All scale-free networks are sparse

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Complex networks

Roads, power-grids, gene regulation, databases, airline connections, the Internet, epidemics, metabolism: all can be modelled as networks.

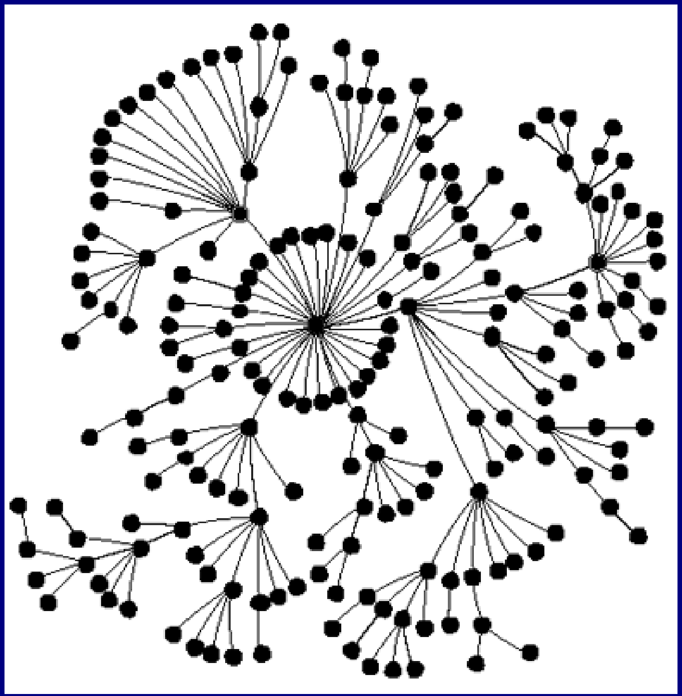
In the last decade, the network model has proved a successful tool in gaining deeper insight into complex systems.

A fundamental characteristic of a complex network is the distribution of the degree of its nodes $P(k)$, that is the probability of having a node with k edges.

Scale-free networks

Considerable attention has been paid to scale-free networks, in which the degree distribution follows a power-law $k^{-\gamma}$.

To study scale-free networks, many generative models have been proposed, but no models creating networks with $\gamma \leq 2$ have been found.



Also, $\gamma \leq 2$ is only observed in networks that are either small, or in which the degree distribution has some cutoff.

Scale-free networks (cont.)

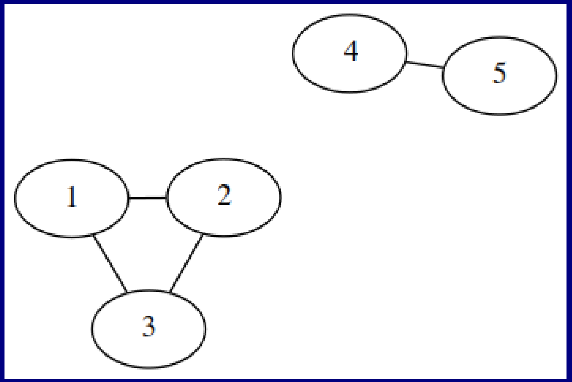
Notably, the mean degree of a scale-free network with $\gamma \leq 2$ would diverge when the number of nodes N becomes infinite.

Thus, in scale-free networks with $\gamma \leq 2$ the number of links would scale like the square of the number of nodes, making the network dense. Networks with $\gamma > 2$ are instead sparse.

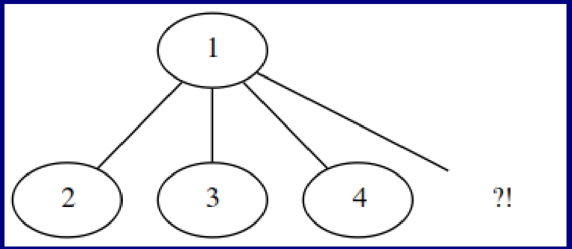
The absence of these scale-free networks is explained by a discontinuous transition in their very realizability.

Graphicality

Not all the sequences of integers can be put in a one-to-one correspondence with the degrees of the nodes of a simple graph. Those that can are said to be graphical.



As an example, take the sequence {2, 2, 2, 1, 1}. It is a graphical sequence because there is at least one simple graph in which the degrees of the nodes are the numbers of the sequence.



Conversely, there are sequences, such as {4, 1, 1, 1}, that are not graphical.

The Erdős-Gallai theorem

A non-increasing sequence of integers $D=\{d_0, d_1, \ldots, d_{N-1}\}$ is graphical if and only if the sum of the elements is even and, for all $0 \leq k < N-1$,

$$L_k = \sum_{i=0}^{k} d_i \leq k(k+1) + \sum_{i=k+1}^{N-1} \min\{k+1, d_i\} = R_k .$$

An intuitive understanding of the theorem is possible by noticing that the terms in the inequalities are the total number of stubs, the stubs in a complete graph of order $k+1$, and the maximum number of extra stubs that can be connected.

A new formulation

An efficient implementation of the theorem is possible by proving recurrence relations for L_k and R_k.

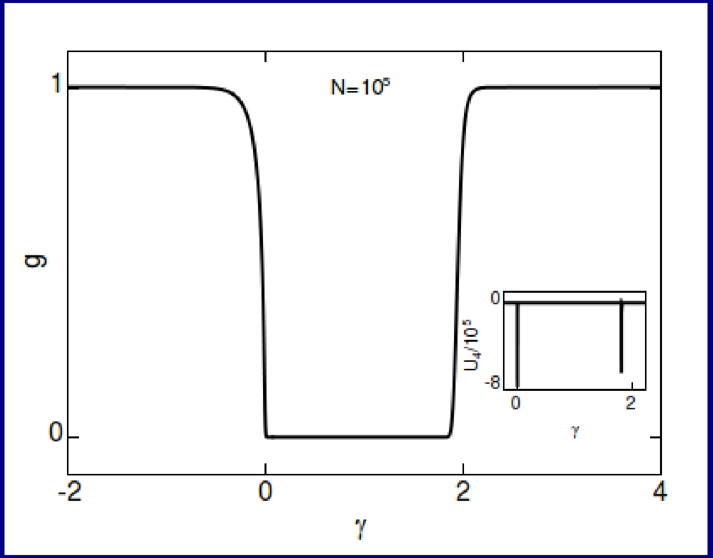
Then, defining $x_k=\min\{i: d_i<k+1\}$, and $k^*=\min\{i: x_i<i+1\}$, L_k and R_k are given by:

$$L_0 = d_0 \qquad R_0 = N-1$$

$$L_k = L_{k-1} + d_k \qquad R_k = \begin{cases} R_{k-1} + x_k - 1 & \forall k < k^* \\ R_{k-1} + 2\mathrm{k} - d_k & \forall k \geqslant k^* \end{cases}$$

Graphicality transitions

The transitions can be observed via numeric simulations, using a fast implementation of the theorem based on the new, recurrence-relation based formulation.



The scaling properties of the largest degrees and of the number of lowest degree nodes in power-law sequences provide a theoretical explanation to the observation.

Scaling properties

To investigate the scaling of the degrees one can use extreme value arguments to find the expression for the expected maximum j^{th} degree of a proper scale-free degree sequence:

$$\hat{d}_j = \max\left\{ x : N \sum_{k=x}^{N-1} \frac{k^{-\gamma}}{H_{N-1,\gamma}} \geqslant j \right\},$$

Where $H_{N\text{-}1,\gamma}$ is the (N-1)$^{\text{th}}$ generalized harmonic number of exponent γ.

Scaling properties (cont.)

The expression can be used to estimate the scaling of the two largest degrees in a sequence when the exponent belongs to different ranges of values. The number of nodes with unitary degree can be easily found from the degree distribution.

Then, exploiting the properties of the new formulation of the Erdős-Gallai theorem, one can directly write the second Erdős-Gallai inequality and study its satisfiability.

$$\left[\frac{(\gamma-1)H_{N-1,\gamma}}{N}+(N-1)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}+\left[\frac{2(\gamma-1)H_{N-1,\gamma}}{N}+(N-1)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}\leq 2\mathrm{N}-2-\frac{N}{H_{N-1,\gamma}}$$

Transition mechanism

Suppose we tried to construct a scale-free network with γ between 0 and 2, by placing the required number of edges sequentially from the highest to the lowest degree node.

Then, $O(N)$ nodes of degree 1 would be used to place the connections involving the first node, and there would be no way to place all the needed edges involving the second node.

Instead, for $\gamma<0$, all but a vanishingly small fraction of nodes have degree of order N, and all the edges can be placed.

Conclusions

In summary, we have found that the graphicality of power-law distributed sequences undergoes two discontinuous transitions at the values 0 and 2 of the exponent γ.

In the limit of a large number of nodes, no network with a power-law degree distribution with $0 \leq \gamma \leq 2$ can exist.

This result arises directly from fundamental mathematical constraints on the node degrees and is independent of the procedure used to generate the networks.

Conclusions

This explains why all the scale-free networks observed in nature have $\gamma>2$, or have a cutoff in their degree distribution.

Thus, all scale-free networks are sparse, either because $\gamma>2$, or because they feature a cutoff.

This result is reassuring, as it implies that numerical methods, often needed in this kind of study, will continue to scale favourably with increasing system size.

Thank you!