Graphical degree sequences and realizations

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MAPCON'12
MPIPKS - Dresden,
May 15, 2012

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Joint work with Zoltán Király and István Miklós

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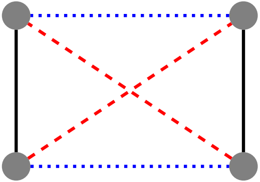

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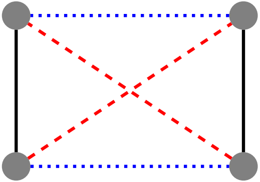

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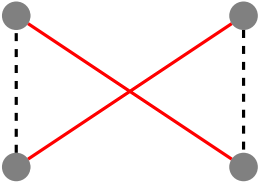

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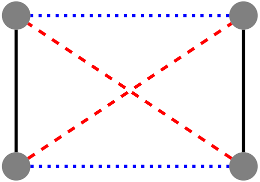

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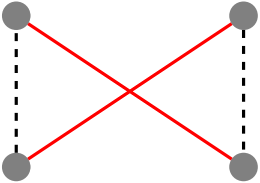

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Theorem (Havel's Lemma, 1955)

If $H \subset V \setminus \{v\}$ and $|H| = |N_G(v)|$ and $N_G(v) \preceq H$ then there exists realization G' such that $N_{G'}(v) = H$.

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Transforming one realization into an other one

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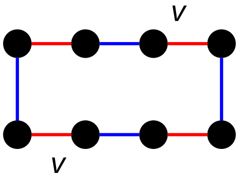

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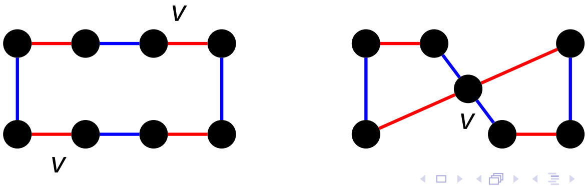

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Let $C_1, \ldots, C_\ell$ be a max. size circuit decomposition of G.
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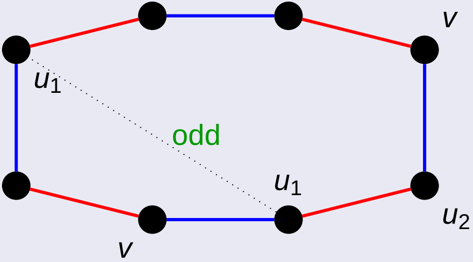

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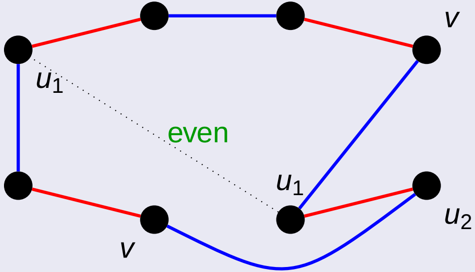

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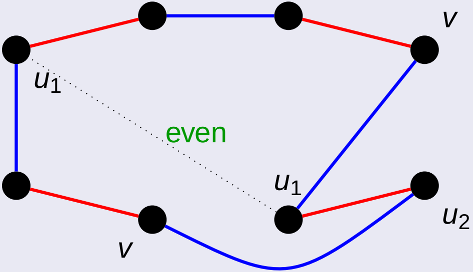

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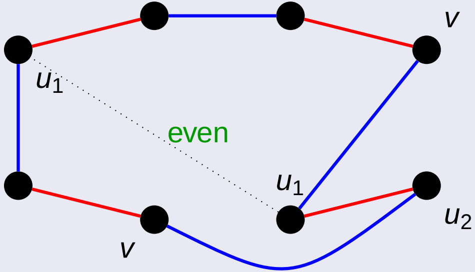

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Realizations G_1 and G_2 of $\mathbf{d}$,

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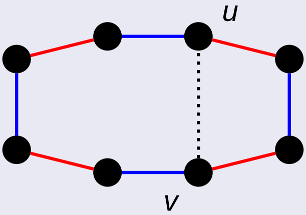

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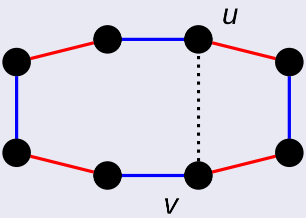

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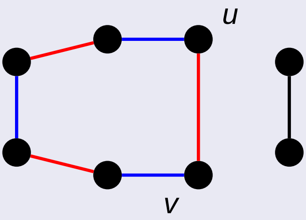

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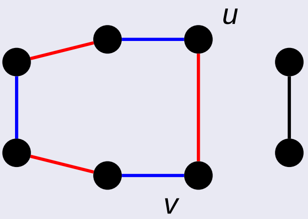

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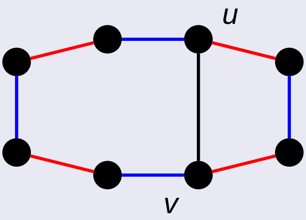

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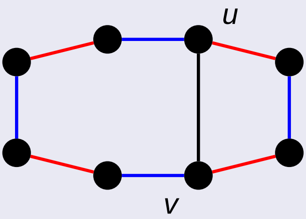

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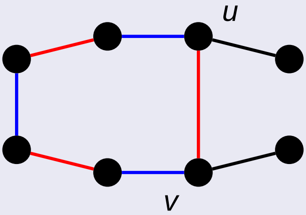

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New upper bound:

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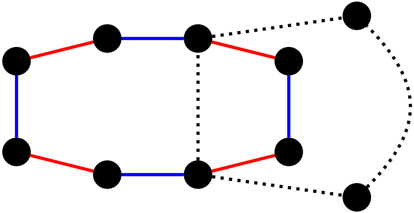

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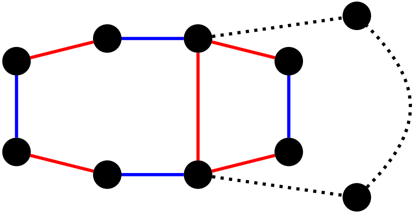

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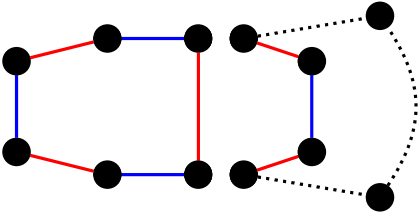

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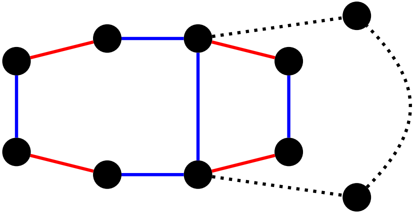

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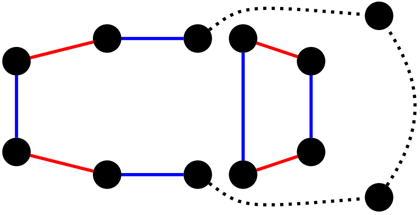

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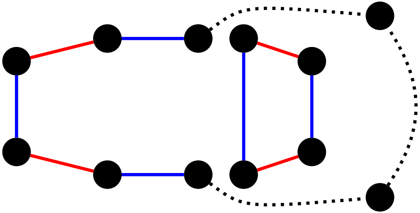

$\#$ of circuits unchanged, $\exists$ shorter circuit - contradiction

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1. consider the (actual) symmetric difference,
2. find a maximal circuit decomposition with a shortest elementary circuit,
3. apply the procedure of one elementary circuit,
4. repeat the whole process with the new (and smaller) symmetric difference.

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Swap-distances
P.L. Erdös
Definitions and History
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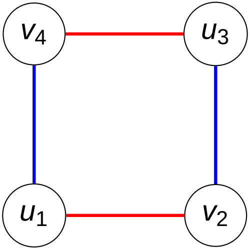

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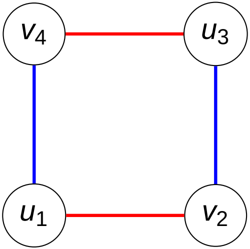

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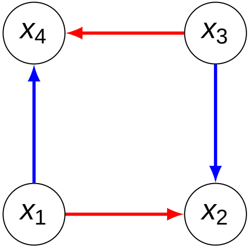

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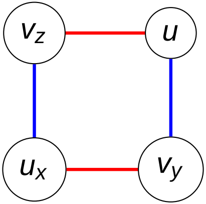

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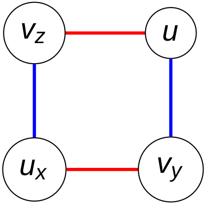

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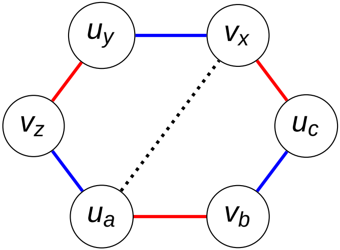

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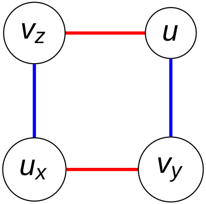

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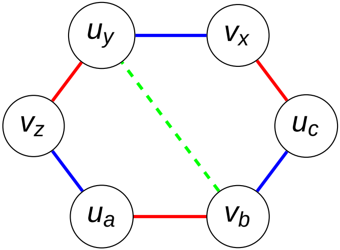

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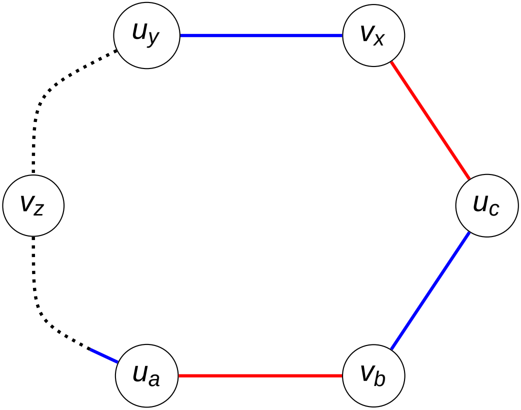

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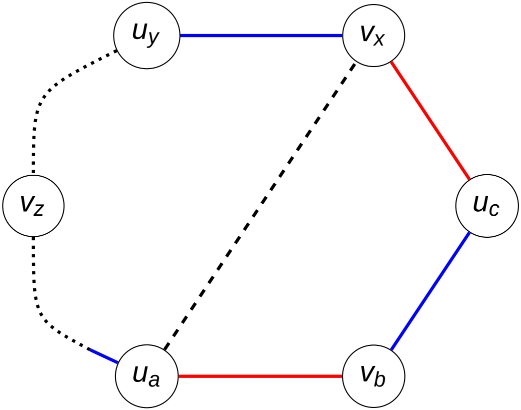

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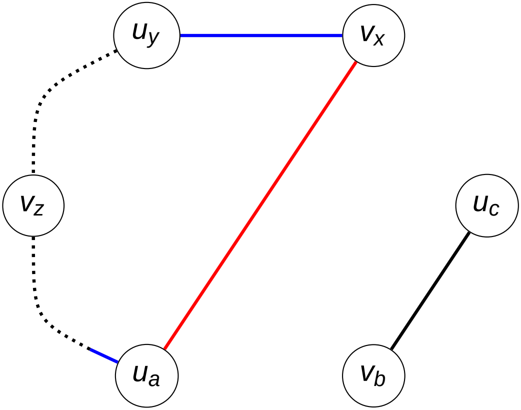

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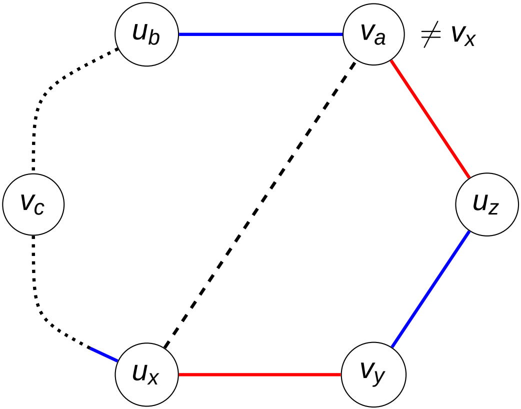

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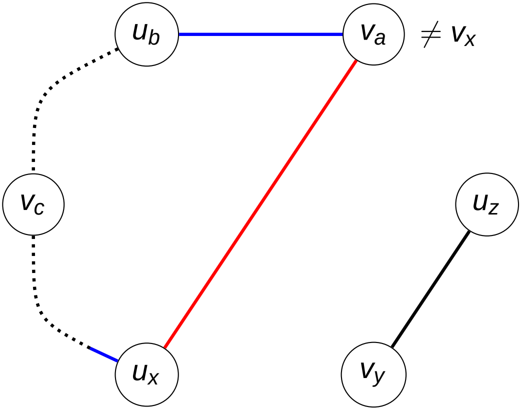

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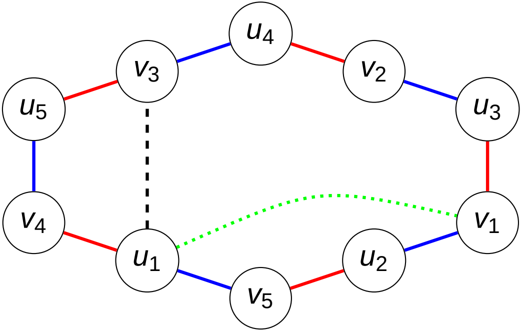

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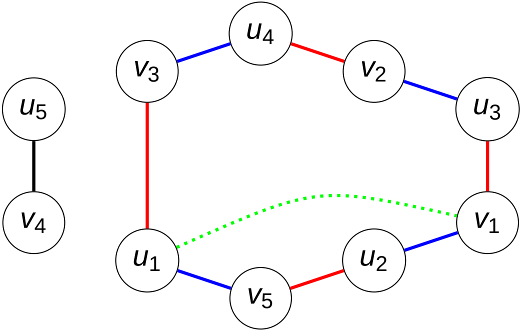

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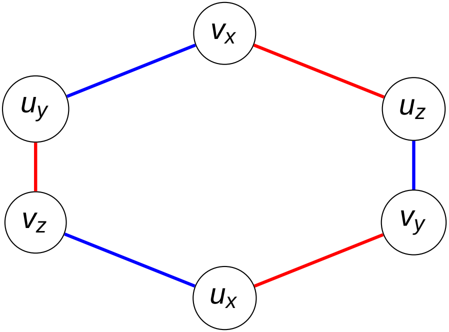

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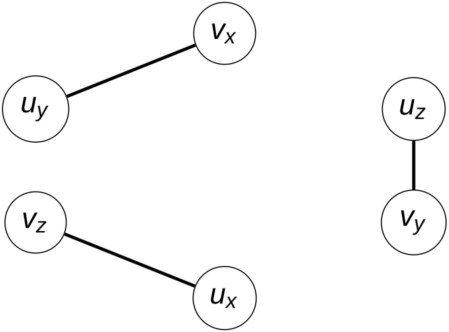

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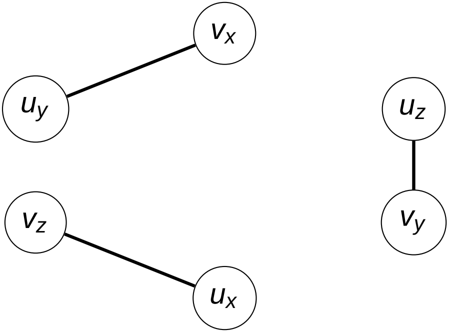

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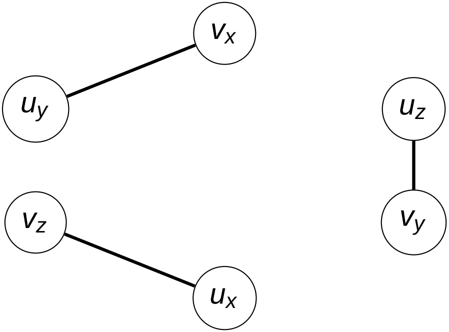

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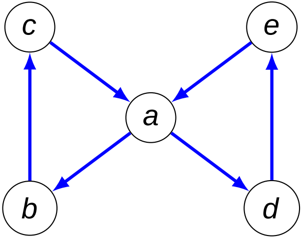
triangular C_6-swaps can be necessary

Analyzing triangular C_6



$\vec{G}_1$ and $\vec{G}_1$ on common set of vertices

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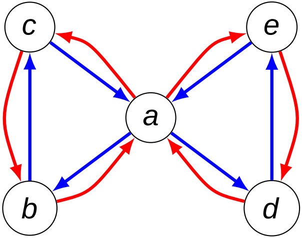

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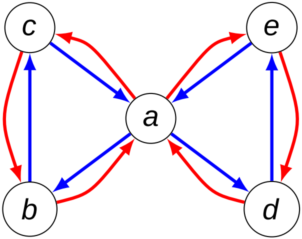

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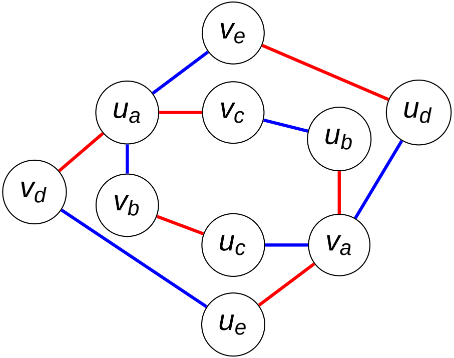

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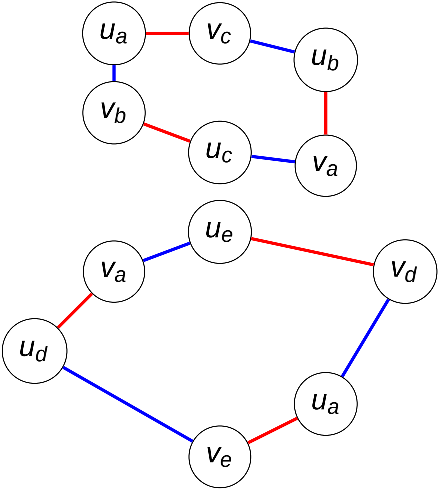

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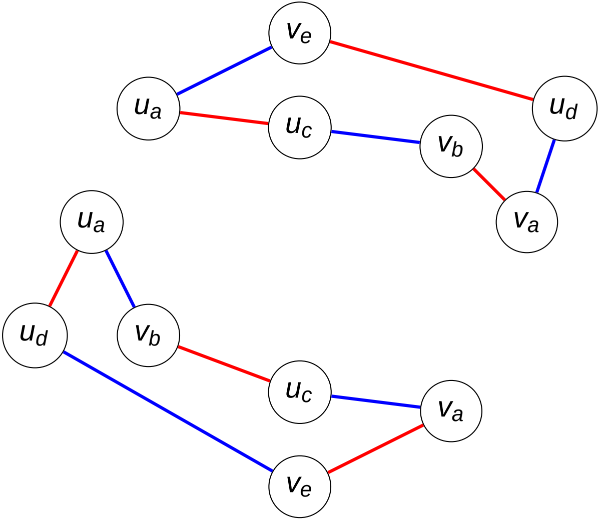

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Theorem

*Let **dd** be a directed degree sequence with $\vec{G}_1$ and $\vec{G}_2$ realizations. Then*

$$\mathbf{dist}_d(\vec{G}_1, \vec{G}_2) = \frac{|E_1 \Delta E_2|}{2} - \mathbf{maxC}_d(G_1, G_2).$$

M.Drew LaMar's result

Directed 3-Cycle Anchored Digraphs And Their Application In The Uniform Sampling Of Realizations From A Fixed Degree Sequence, in *ACM 2011 Winter Simulation Conference* (2011), 1–12.

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Catherine Greenhill's result

A polynomial bound on the mixing time of a Markov chain for sampling regular directed graphs, *arXiv* **1105.0457v4** (2011), 1–48.

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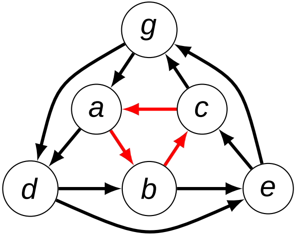

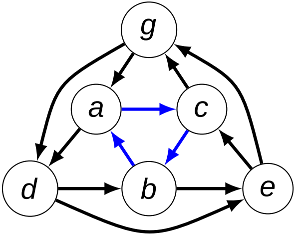

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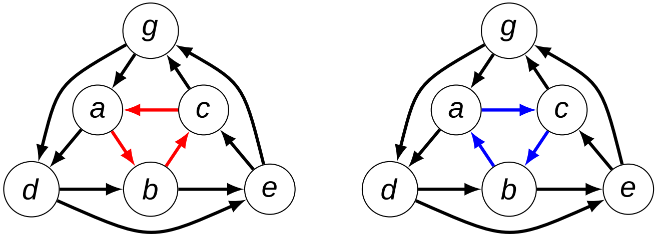

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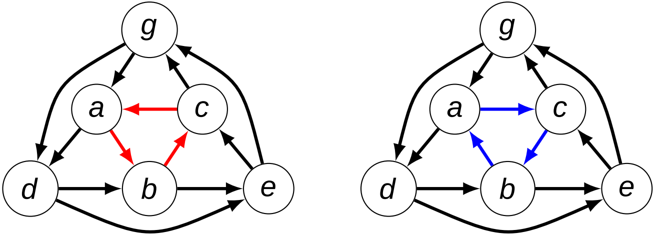

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of course this was never a requirement