Topological stability criteria for collective dynamics in complex networks

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Structure dependent dynamics

- Collective dynamics of coupled dynamical units are often intimately related to the structure of the coupling network.
- Statistical methods have revealed that structural properties on all scales from individual nodes to the entire network may play a role.

Paradigmatic example: Synchronization of phase-oscillators

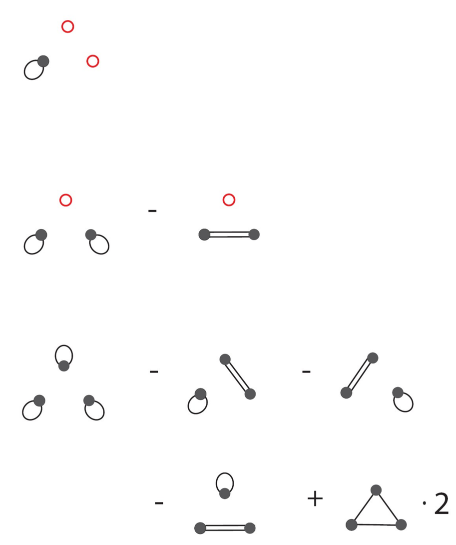
J contains information about Dynamics *and* Structure

- e.g., heterogeneous Kuramoto model $\dot{x}_i = \omega_i + \sum_j A_{ij} \cdot \sin(x_j - x_i)$
- Stability of phase locked states determined by spectrum of Jacobian matrix $\mathbf{J} \in \mathbb{R}^{N \times N}$: $J_{ik} = \partial \dot{x}_i / \partial x_k$ = 0 if x_i, x_k not coupled

$D_{1,S} = J_{11}$
$= \times$

$D_{2,S} = J_{11} J_{22} - J_{12}^2$
$= \times \cdot \times - |$

$D_{3,S} = J_{11} J_{22} J_{33} - (J_{11} J_{23}^2 + J_{22} J_{13}^2 + J_{33} J_{12}^2)$
$+ 2 \cdot J_{12} J_{23} J_{13}$
$= \times \cdot \times \cdot \times - \times \cdot | + 2 \cdot \triangle$

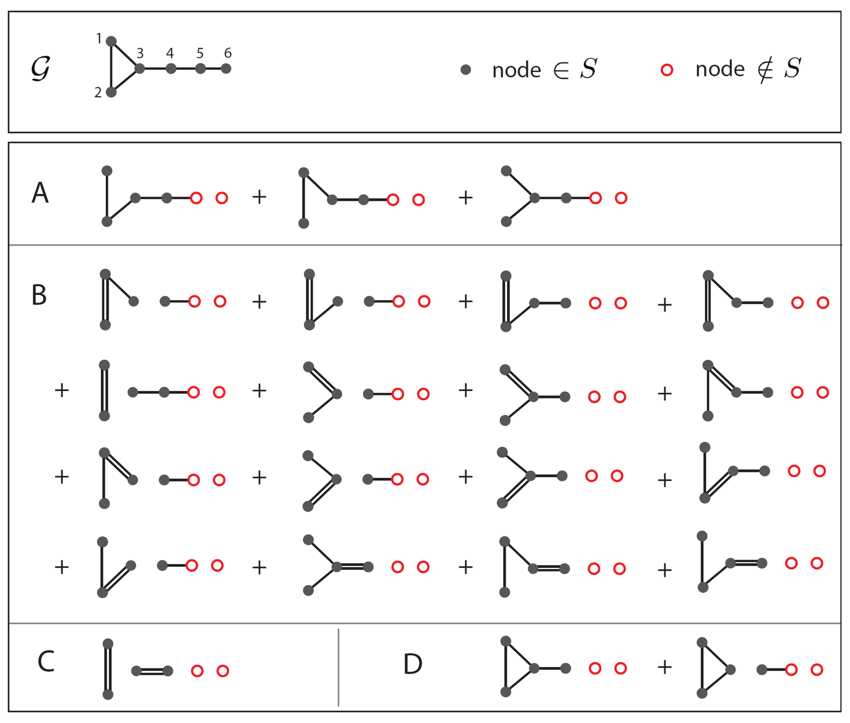


$$\mathbf{J} = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{12} & J_{22} & J_{23} \\ J_{13} & J_{23} & J_{33} \end{pmatrix}$$

Minors of the matrix **J** in algebraic and graphical notation with corresponding subgraphs of the three-node graph *G*.

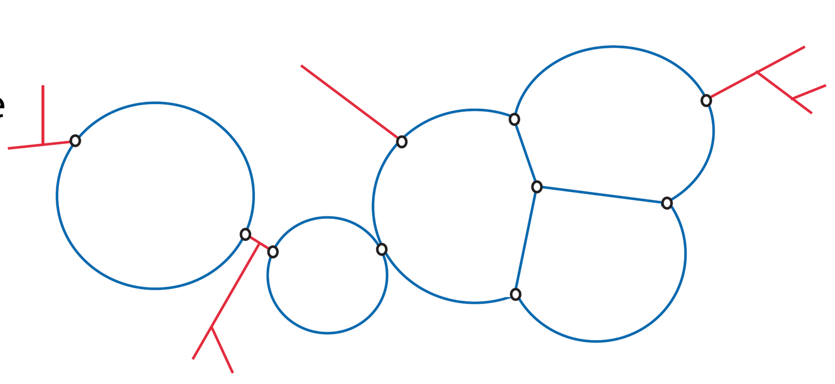
$D_{1,S} = \times$
$D_{2,S} = \times \cdot \times - |$
$D_{3,S} = \times \cdot \times \cdot \times - \times \cdot | + 2\triangle$
$D_{4,S} = \times \cdot \times \cdot \times \cdot \times - \times \cdot \times \cdot | + | \cdot | + 2 \times \cdot \triangle - 2\square$

Graphical notation of minors. The minor of order |*S*|=7 has 11 terms in the graphical, but 720 in the algebraic notation.



Consider the minor D_4 of the graph *G* sketched alongside. If *S* is chosen as the set of vertices plotted in grey, the terms can be written as
$\times \cdot \times \cdot \times \cdot \times = J_{11} J_{22} J_{33} J_{44} = (-1)^4 (J_{12}+J_{13})(J_{21}+J_{23})(J_{31}+J_{32}+J_{34})(J_{43}+J_{45}) =$ A+B+C+2D, $-\times \cdot \times \cdot | = -(B+2C)$, $| \cdot | = C$, $2 \times \cdot \triangle = -2D$ and $-2\square = 0$.
We can immediately read off that $D_4 = A = \Phi_S$,

Decomposition of a graph *G* in acyclic parts (red), segments of cycles (blue) and branching points (open black). Lines represent paths of *G* that may contain several vertices. Stability requires that (i) the acyclic parts only contain links with positive weights; (ii) any unbranched segment of a cycle contains at most one link with negative weight; (iii) the number of links with negative weight does not exceed the number of independent cycles (here, 5).



Jacobi's Signature Criterion

Stability conditions on minors of J

A symmetric matrix **J** $\in \mathbb{R}^{N \times N}$ is negative semi-definite iff all its principal minors $D_{|S|} = \det(J_{ik})$, i,k ∈ S, S ⊂ {1,...,N} satisfy sign ($D_{|S|}$) = $(-1)^{|S|}$

Topological Interpretation

Stability conditions on subgraphs of *G*

Interprete **J** as adjacency matrix of a graph *G*

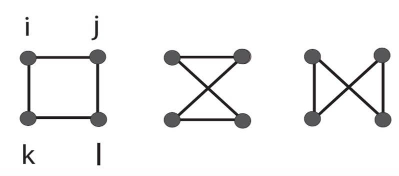
- Entry J_{ij} ⟶ Weight of a link *ij*
- Term $J_{ij} J_{jk}$ ⟶ Subgraph of *G* spanned by the links *ij* and *jk*
- Minor $D_{|S|}$ ⟶ Sum over subgraphs with |*S*| links

Graphical notation and calculus

Handling of large |S|

- Basis of symbols denoting cycles of lengths *m = 1,2,3,4,5...*: $\times, |, \triangle, \square, ⬠, \ldots$
- Summation convention: symbol in $D_{|S|}$ denotes the sum over all non-equivalent possibilities to build the depicted subgraph with the nodes in *S*

S = { i, j, k, l } $\square := J_{ij} J_{jl} J_{lk} J_{ki} + J_{ij} J_{jk} J_{kl} J_{li} + J_{ik} J_{kj} J_{jl} J_{li}$



Zero row sum

Generalization of Kirchhoff 's Theorem

In many systems, including the Kuramoto model, **J** has zero row sums (force balance along links). In these systems

$D_{|S|} = (-1)^{|S|} \Sigma$ all acyclic subgraphs of *G* with |S| links and no two nodes $\notin S$ in the same component
$=: (-1)^{|S|} \Phi_S$

Necessary topological stability criteria

Stuctural properties necessary for stability

Stability requires that $\Phi_S > 0$ for all *S*. We can show that this restricts the number, position and strength of potential links with negative weight: Every component of *G* has to have a spanning tree of links with positive weight. Moreover, negative link weights are subject to a topology dependent lower bound.

Summary

- The topological interpretation of Jacobi's Signature Criterion allows to study the interplay of structure and dynamics in complex networks.
- The derived topological stability criteria pertain to structures on all scales from single nodes to the entire network. They apply to all systems with symmetric or Hermitian Jacobian, i.e., in particular to networks of symmetrically coupled units.

Do, Boccaletti, Gross, *Phys. Rev. Lett.* **108**, 194102 (2012)