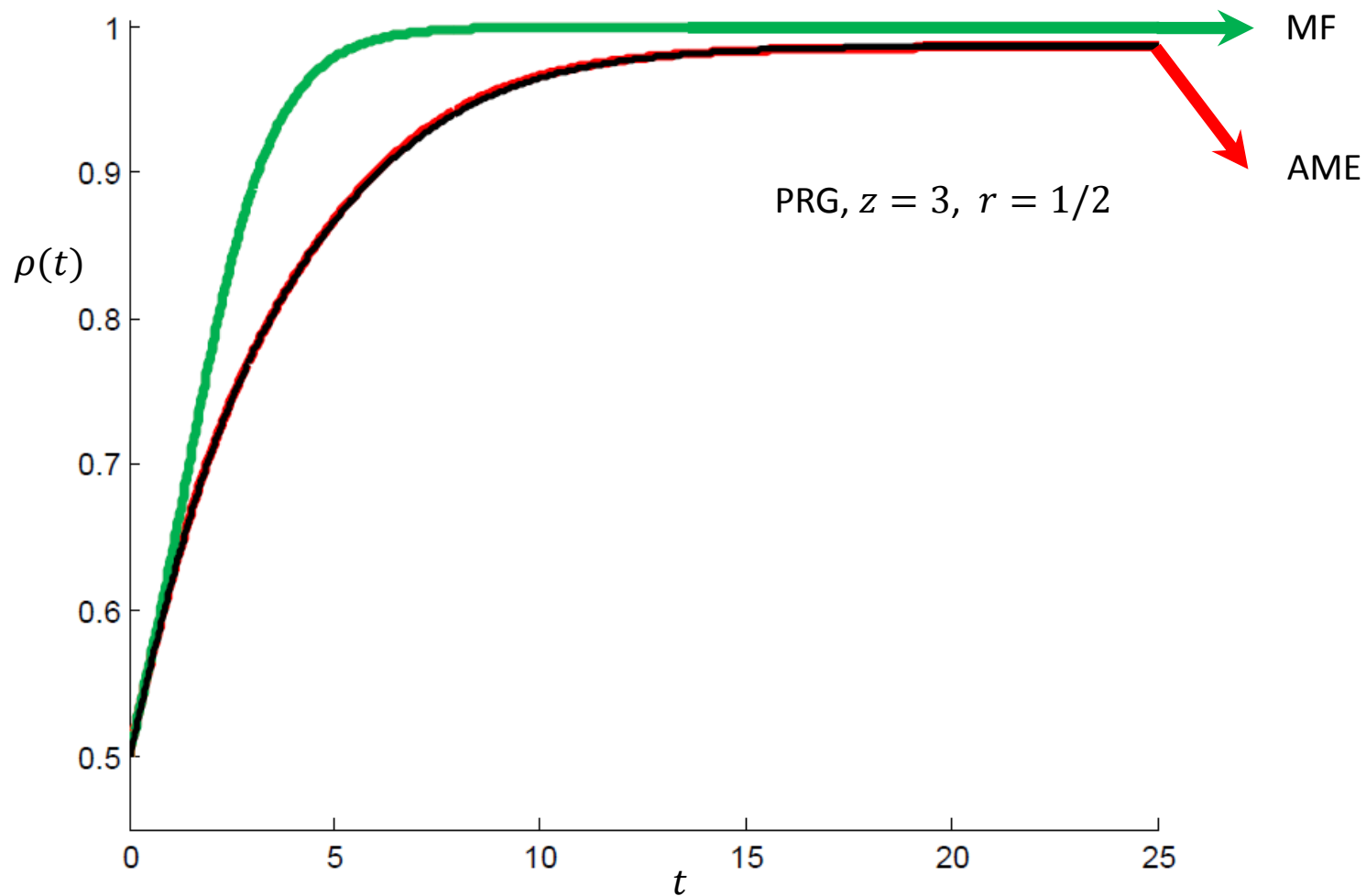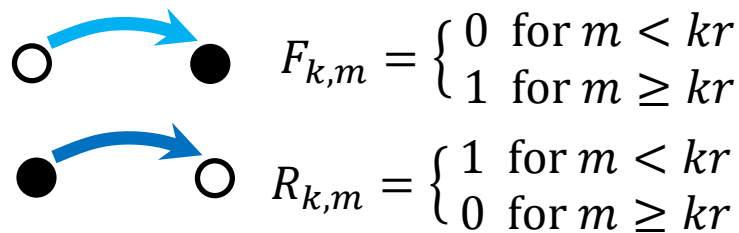$$\beta^i = \frac{\sum P_k \sum_{m=0}^{k}mF_{k,m}s_{k,m}}{\sum P_k \sum_{m=0}^{k}m\,s_{k,m}}$$

$$\gamma^i = \frac{\sum P_k \sum_{m=0}^{k}mR_{k,m}i_{k,m}}{\sum P_k \sum_{m=0}^{k}m\,i_{k,m}}$$

$$\rho = \sum_k P_k \sum_{m=0}^{k}i_{k,m}$$

Non-monotone threshold model

Poisson degree distribution

$$F_{k,m} = \begin{cases} 0 & \text{for } m < kr \\ 1 & \text{for } m \geq kr \end{cases}$$

$$R_{k,m} = \begin{cases} 1 & \text{for } m < kr \\ 0 & \text{for } m \geq kr \end{cases}$$
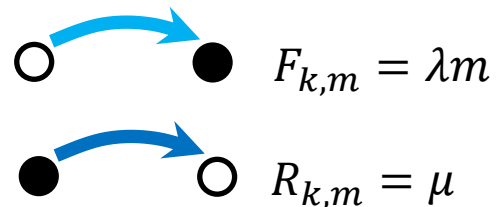

PRG, $z = 3$, $r = 1/2$

MF

AME

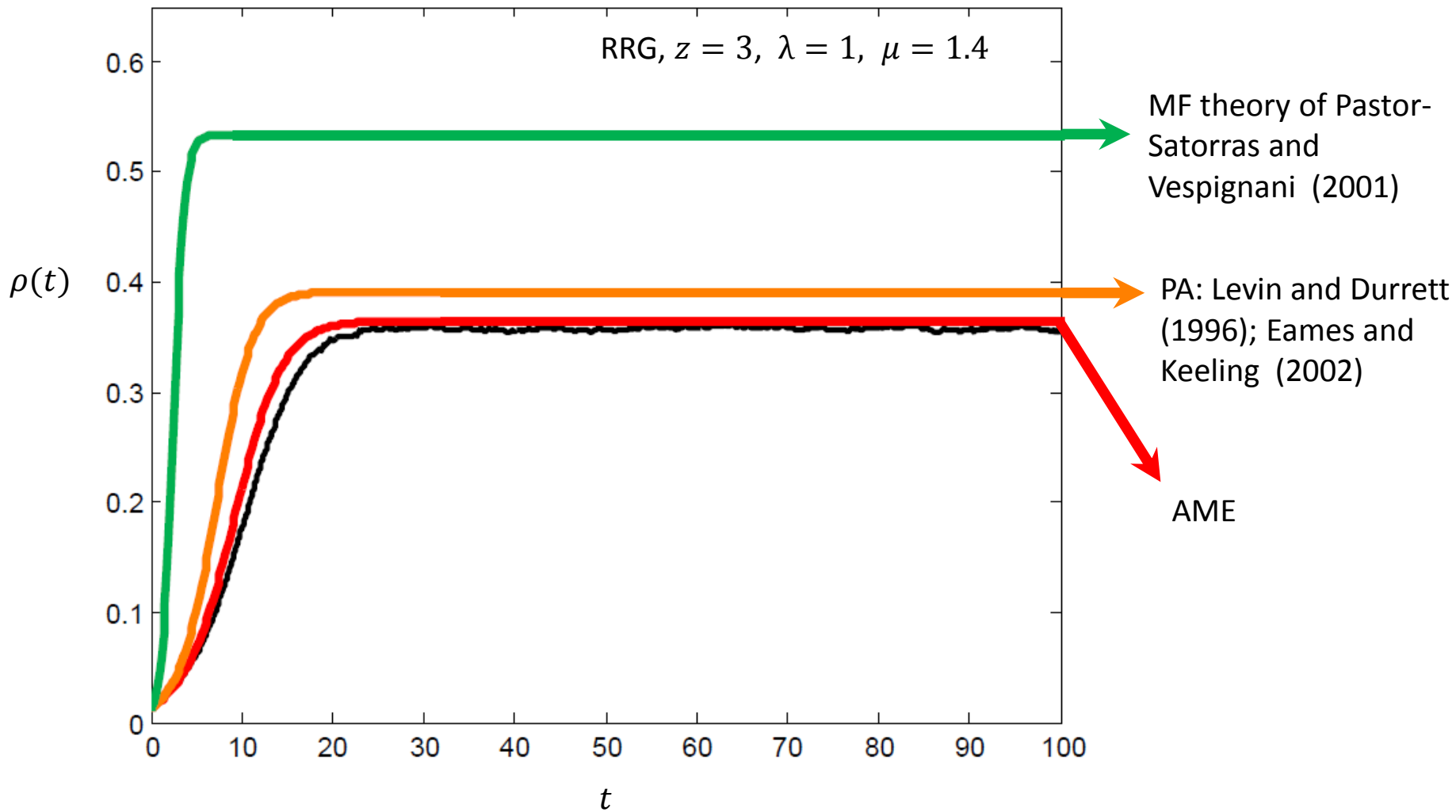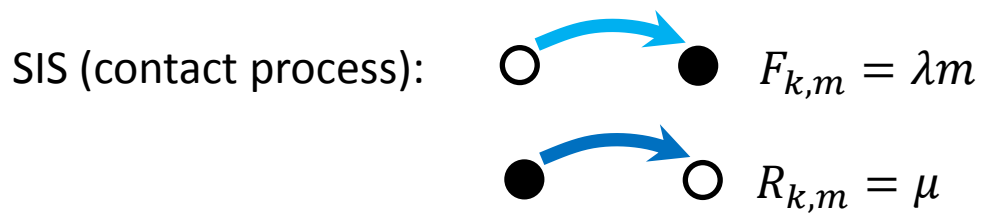**SIS (susceptible-infected-susceptible) model for disease spread**
Each node is either infected or susceptible.
Infected nodes become susceptible at rate μ;
an infected node infects each of its susceptible neighbours at rate λ.

$$F_{k,m} = \lambda m$$

$$R_{k,m} = \mu$$

[cf. Marceau et al, PRE (2010),
Lindquist et al, J. Math. Biol. (2011)]

SIS (contact process): ○ ⟶ ● $F_{k,m} = \lambda m$

● ⟶ ○ $R_{k,m} = \mu$

RRG, $z = 3$, $\lambda = 1$, $\mu = 1.4$

MF theory of Pastor-Satorras and Vespignani (2001)

PA: Levin and Durrett (1996); Eames and Keeling (2002)

AME

SIS (contact process): $F_{k,m} = \lambda m$

$R_{k,m} = \mu$

RRG, $z = 3$, $\lambda = 1$, $\mu = 1.7$

MF theory of Pastor-Satorras and Vespignani (2001)

PA: Levin and Durrett (1996); Eames and Keeling (2002)

AME
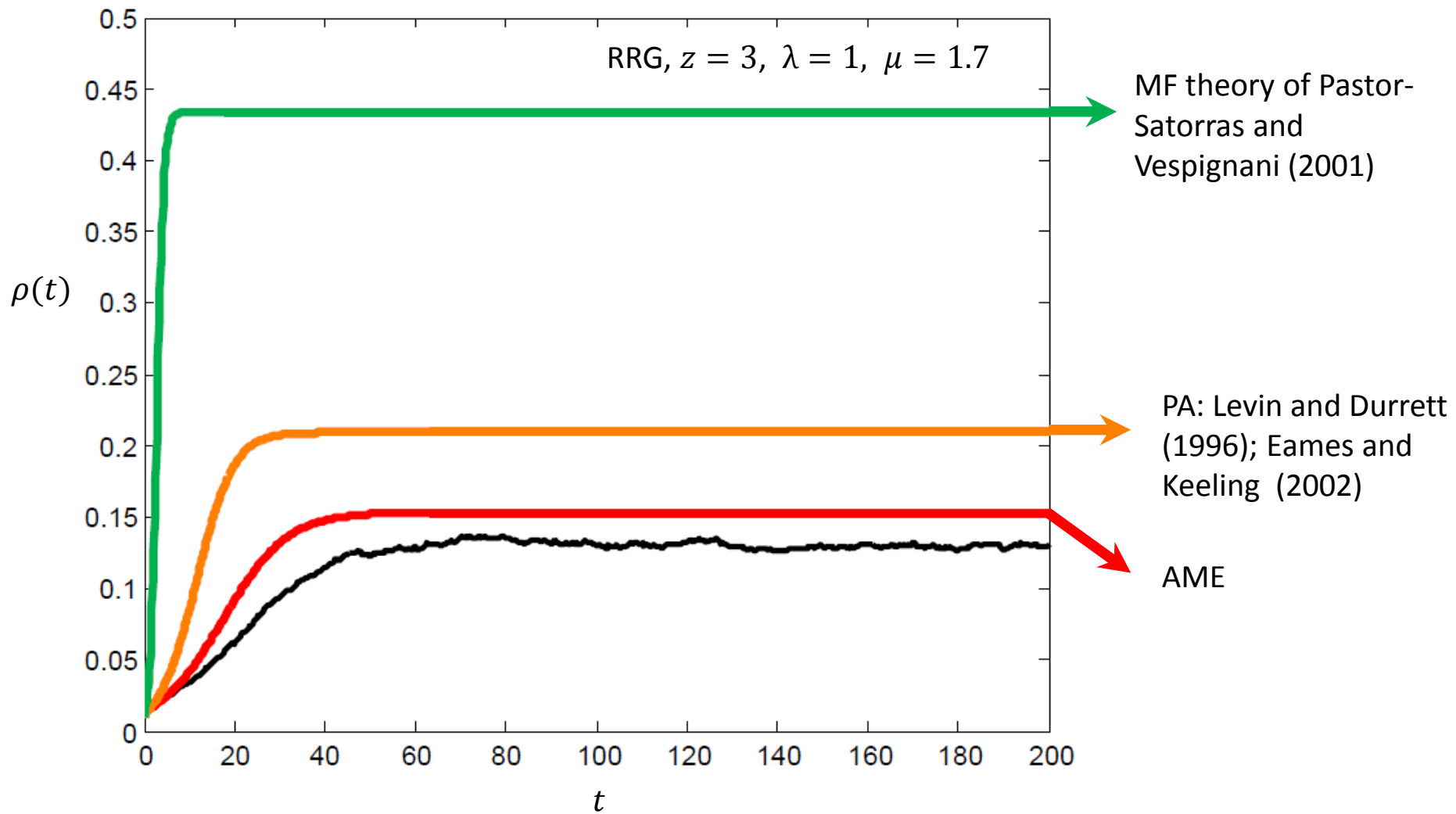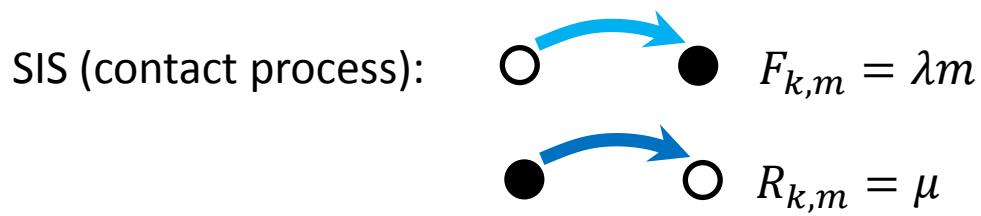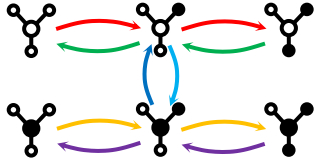
$$\frac{d}{dt}s_{k,m} = -F_{k,m}s_{k,m} + R_{k,m}i_{k,m} - (\gamma^s m + \beta^s(k-m))s_{k,m} + \beta^s(k-m+1)s_{k,m-1} + \gamma^s(m+1)s_{k,m+1}$$

$$\frac{d}{dt}i_{k,m} = -R_{k,m}i_{k,m} + F_{k,m}s_{k,m} - (\gamma^i m + \beta^i(k-m))i_{k,m} + \beta^i(k-m+1)i_{k,m-1} + \gamma^i(m+1)i_{k,m+1}$$

Pair Approximation: using the binomial ansatz

$$s_{k,m}(t) = \left(1-\rho_k(t)\right)B_{k,m}\left(p(t)\right),$$
$$i_{k,m}(t) = \rho_k(t)B_{k,m}\left(q(t)\right),$$

moments of the approximate master equation give equations for $\rho_k(t)$, $q(t)$ and $p(t)$.

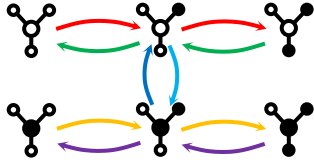Note: in general, this does *not* give an exact solution of the AME.

$$\frac{d}{dt}\rho_k = -\rho_k\sum_m R_{k,m}B_{k,m}(q) + (1-\rho_k)\sum_m F_{k,m}B_{k,m}(p)$$

$$\frac{d}{dt}p = \frac{1}{1-\omega}\sum_k \frac{k}{z}P_k\sum_m\left(1+p-2\frac{m}{k}\right)\left((1-\rho_k)F_{k,m}B_{k,m}(p) - \rho_k R_{k,m}B_{k,m}(q)\right)$$

$$\omega = \sum_k \frac{k}{z}P_k\rho_k \qquad (1-q)\omega = p(1-\omega) \qquad \rho = \sum_k P_k\rho_k$$

Further approximating $p(t)$ and $q(t)$ by $\omega(t)$ gives a Mean Field approximation:

$$\frac{d}{dt}\rho_k = -\rho_k\sum_m R_{k,m}B_{k,m}(\omega) + (1-\rho_k)\sum_m F_{k,m}B_{k,m}(\omega)$$

Number of differential equations, if $P_k \neq 0$ for $k = 0,1,2,\ldots,K$:

$$\frac{d}{dt}s_{k,m} = -F_{k,m}s_{k,m} + R_{k,m}i_{k,m} - (\gamma^s m + \beta^s(k-m))s_{k,m} + \beta^s(k-m+1)s_{k,m-1} + \gamma^s(m+1)s_{k,m+1}$$

$$\frac{d}{dt}i_{k,m} = -R_{k,m}i_{k,m} + F_{k,m}s_{k,m} - (\gamma^i m + \beta^i(k-m))i_{k,m} + \beta^i(k-m+1)i_{k,m-1} + \gamma^i(m+1)i_{k,m+1}$$

$$(K+2)(K+1)$$

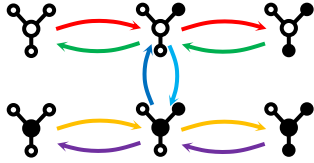$$\frac{d}{dt}\rho_k = -\rho_k \sum_m R_{k,m}B_{k,m}(q) + (1-\rho_k)\sum_m F_{k,m}B_{k,m}(p)$$

$$\frac{d}{dt}p = \frac{1}{1-\omega}\sum_k \frac{k}{z}P_k \sum_m \left(1 + p - 2\frac{m}{k}\right)\left((1-\rho_k)F_{k,m}B_{k,m}(p) - \rho_k R_{k,m}B_{k,m}(q)\right)$$

$$\omega = \sum_k \frac{k}{z}P_k\rho_k \qquad (1-q)\omega = p(1-\omega) \qquad \rho = \sum_k P_k\rho_k$$

$$K+2$$

$$\frac{d}{dt}\rho_k = -\rho_k \sum_m R_{k,m}B_{k,m}(\omega) + (1-\rho_k)\sum_m F_{k,m}B_{k,m}(\omega)$$

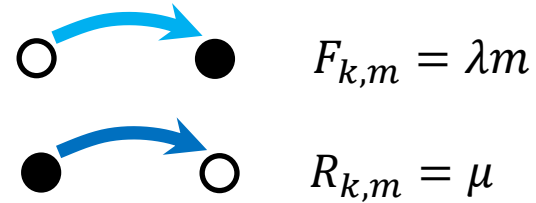$$K+1$$

$$\frac{d}{dt}s_{k,m} = -F_{k,m}s_{k,m} + R_{k,m}i_{k,m} - (\gamma^s m + \beta^s(k-m))s_{k,m} + \beta^s(k-m+1)s_{k,m-1} + \gamma^s(m+1)s_{k,m+1}$$

$$\frac{d}{dt}i_{k,m} = -R_{k,m}i_{k,m} + F_{k,m}s_{k,m} - (\gamma^i m + \beta^i(k-m))i_{k,m} + \beta^i(k-m+1)i_{k,m-1} + \gamma^i(m+1)i_{k,m+1}$$

$$F_{k,m} = \lambda m$$

$$R_{k,m} = \mu$$

$$\frac{d}{dt}\rho_k = -\rho_k \sum_m R_{k,m}B_{k,m}(q) + (1-\rho_k)\sum_m F_{k,m}B_{k,m}(p)$$

$$\frac{d}{dt}p = \frac{1}{1-\omega}\sum_k \frac{k}{z}P_k \sum_m \left(1 + p - 2\frac{m}{k}\right)\left((1-\rho_k)F_{k,m}B_{k,m}(p) - \rho_k R_{k,m}B_{k,m}(q)\right)$$

$$\omega = \sum_k \frac{k}{z}P_k\rho_k \qquad (1-q)\omega = p(1-\omega) \qquad \rho = \sum_k P_k\rho_k$$

$$\frac{d}{dt}\rho_k = -\rho_k \sum_m R_{k,m}B_{k,m}(\omega) + (1-\rho_k)\sum_m F_{k,m}B_{k,m}(\omega)$$

$$\frac{d}{dt}s_{k,m} = -F_{k,m}s_{k,m} + R_{k,m}i_{k,m} - (\gamma^s m + \beta^s(k-m))s_{k,m} + \beta^s(k-m+1)s_{k,m-1} + \gamma^s(m+1)s_{k,m+1}$$

$$\frac{d}{dt}i_{k,m} = -R_{k,m}i_{k,m} + F_{k,m}s_{k,m} - (\gamma^i m + \beta^i(k-m))i_{k,m} + \beta^i(k-m+1)i_{k,m-1} + \gamma^i(m+1)i_{k,m+1}$$
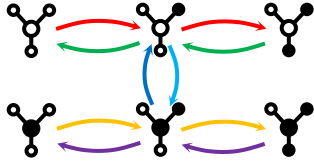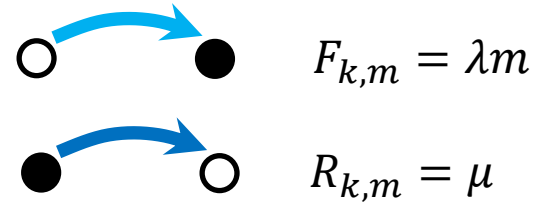
SIS (contact process):

$$F_{k,m} = \lambda m$$

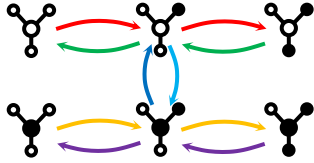$$R_{k,m} = \mu$$

$$\frac{d}{dt}\rho_k = -\mu\rho_k + \lambda(1-\rho_k)kp$$

$$\frac{d}{dt}p = -2\lambda p(1-p) + \frac{1}{1-\omega}[\lambda p(1-p)\omega_2 + \mu(\omega + p\omega - 2p)]$$

$$\omega = \sum_k \frac{k}{z}P_k\rho_k \qquad \omega_2 = \sum_k \frac{k^2}{z}P_k(1-\rho_k)$$

PA of House and Keeling (2010)

$$\frac{d}{dt}\rho_k = -\mu\rho_k + \lambda(1-\rho_k)k\omega$$

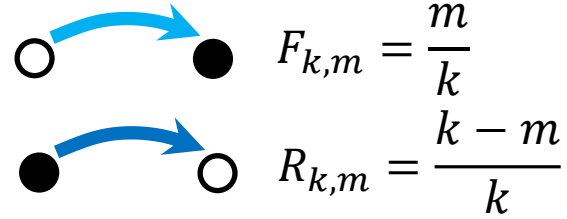MF theory of Pastor-Satorras and Vespignani (2001)

Voter model:

$$F_{k,m} = \frac{m}{k}$$

$$R_{k,m} = \frac{k-m}{k}$$

$$\frac{d}{dt}s_{k,m} = -F_{k,m}s_{k,m} + R_{k,m}i_{k,m} - (\gamma^s m + \beta^s(k-m))s_{k,m} + \beta^s(k-m+1)s_{k,m-1} + \gamma^s(m+1)s_{k,m+1}$$

$$\frac{d}{dt}i_{k,m} = -R_{k,m}i_{k,m} + F_{k,m}s_{k,m} - (\gamma^i m + \beta^i(k-m))i_{k,m} + \beta^i(k-m+1)i_{k,m-1} + \gamma^i(m+1)i_{k,m+1}$$
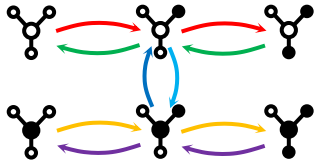
$$\frac{d}{dt}\rho_k = -\rho_k \sum_m R_{k,m}B_{k,m}(q) + (1-\rho_k)\sum_m F_{k,m}B_{k,m}(p)$$

$$\frac{d}{dt}p = \frac{1}{1-\omega}\sum_k \frac{k}{z}P_k \sum_m \left(1+p-2\frac{m}{k}\right)\left((1-\rho_k)F_{k,m}B_{k,m}(p) - \rho_k R_{k,m}B_{k,m}(q)\right)$$

$$\omega = \sum_k \frac{k}{z}P_k\rho_k \qquad (1-q)\omega = p(1-\omega) \qquad \rho = \sum_k P_k\rho_k$$

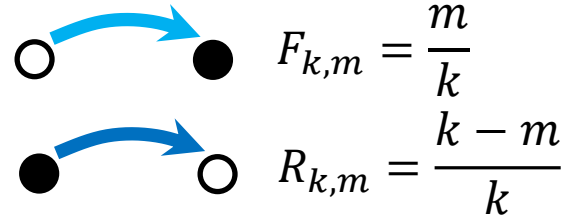$$\frac{d}{dt}\rho_k = -\rho_k \sum_m R_{k,m}B_{k,m}(\omega) + (1-\rho_k)\sum_m F_{k,m}B_{k,m}(\omega)$$

Voter model:

$$F_{k,m} = \frac{m}{k}$$

$$R_{k,m} = \frac{k-m}{k}$$

$$\frac{d}{dt}s_{k,m} = -F_{k,m}s_{k,m} + R_{k,m}i_{k,m} - (\gamma^s m + \beta^s(k-m))s_{k,m} + \beta^s(k-m+1)s_{k,m-1} + \gamma^s(m+1)s_{k,m+1}$$

$$\frac{d}{dt}i_{k,m} = -R_{k,m}i_{k,m} + F_{k,m}s_{k,m} - (\gamma^i m + \beta^i(k-m))i_{k,m} + \beta^i(k-m+1)i_{k,m-1} + \gamma^i(m+1)i_{k,m+1}$$
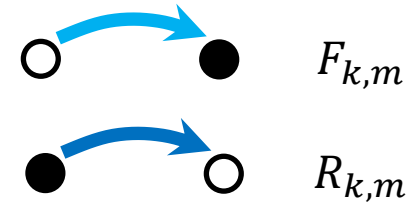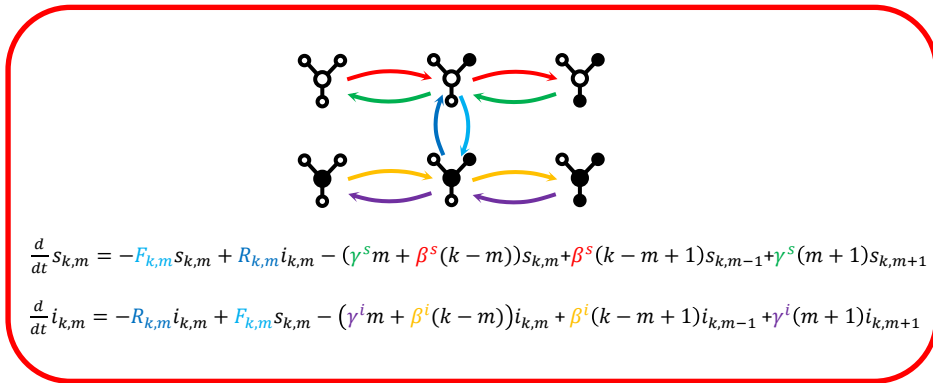
$$\frac{d}{dt}\rho_k = \frac{p}{\omega}(\omega - \rho_k)$$

$$\frac{d}{dt}p = -\frac{2p}{z\omega}(p(z-1) - (z-2)\omega)$$
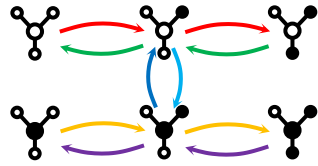
PA of Vazquez and Eguíluz (2008)

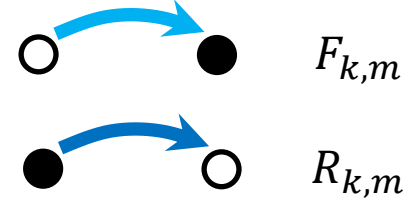$$\frac{d}{dt}\rho_k = -\rho_k + \rho(0)$$

MF theory of Sood and Redner (2005)

$$\frac{d}{dt} s_{k,m} = -F_{k,m}s_{k,m} + R_{k,m}i_{k,m} - (\gamma^s m + \beta^s(k-m))s_{k,m} + \beta^s(k-m+1)s_{k,m-1} + \gamma^s(m+1)s_{k,m+1}$$

$$\frac{d}{dt} i_{k,m} = -R_{k,m}i_{k,m} + F_{k,m}s_{k,m} - (\gamma^i m + \beta^i(k-m))i_{k,m} + \beta^i(k-m+1)i_{k,m-1} + \gamma^i(m+1)i_{k,m+1}$$

$F_{k,m}$

$R_{k,m}$

Other example of binary-state dynamics:

Process	$F_{k,m}$	$R_{k,m}$
SIS	λm	μ
Voter model	m/k	$1 - F_{k,m}$
Glauber dynamics	$[1 + \exp(\frac{2J}{T}(k-2m))]^{-1}$	$1 - F_{k,m}$
Majority-vote	$\begin{cases} Q & \text{if } m < k/2 \\ 1/2 & \text{if } m = k/2 \\ 1-Q & \text{if } m > k/2 \end{cases}$	$1 - F_{k,m}$

$$\frac{d}{dt} s_{k,m} = -F_{k,m} s_{k,m} + R_{k,m} i_{k,m} - (\gamma^s m + \beta^s(k-m)) s_{k,m} + \beta^s(k-m+1) s_{k,m-1} + \gamma^s(m+1) s_{k,m+1}$$

$$\frac{d}{dt} i_{k,m} = -R_{k,m} i_{k,m} + F_{k,m} s_{k,m} - (\gamma^i m + \beta^i(k-m)) i_{k,m} + \beta^i(k-m+1) i_{k,m-1} + \gamma^i(m+1) i_{k,m+1}$$

$F_{k,m}$

$R_{k,m}$

Approximate master equation approach gives high-accuracy approximations for a range of non-monotone binary dynamics (defined by $F_{k,m}$ and $R_{k,m}$).

Moreover, it:

"Automatically" generates pair approximation and mean-field equations.
Enables dynamical systems analysis (e.g. bifurcation theory).
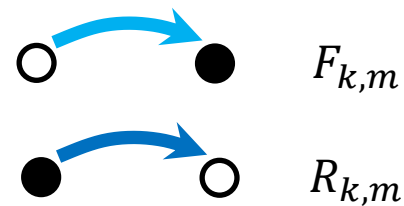Allows extensions to coevolving dynamics and networks.
[Durrett et al. (2012)]

PRL 107, 068701 (2011)

PNAS 109, 3682 (2012)

$$\frac{d}{dt} \rho_k = -\rho_k \sum_m R_{k,m} B_{k,m}(q) + (1-\rho_k) \sum_m F_{k,m} B_{k,m}(p)$$

$$\frac{d}{dt} p = \frac{1}{1-\omega} \sum_k \frac{k}{z} P_k \sum_m \left(1 + p - 2\frac{m}{k}\right) \left((1-\rho_k) F_{k,m} B_{k,m}(p) - \rho_k R_{k,m} B_{k,m}(q)\right)$$

$$\omega = \sum_k \frac{k}{z} P_k \rho_k \qquad (1-q)\omega = p(1-\omega) \qquad \rho = \sum_k P_k \rho_k$$

Further results (in progress)

$F_{k,m}$

$R_{k,m}$

- Matlab m-files for solving the approximate master equations, pair approximation, and mean-field theory equations for given P_k, $F_{k,m}$ and $R_{k,m}$:
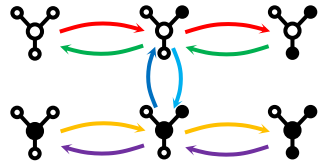
  now available to download from www.ul.ie/gleesonj

- Pair approximation solutions and master equation solutions are identical for all time if:

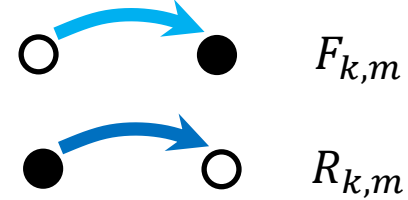$$R_{k,m} = 0 \quad \text{and} \quad F_{k,m} = A(k) + B(k)m$$

  e.g., SI disease-spread model ($A = 0$). Note B may be negative...

- Spin systems: pair approximation solutions and master equation solutions are identical in the limit $t \to \infty$ for Ising model Glauber dynamics, but not for other (non-equilibrium) spin systems.

$$\frac{d}{dt}s_{k,m} = -F_{k,m}s_{k,m} + R_{k,m}i_{k,m} - (\gamma^s m + \beta^s(k-m))s_{k,m} + \beta^s(k-m+1)s_{k,m-1} + \gamma^s(m+1)s_{k,m+1}$$

$$\frac{d}{dt}i_{k,m} = -R_{k,m}i_{k,m} + F_{k,m}s_{k,m} - (\gamma^i m + \beta^i(k-m))i_{k,m} + \beta^i(k-m+1)i_{k,m-1} + \gamma^i(m+1)i_{k,m+1}$$

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$$\frac{d}{dt}\rho_k = -\rho_k \sum_m R_{k,m}B_{k,m}(q) + (1-\rho_k)\sum_m F_{k,m}B_{k,m}(p)$$

$$\frac{d}{dt}p = \frac{1}{1-\omega}\sum_k \frac{k}{z}P_k \sum_m \left(1+p-2\frac{m}{k}\right)\left((1-\rho_k)F_{k,m}B_{k,m}(p) - \rho_k R_{k,m}B_{k,m}(q)\right)$$

$$\omega = \sum_k \frac{k}{z}P_k\rho_k \qquad (1-q)\omega = p(1-\omega) \qquad \rho = \sum_k P_k\rho_k$$

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- Mason Porter, Oxford
- Peter Mucha, U. North Carolina
- Rick Durrett, Duke

- Science Foundation Ireland
- MACSI: Mathematics Applications Consortium for Science & Industry
- IRCSET Inspire

Seeking PhD students and (soon) postdoctoral researchers: see www.ul.ie/gleesonj

Beyond mean-field theory: High-accuracy approximation of binary-state dynamics on networks

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