# Lessons from Self-Similarity in Complex Networks

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joint work with

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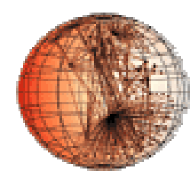

Dmitri Krioukov
CAIDA, UCSD, USA

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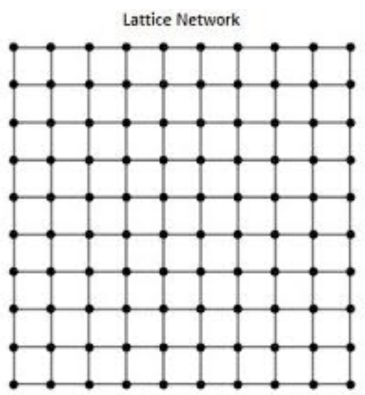


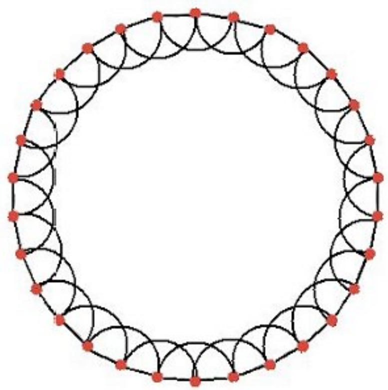

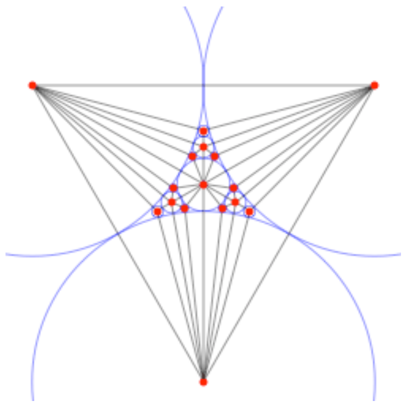

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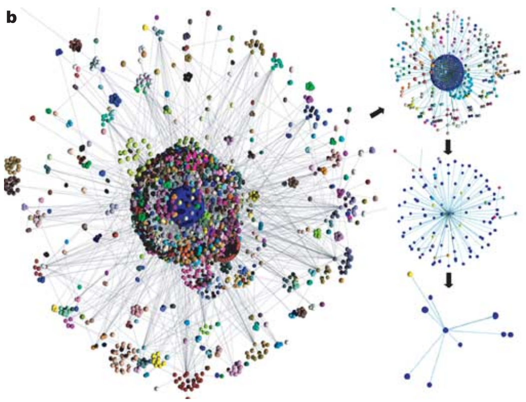
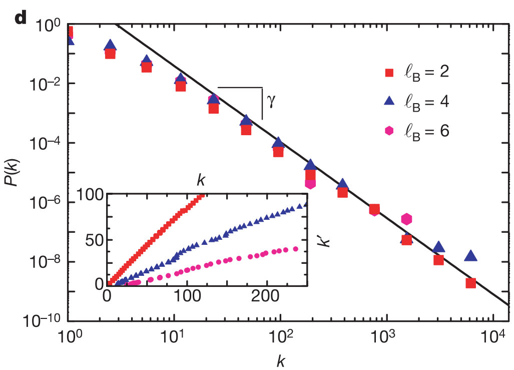
self-similarity




C. Song, S. Havlin, H. A. Makse, Nature 433, 392-395 (2005)
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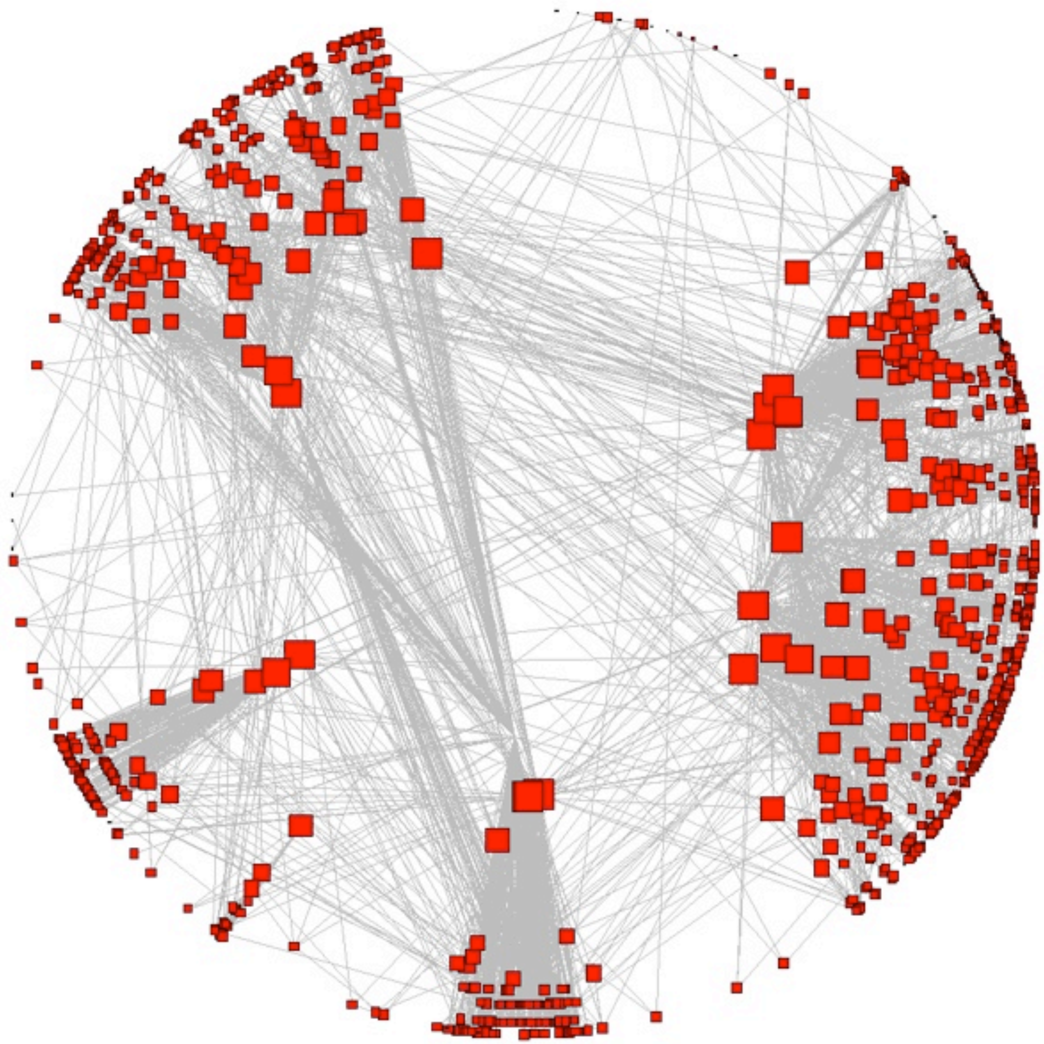
# An alternative approach to self-similarity



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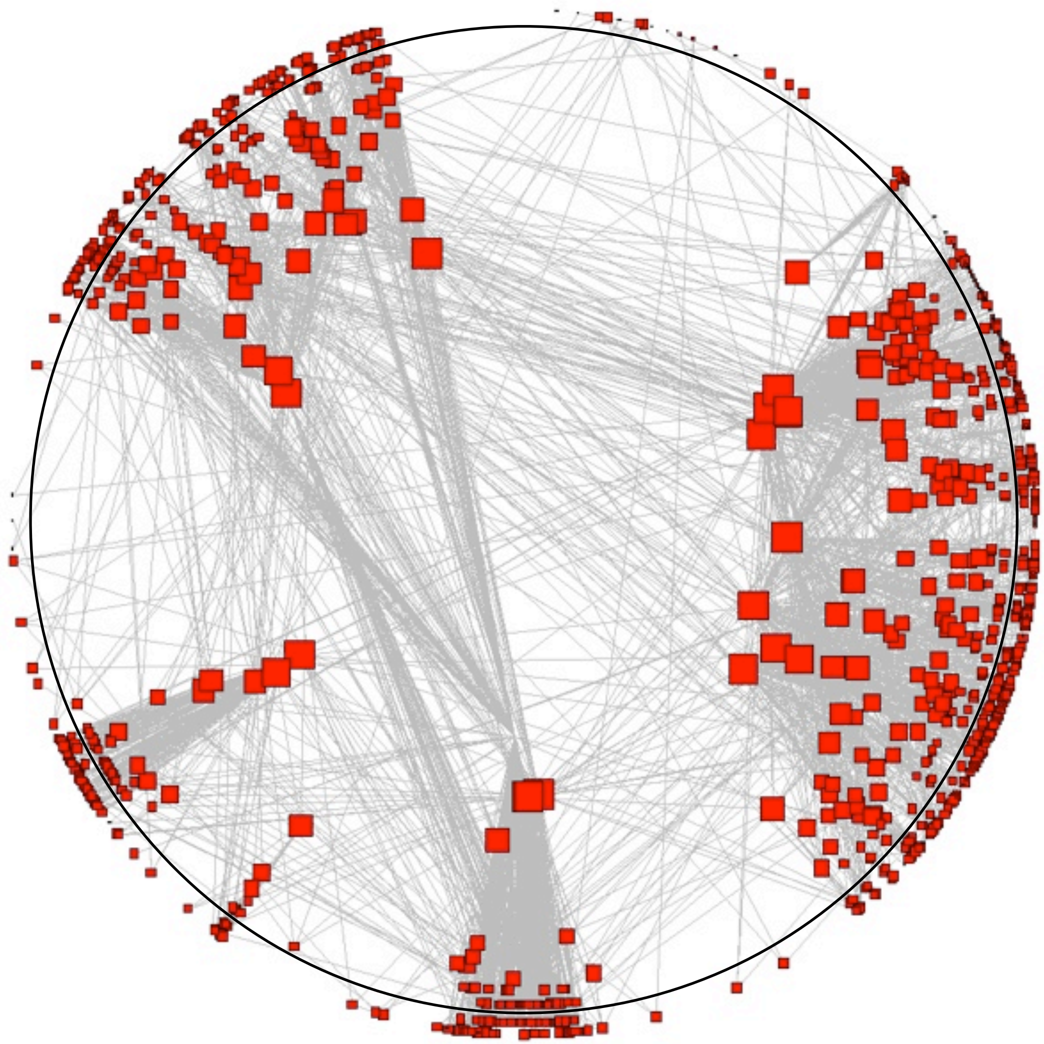

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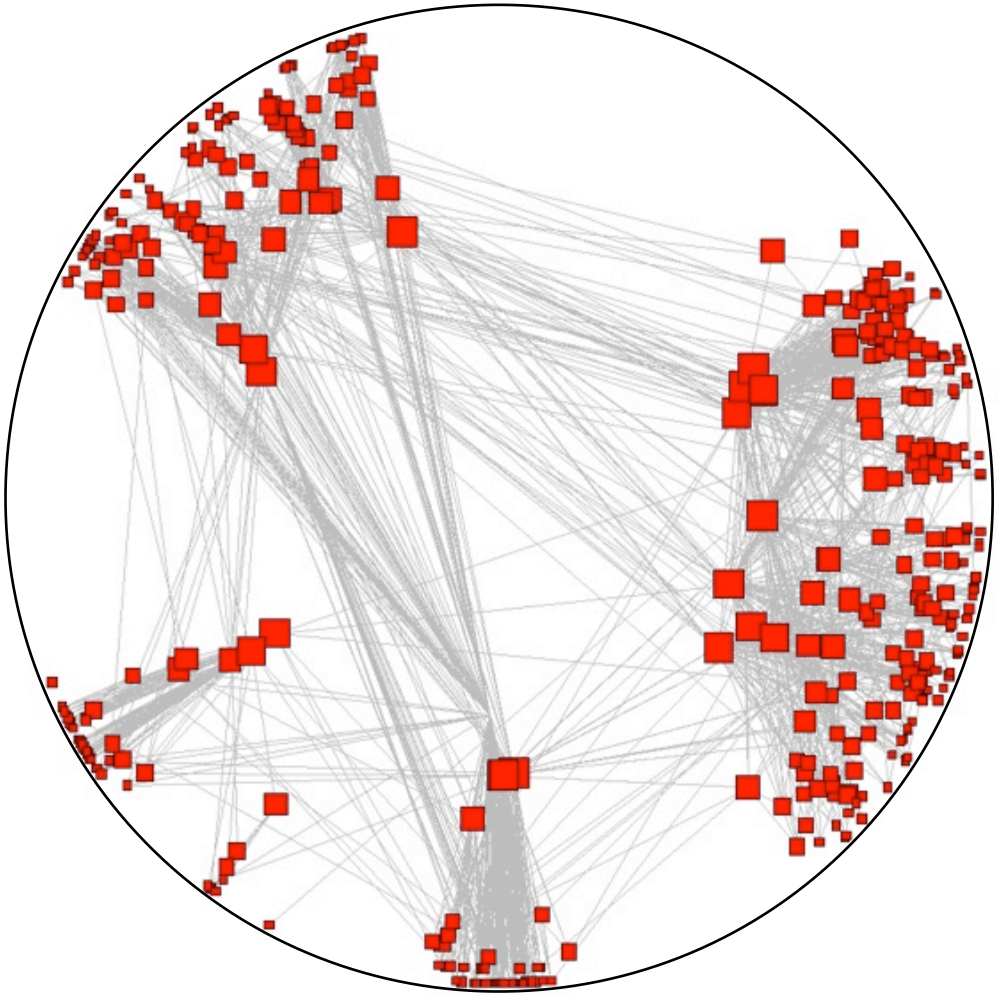

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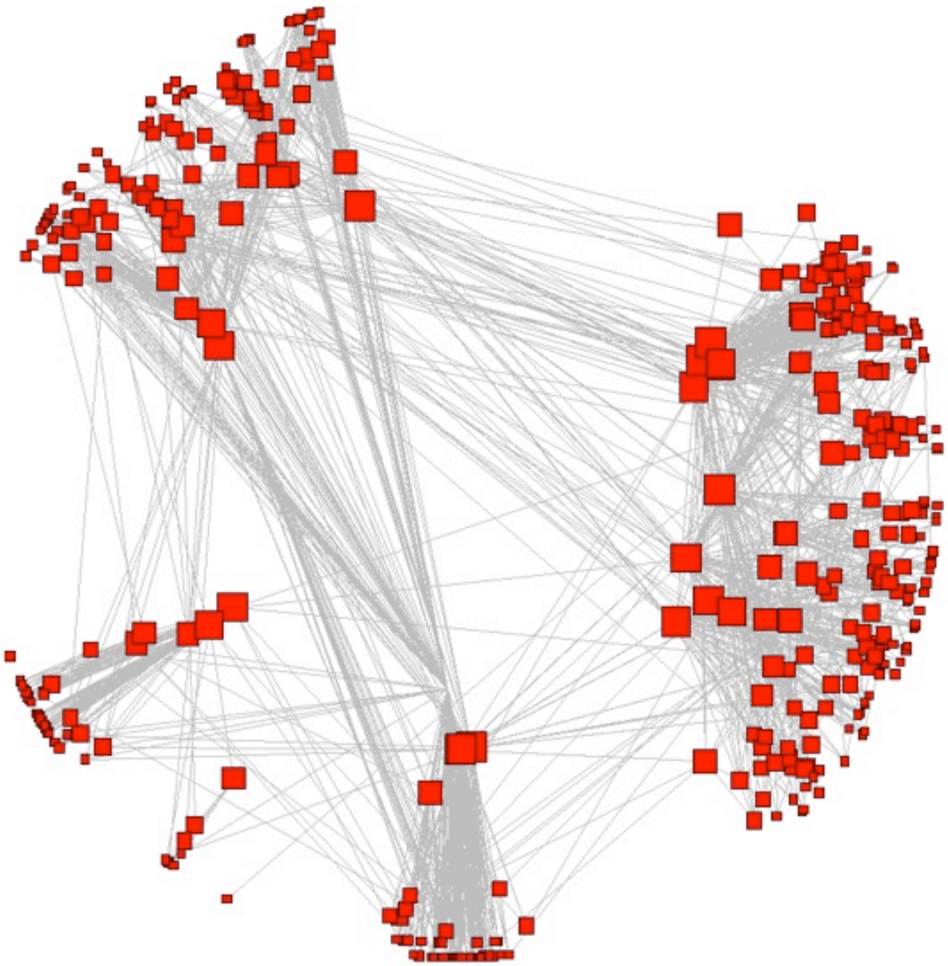

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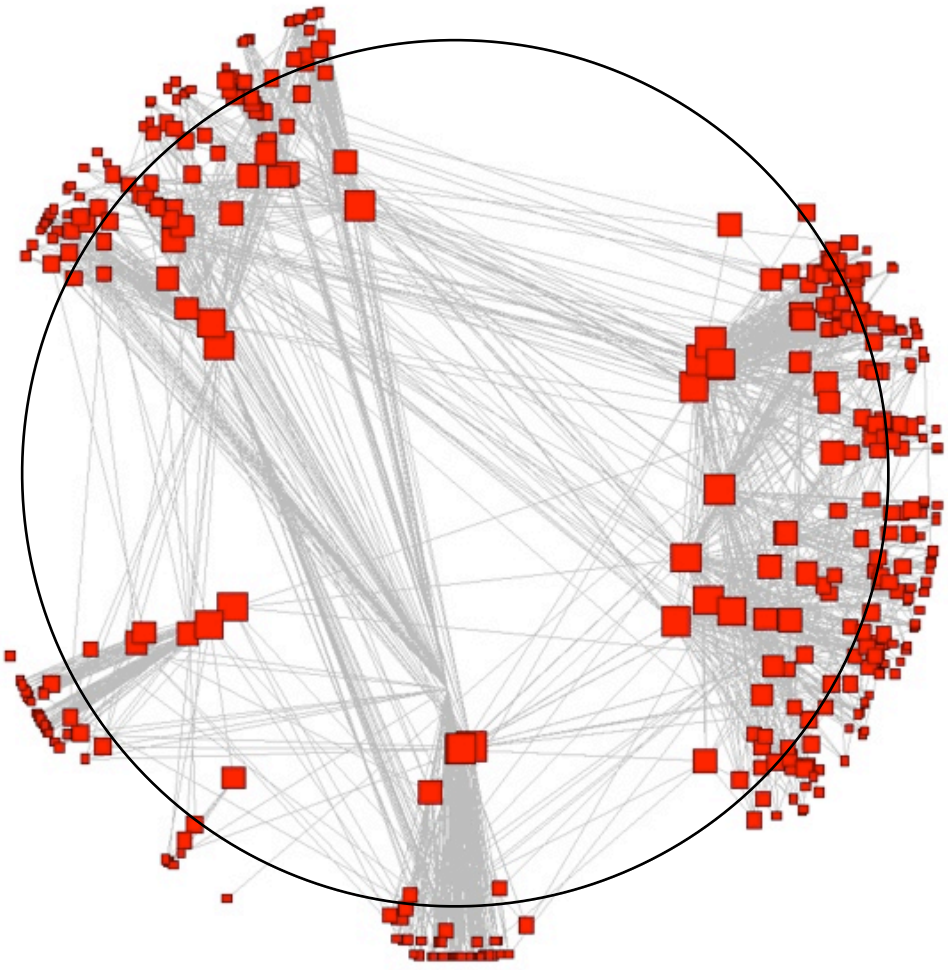

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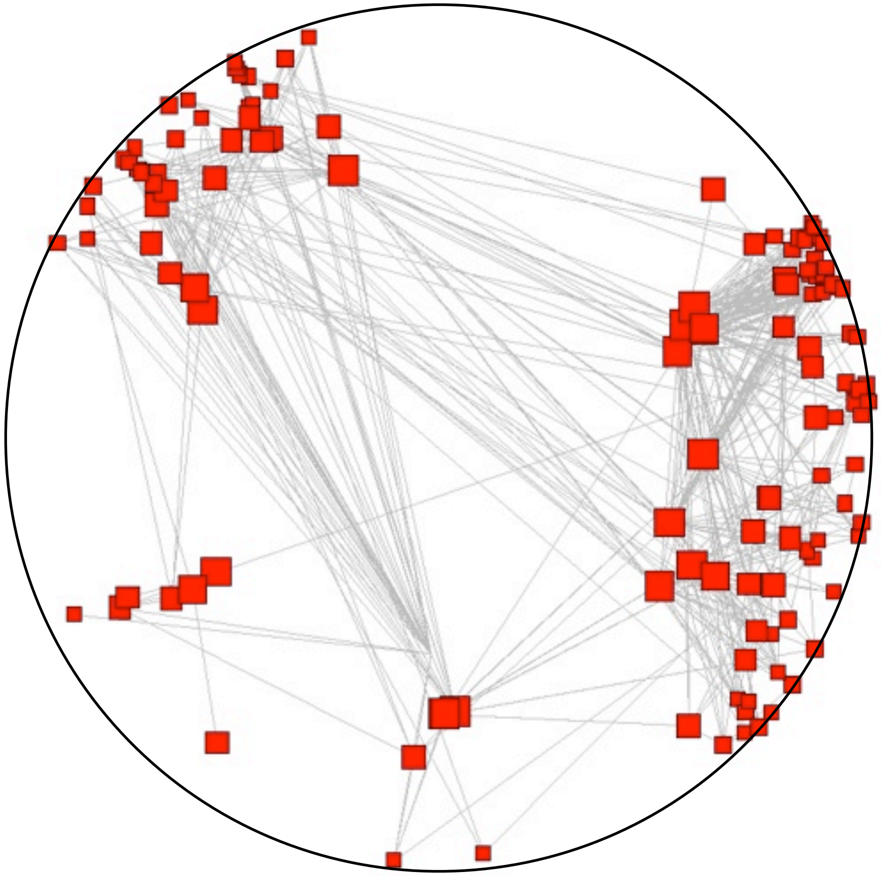

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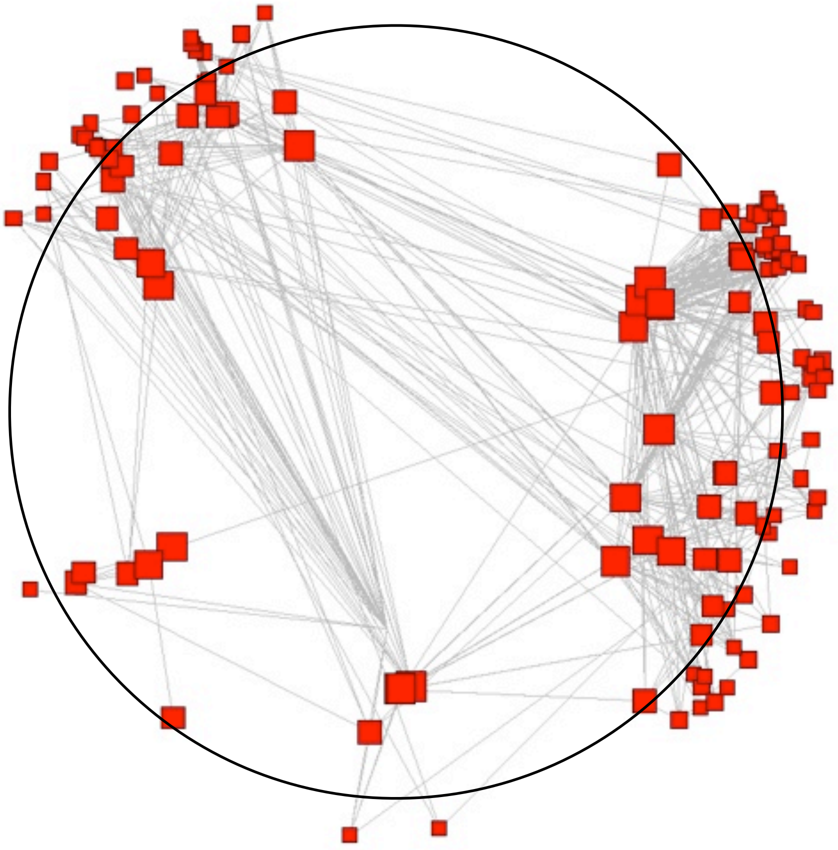

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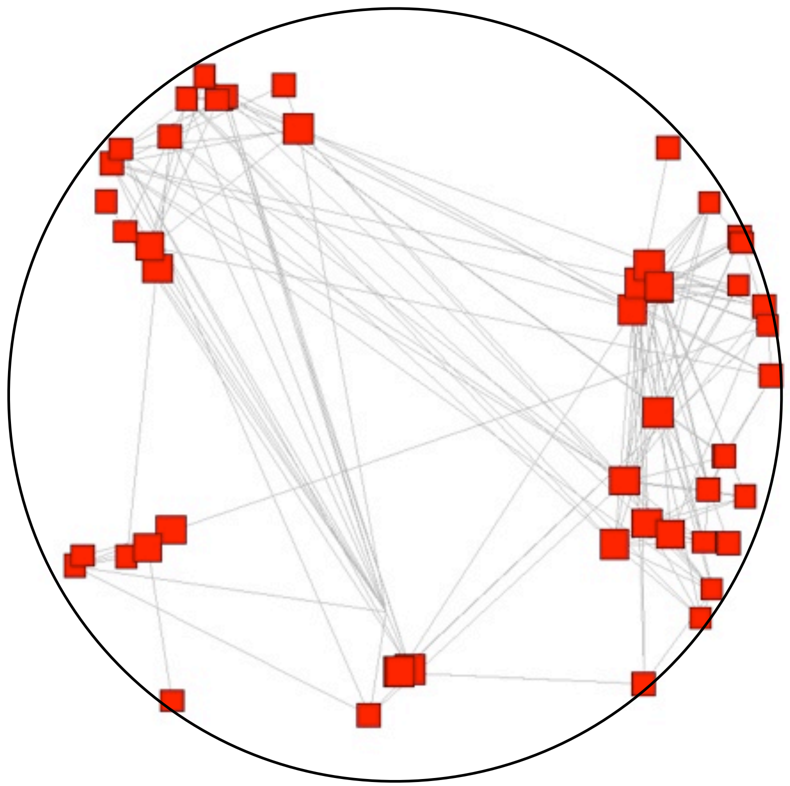

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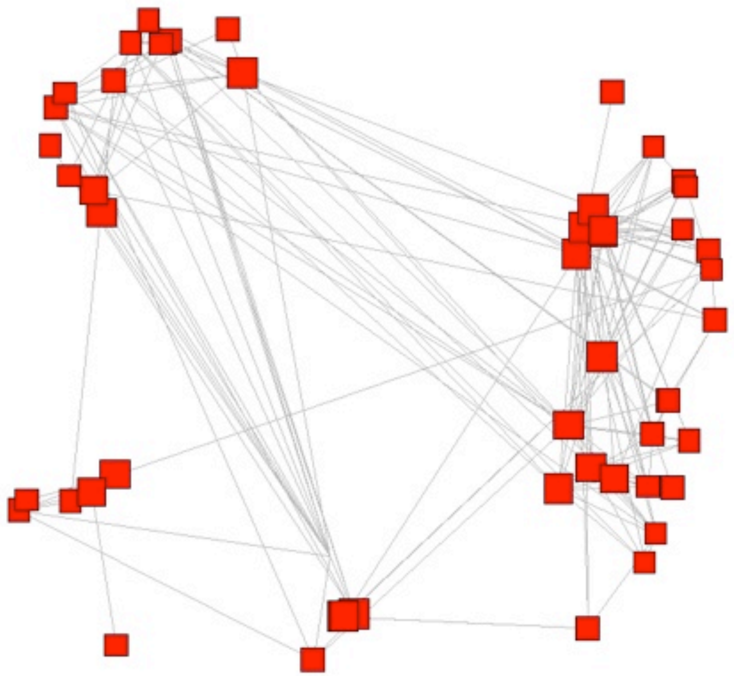
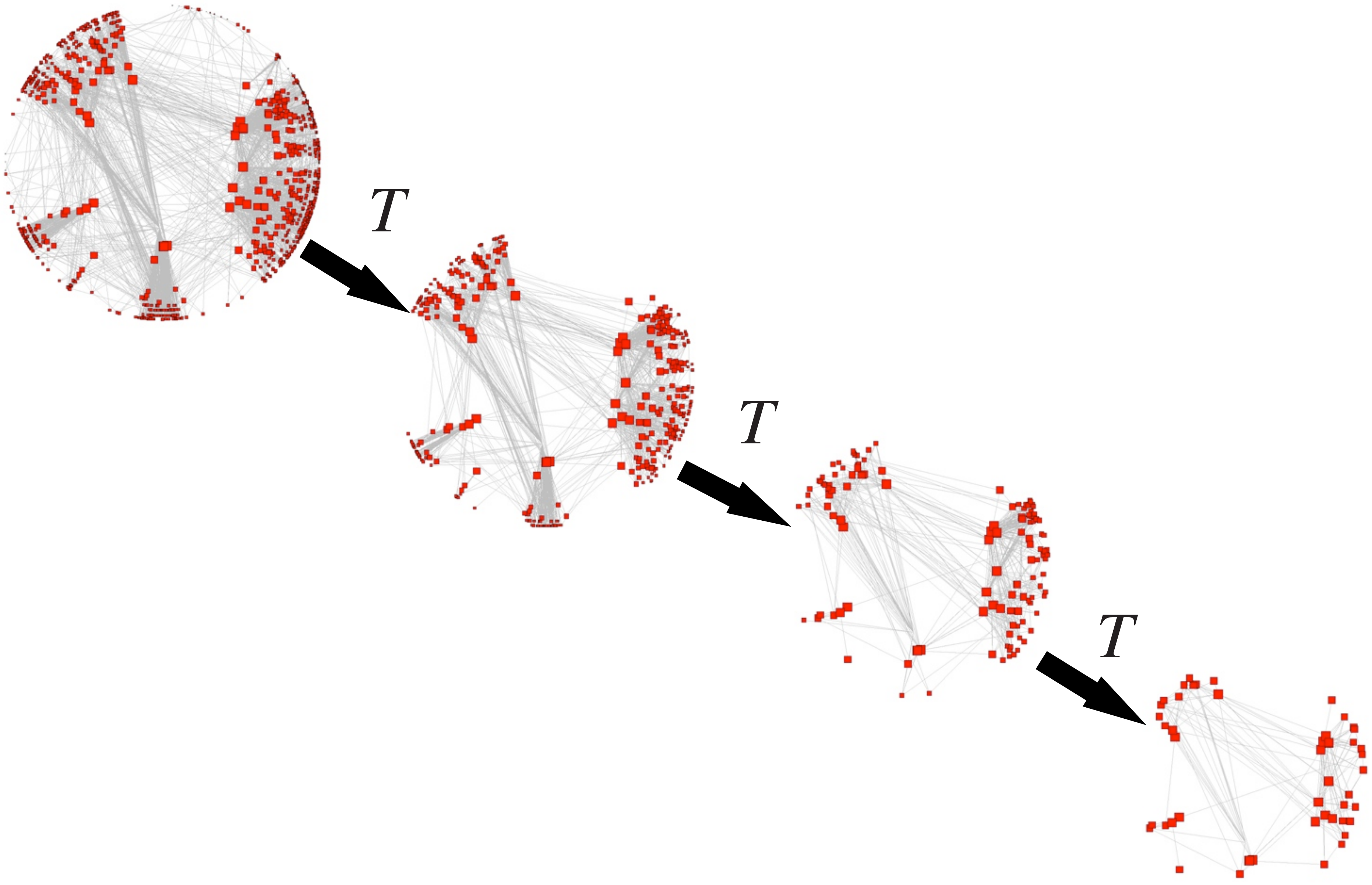


# Self-similar transformations: Russian dolls



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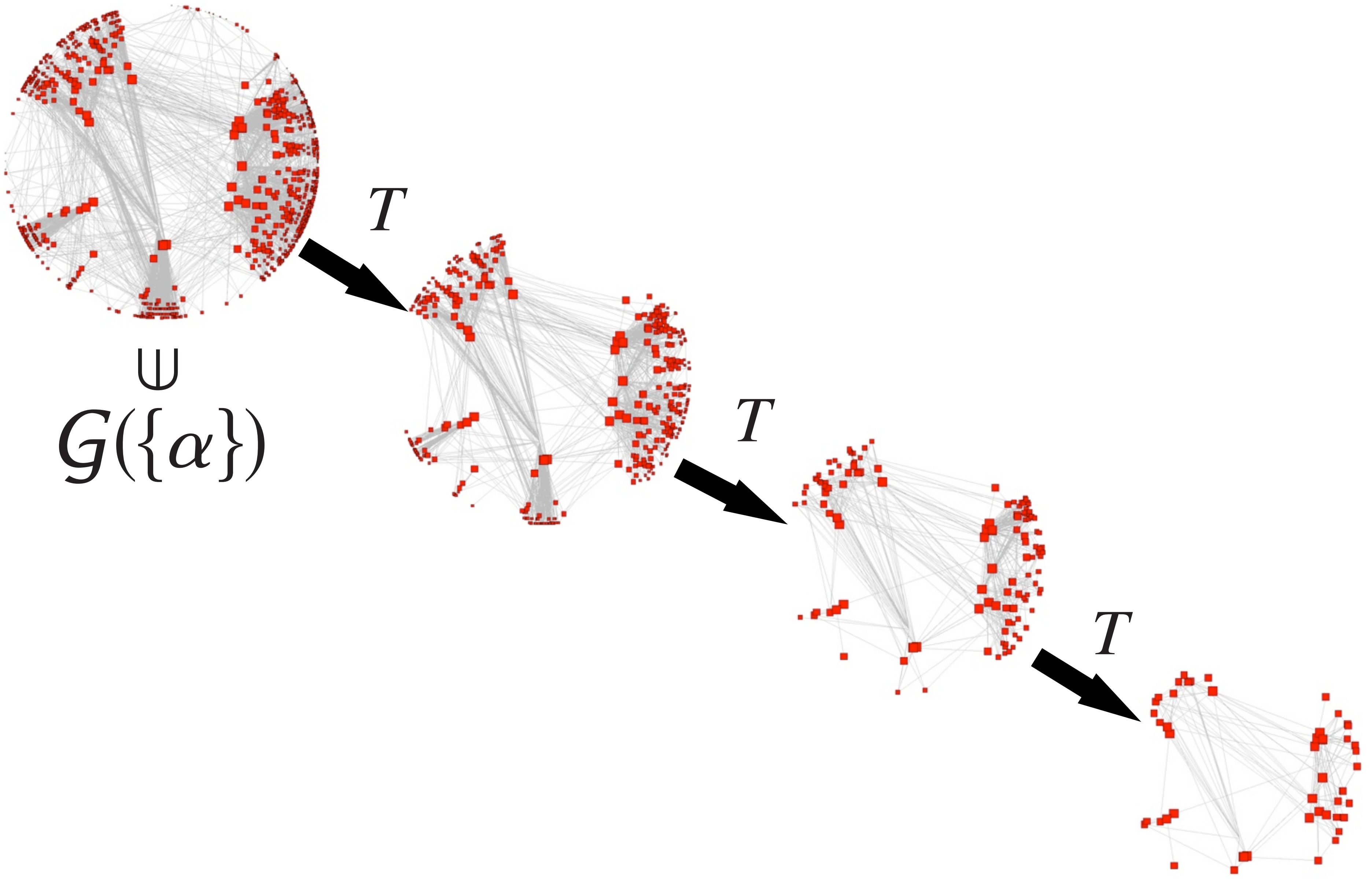

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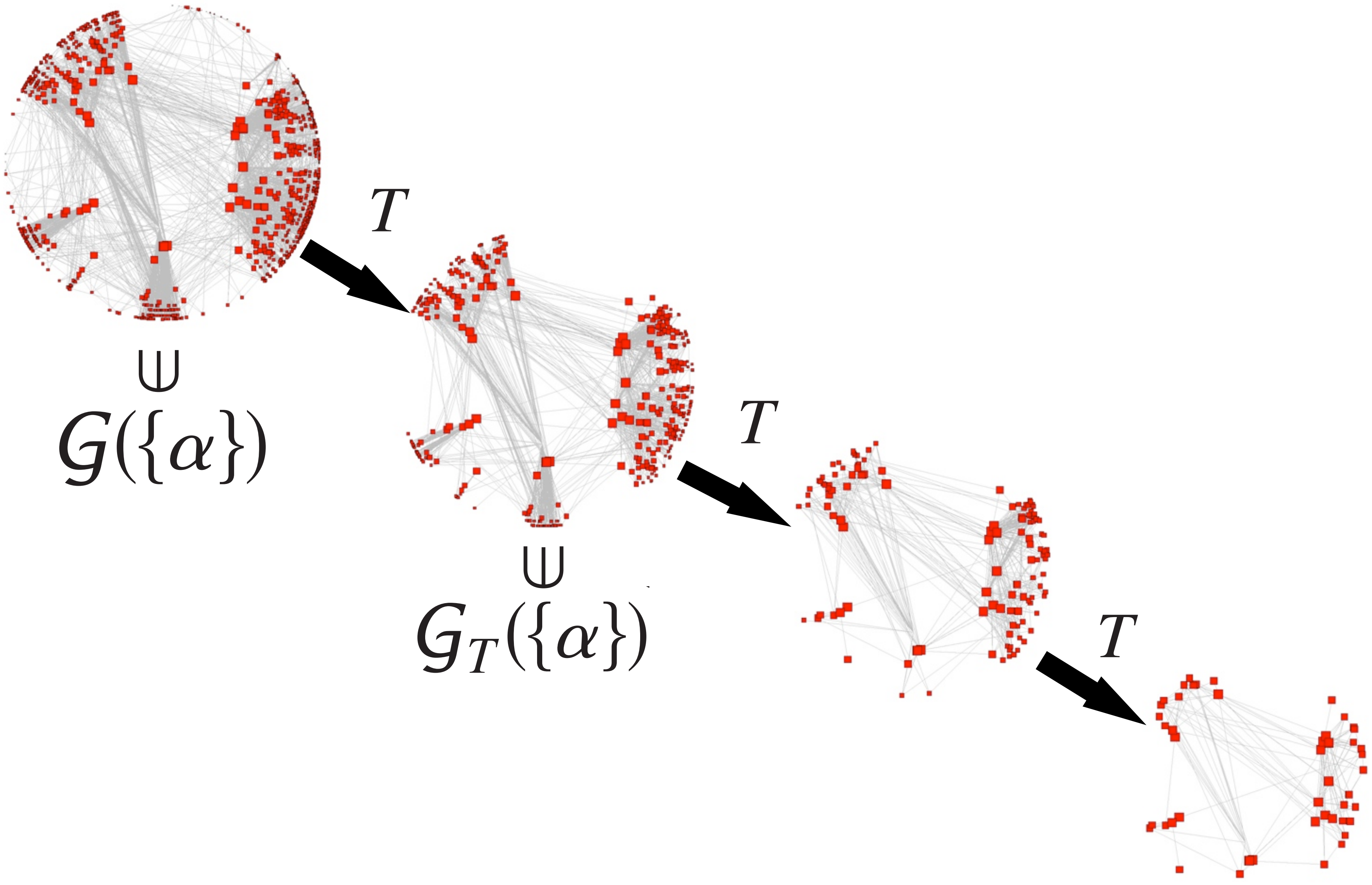
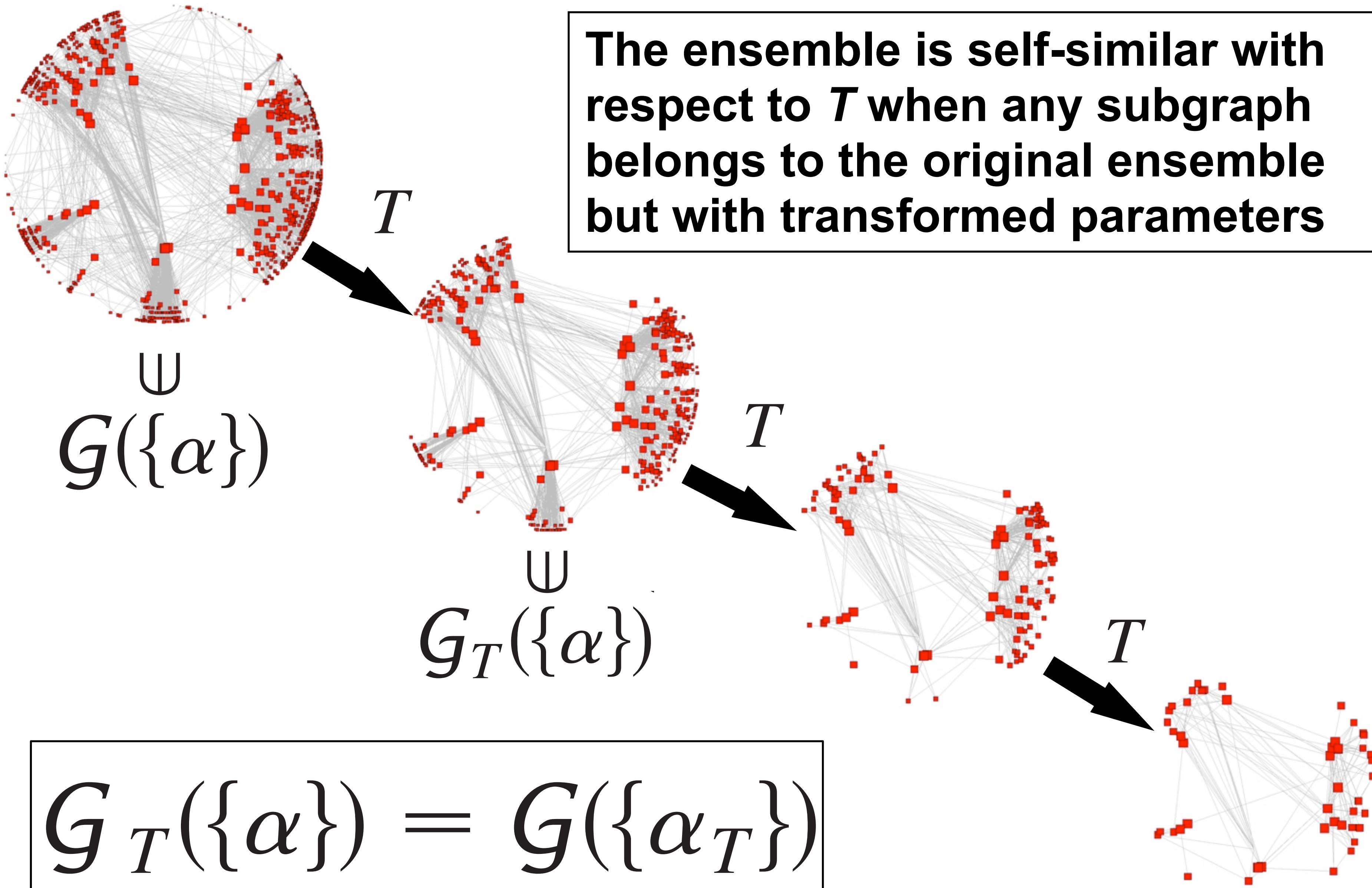


# Self-similar transformations: Russian dolls

**The ensemble is self-similar with respect to *T* when any subgraph belongs to the original ensemble but with transformed parameters**




$$\mathcal{G}_T(\{\alpha\}) = \mathcal{G}(\{\alpha_T\})$$

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1) Assign each node a hidden variable $\kappa$ distributed as

$$\rho(\kappa) = (\gamma - 1)\kappa_0^{\gamma-1}\kappa^{-\gamma}$$

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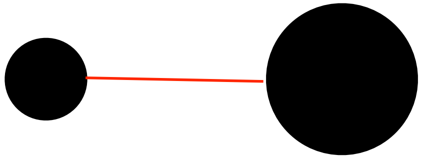

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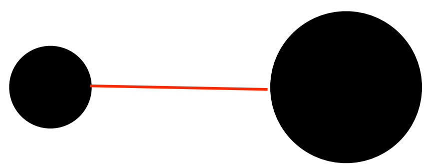

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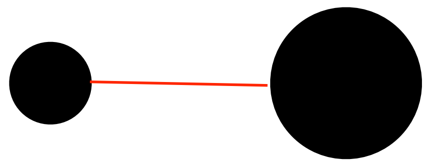


with this choice the net is maximally random

$$r(\kappa, \kappa') = f(\mu\kappa\kappa')$$

$$f(x) = \frac{1}{1 + 1/x}$$

$$f(x) \leq 1$$

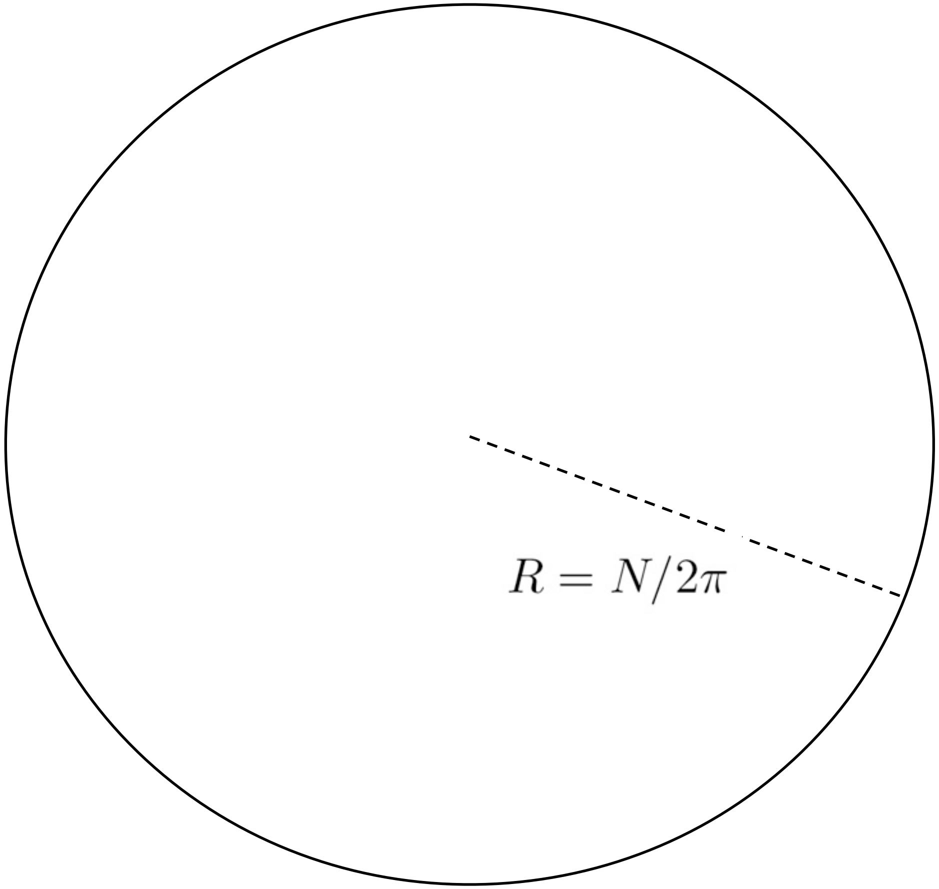
$$f(0) = 0$$

$$r(\kappa, \kappa') \approx \mu\kappa\kappa'$$

canonical version of the configuration model. $\kappa$ is the expected degree of the node

K. Anand, G. Bianconi, and S. Severini, Phys. Rev. E 83, 036109 (2011)

# Models: type II




M. A. Serrano, D. Krioukov, and M. Boguñá, Phys. Rev. Lett. 100, 078701 (2008)

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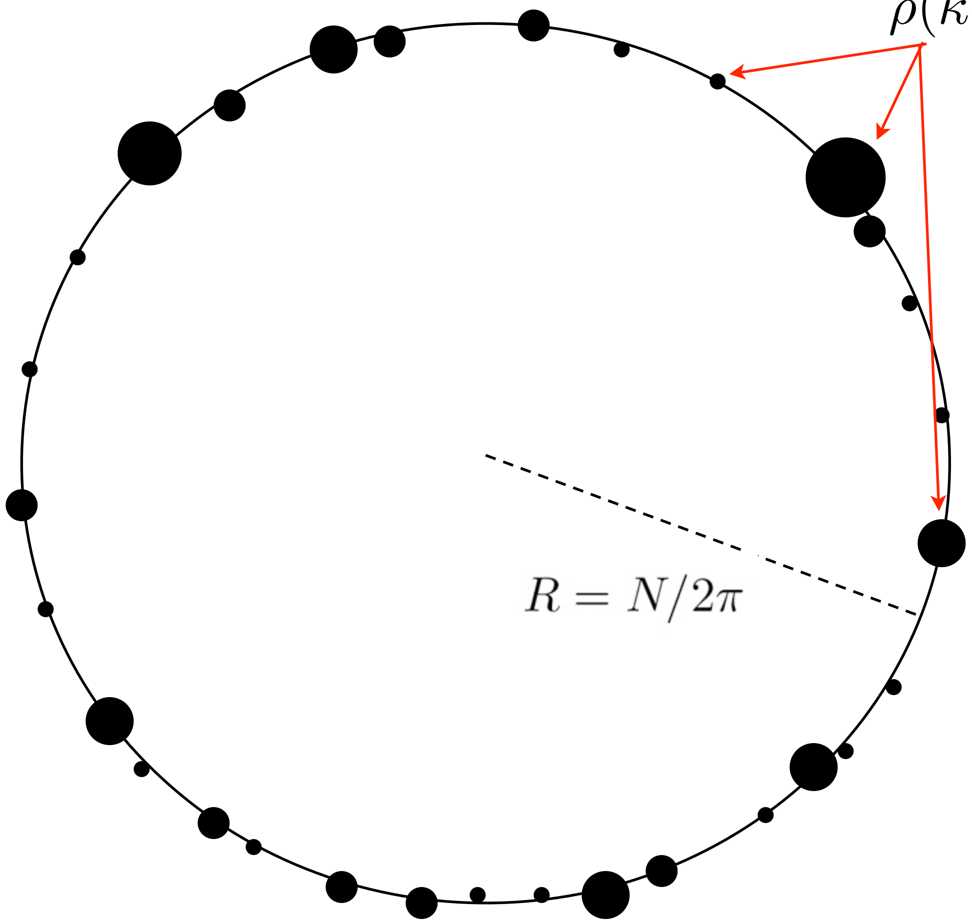
$$\rho(\kappa) = (\gamma - 1)\frac{\kappa_0^{\gamma-1}}{\kappa^\gamma} \quad , \kappa \in [\kappa_0, \infty)$$

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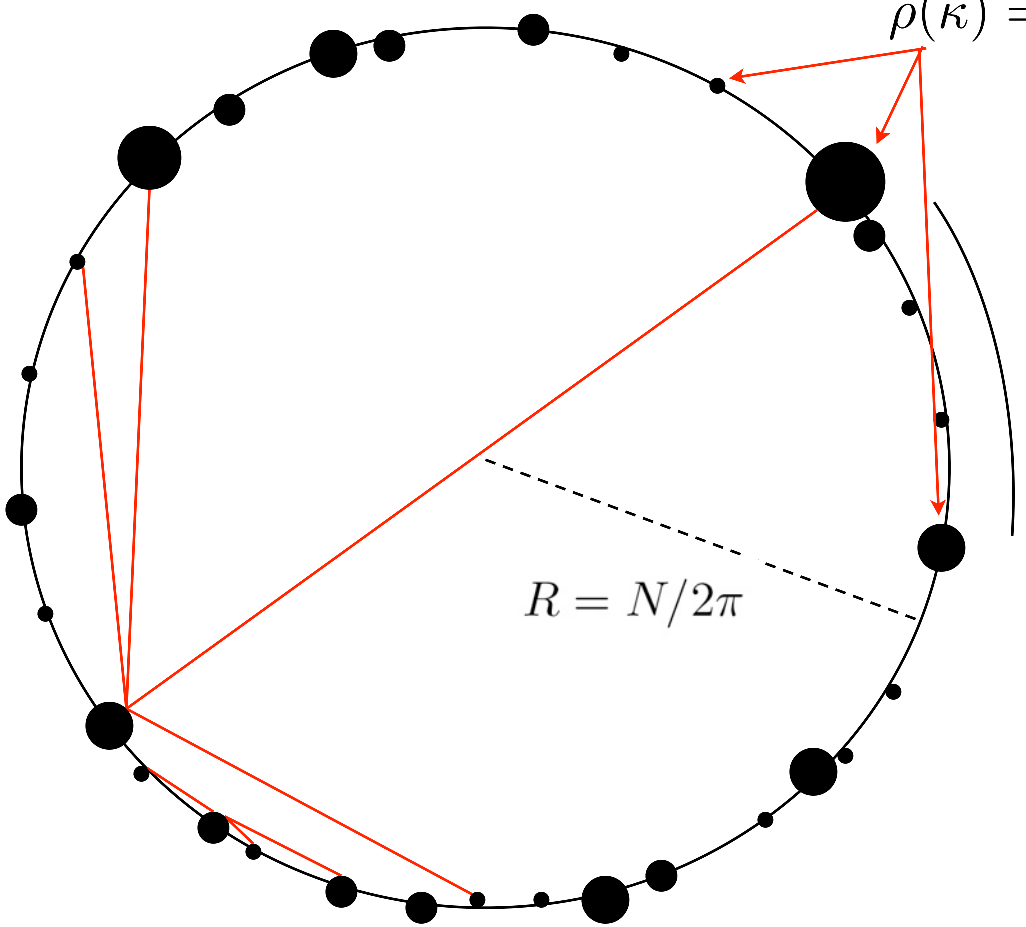


$$r(\theta, \kappa; \theta', \kappa') = h\left(\frac{d}{\mu \kappa \kappa'}\right)$$

connection probability between a pair of nodes

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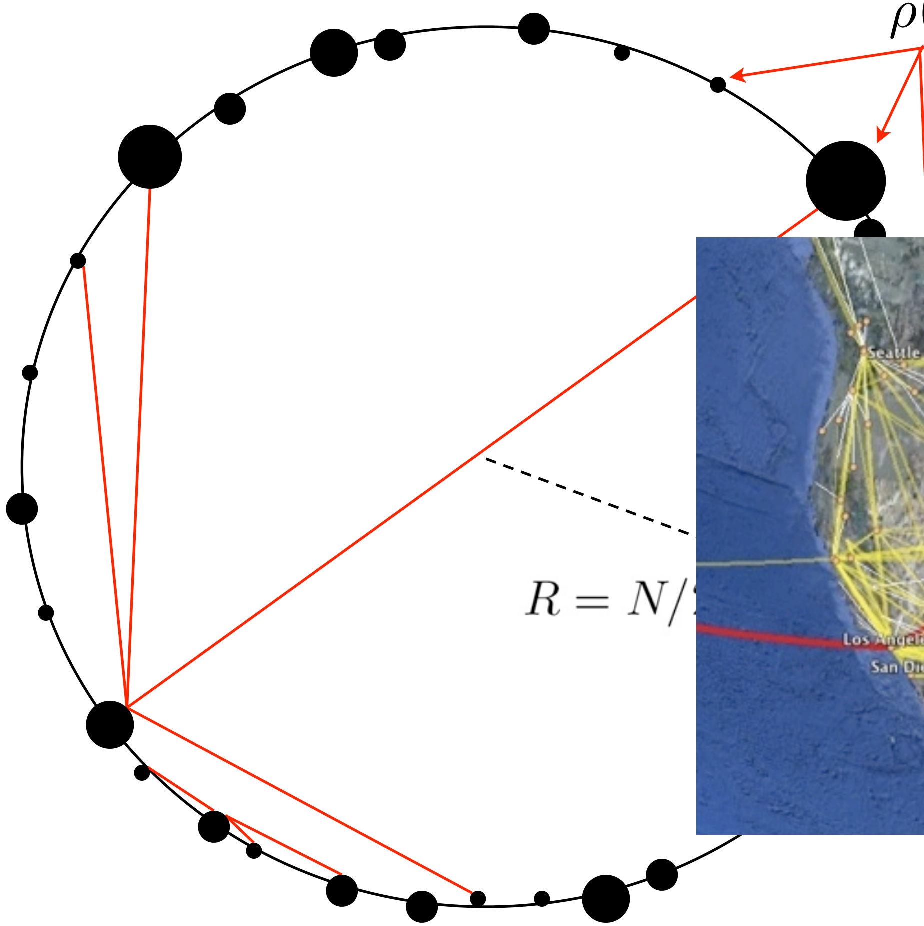

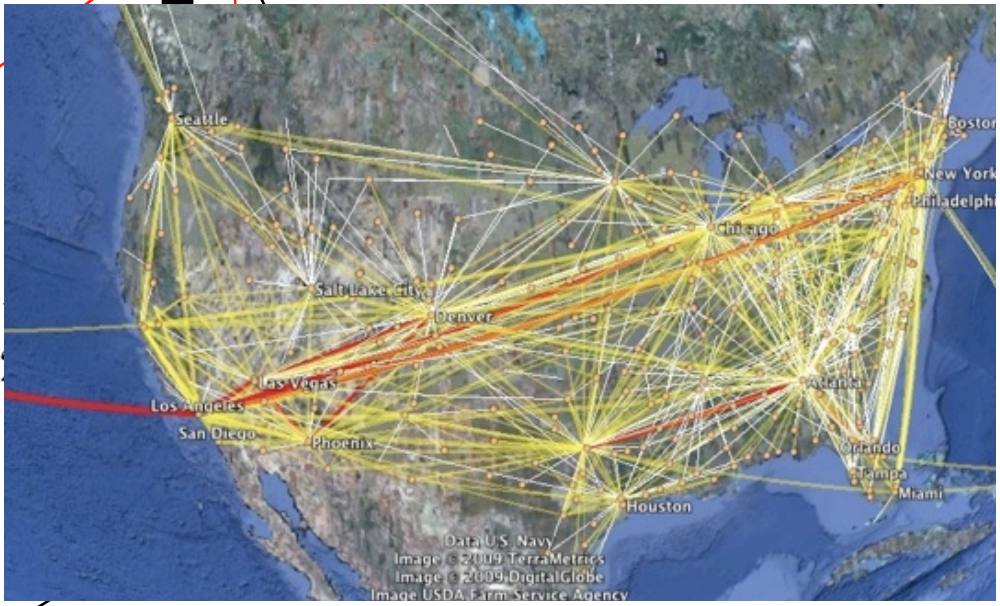


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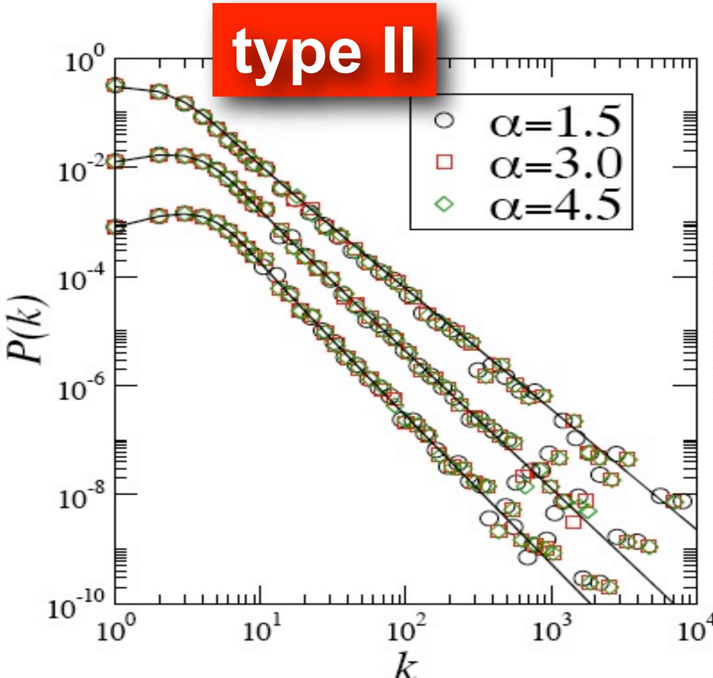
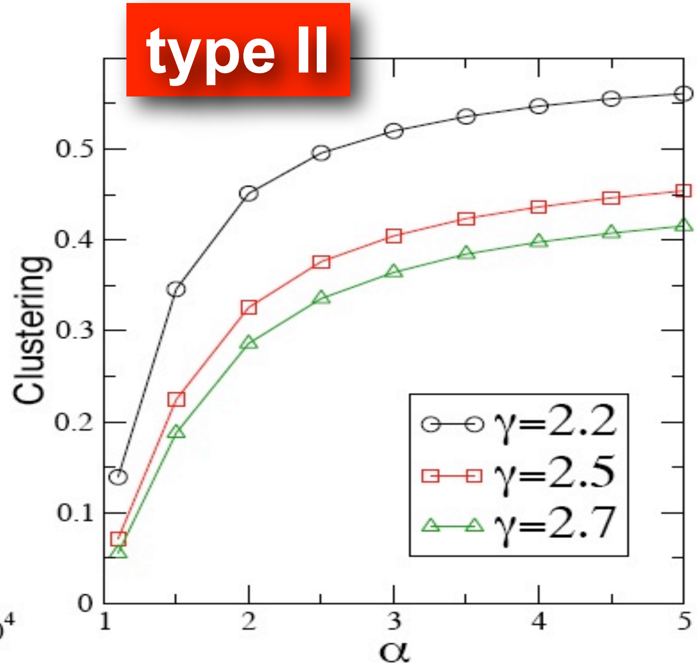
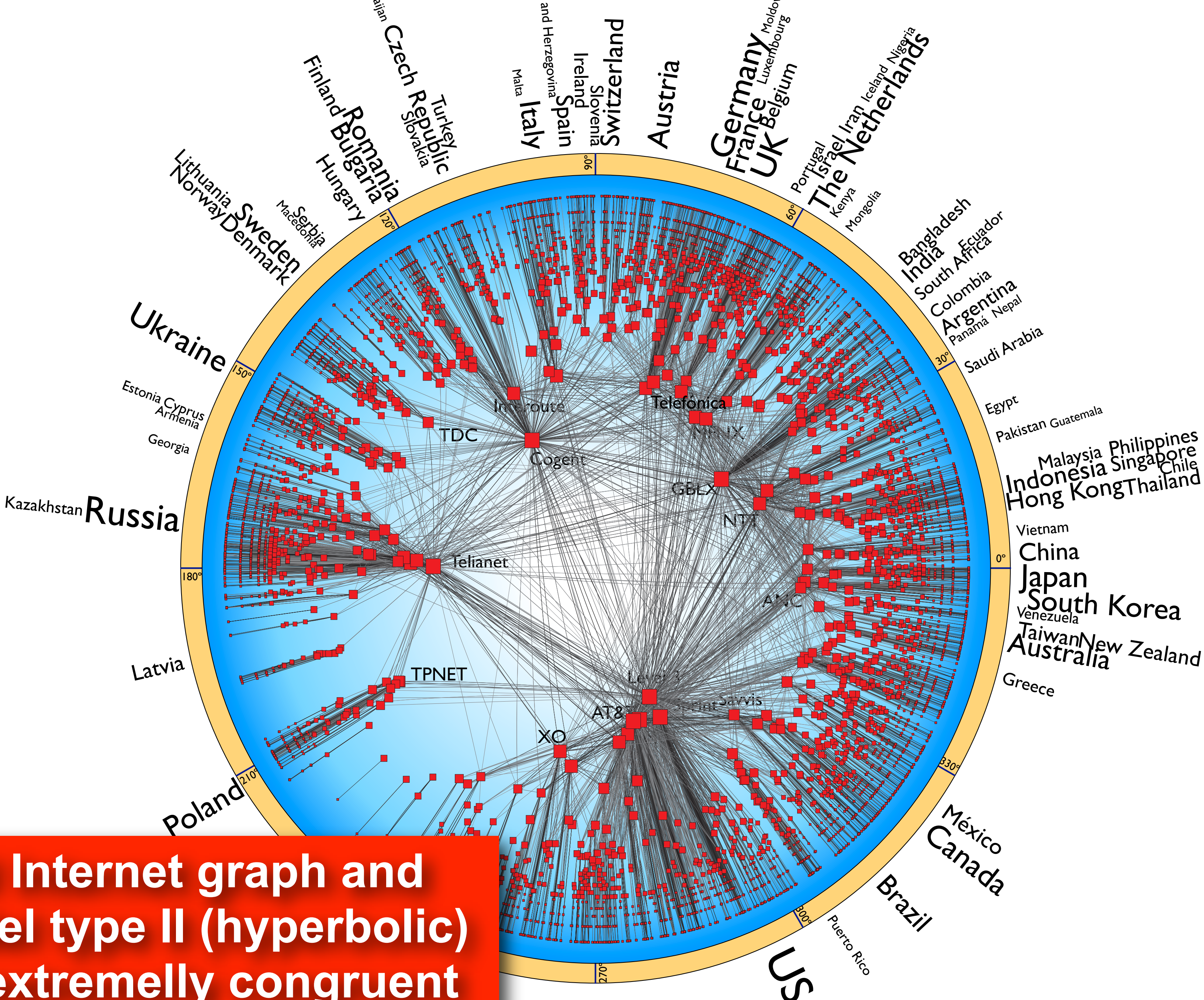


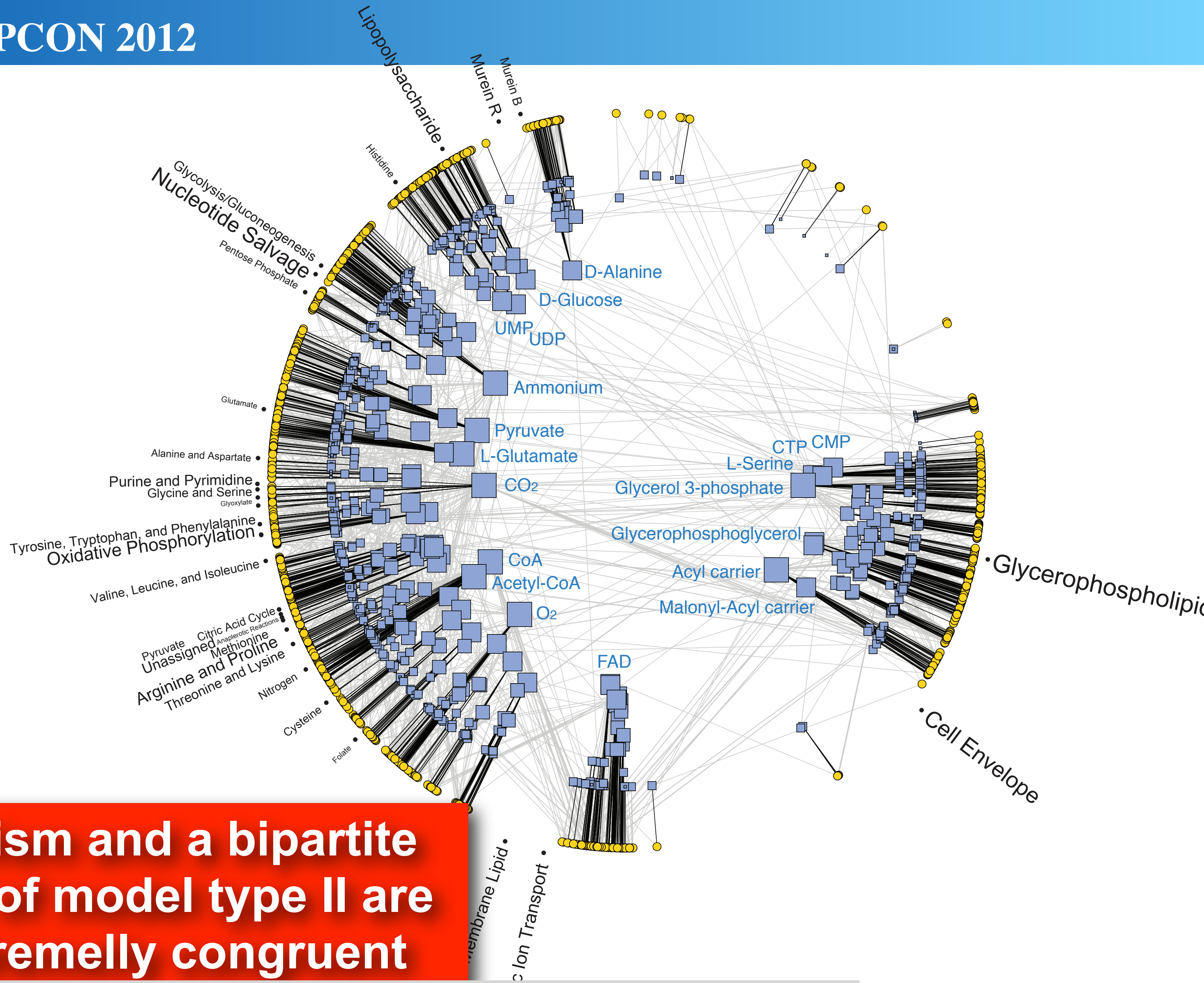






# Real Internet graph and model type II (hyperbolic) are extremelly congruent

M. Boguñá, F. Papadopoulos, and D. Krioukov, Nature Communications 1, 62 (2010)

# MAPCON 2012

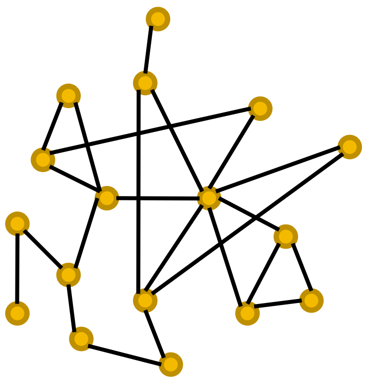



**Metabolism and a bipartite version of model type II are also extremelly congruent**

M. A. Serrano, M. Boguñá, and F. Sagués, Molecular BioSystems 8, 843-850 (2012)

## Transformation $T$

degree-thresholding renormalization procedure: keep only nodes with degrees above a certain threshold, $k > k_T$



$k_T = 0$

$$\mathcal{G}(\{\alpha\})$$

# Renormalizing the ensemble

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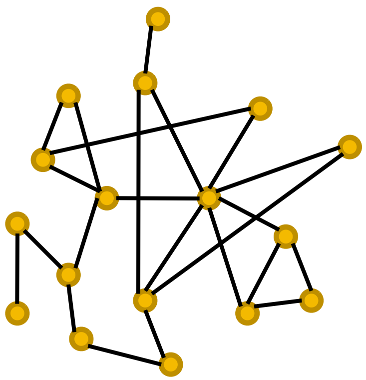

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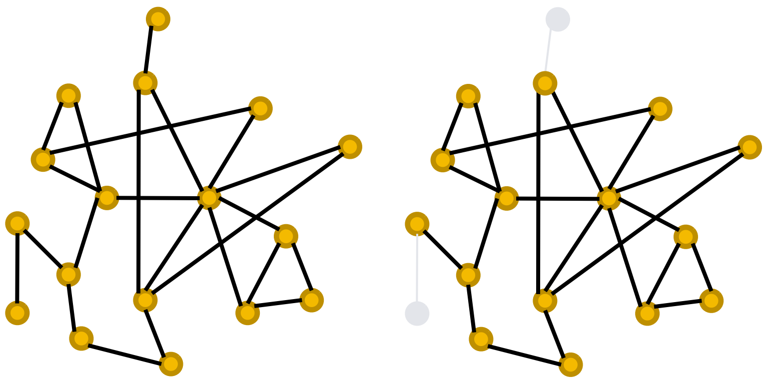

$k_T = 0$ $k_T = 1$

$\mathcal{G}(\{\alpha\})$ $\mathcal{G}_T(\{\alpha\})$

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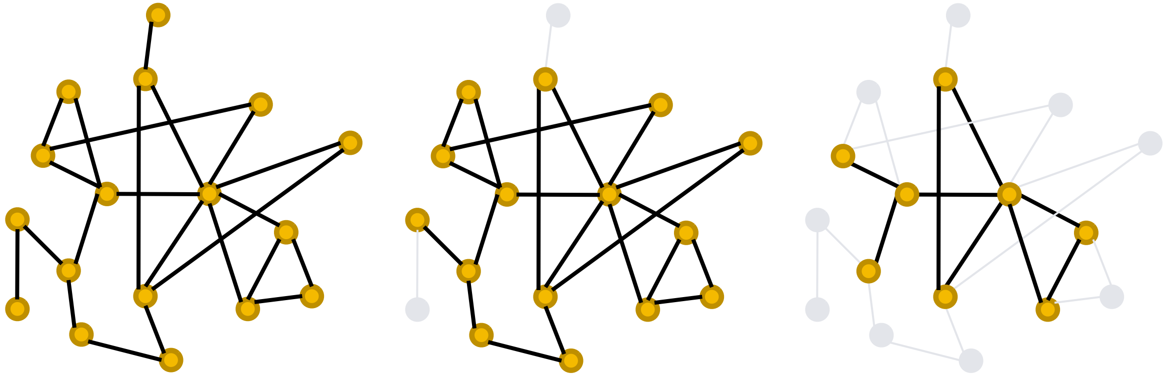

$k_T = 0$ $k_T = 1$ $k_T = 2$

$\mathcal{G}(\{\alpha\})$ $\mathcal{G}_T(\{\alpha\})$

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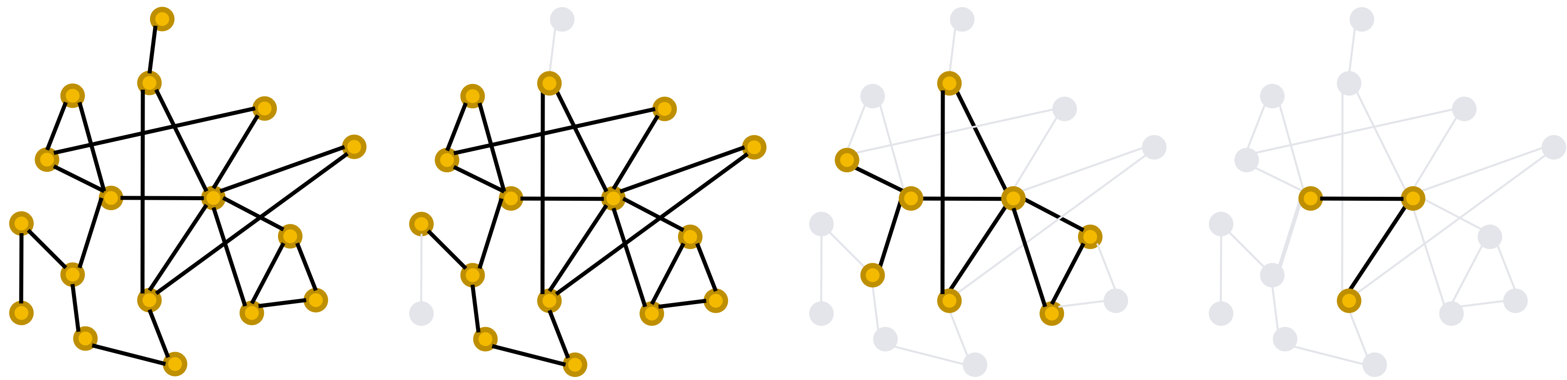
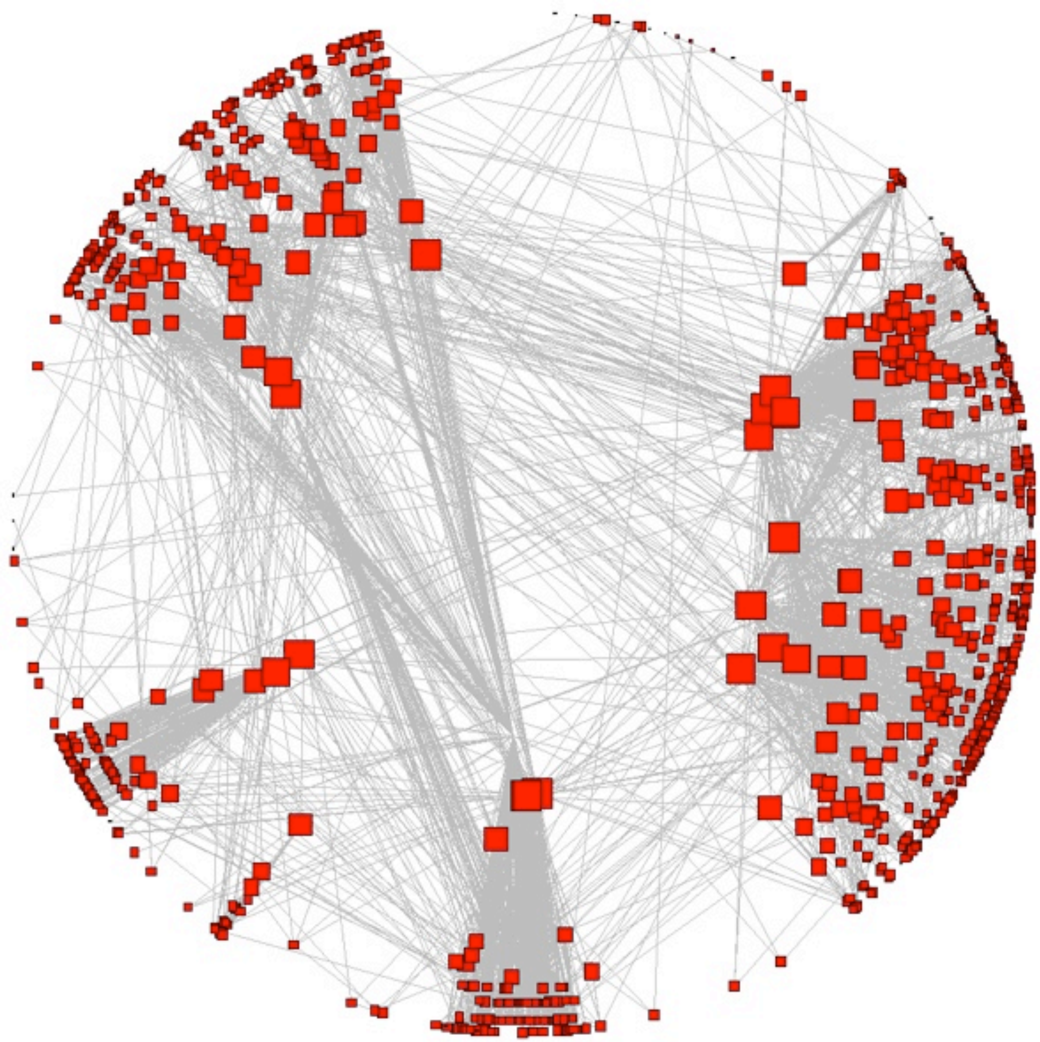


$k_T = 0$ $\quad$ $k_T = 1$ $\quad$ $k_T = 2$ $\quad$ $k_T = 3$

$\mathcal{G}(\{\alpha\})$ $\qquad$ $\mathcal{G}_T(\{\alpha\})$

# self-similarity of type II nets



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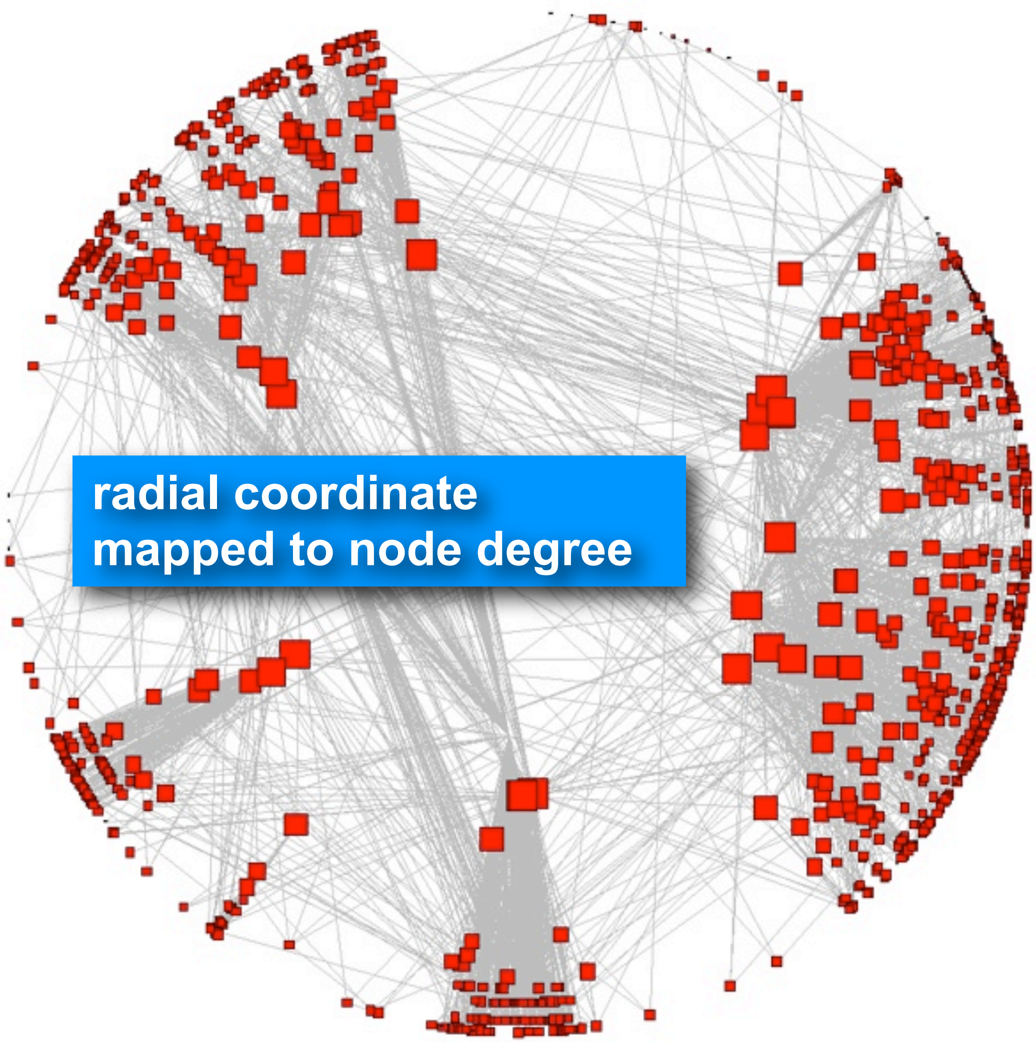

radial coordinate
mapped to node degree

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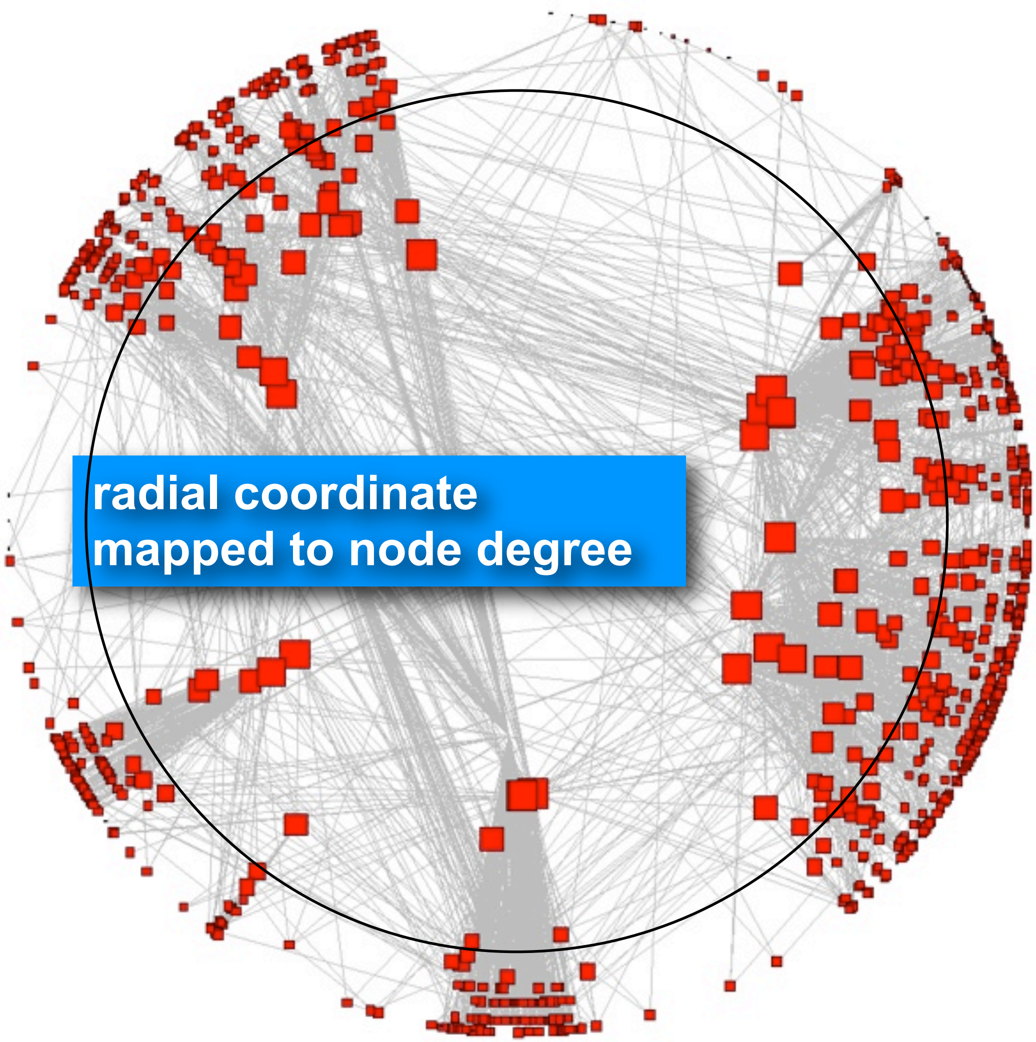

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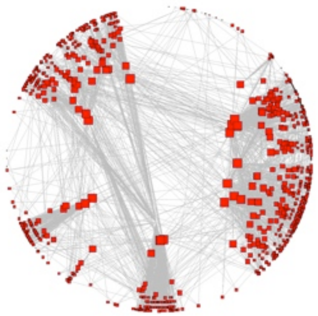

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**model: type II**

Distribution of hidden variables

$$\rho(\kappa) = (\gamma - 1)\kappa_0^{\gamma-1}\kappa^{-\gamma}$$

Connection probability

$$r(\kappa, \theta; \kappa', \theta') = h\left(\frac{d}{\mu\kappa\kappa'}\right)$$

Fixing the average degree

$$\mu_{\mathrm{II}} = \frac{\langle k \rangle}{2\delta I \kappa_0^2}\left(\frac{\gamma - 2}{\gamma - 1}\right)^2$$

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$$\kappa_0 \longrightarrow \kappa_T$$

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**model: type II after T**

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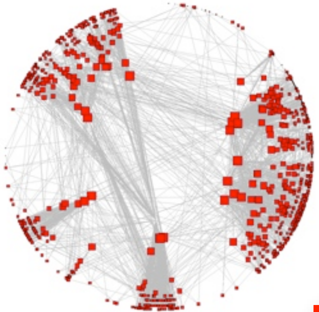
$$r(\kappa,\theta;\kappa',\theta') = h\left(\frac{d}{\mu\kappa\kappa'}\right)$$

Fixing the average degree

$$\mu_{\mathrm{II}} = \frac{\langle k\rangle_T}{2\delta_T I\kappa_T^2}\left(\frac{\gamma-2}{\gamma-1}\right)^2$$

# self-similarity of type II nets



$$\kappa_0 \longrightarrow \kappa_T$$

$$N \longrightarrow N_T = N(\kappa_0/\kappa_T)^{\gamma-1}$$

$$\delta \longrightarrow \delta_T = \delta(\kappa_0/\kappa_T)^{\gamma-1}$$

**model: type II**

Distribution of hidden variables

$$\rho(\kappa) = (\gamma - 1)\kappa_0^{\gamma-1}\kappa^{-\gamma}$$

Conn

$r(\kappa,$

Fixing

$$\mu_{\mathrm{II}} = \frac{\langle k \rangle}{2\delta I \kappa_0^2}\left(\frac{\gamma-2}{\gamma-1}\right)^2$$

$$\langle k \rangle \rightarrow \langle k \rangle_T = \langle k \rangle (\kappa_T/\kappa_0)^{3-\gamma}$$

**model: type II after T**

Distribution of hidden variables

$$\rho_T(\kappa) = (\gamma - 1)\kappa_T^{\gamma-1}\kappa^{-\gamma}$$

Connection probability

$$r(\kappa, \theta; \kappa', \theta') = h\left(\frac{d}{\mu \kappa \kappa'}\right)$$

Fixing the average degree

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# Self-similarity of real networks

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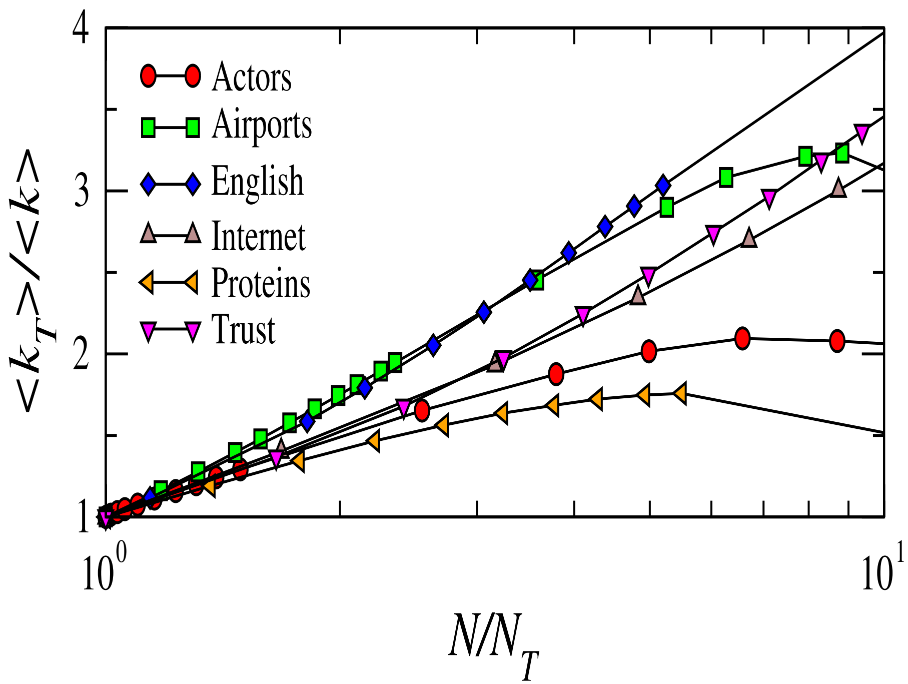
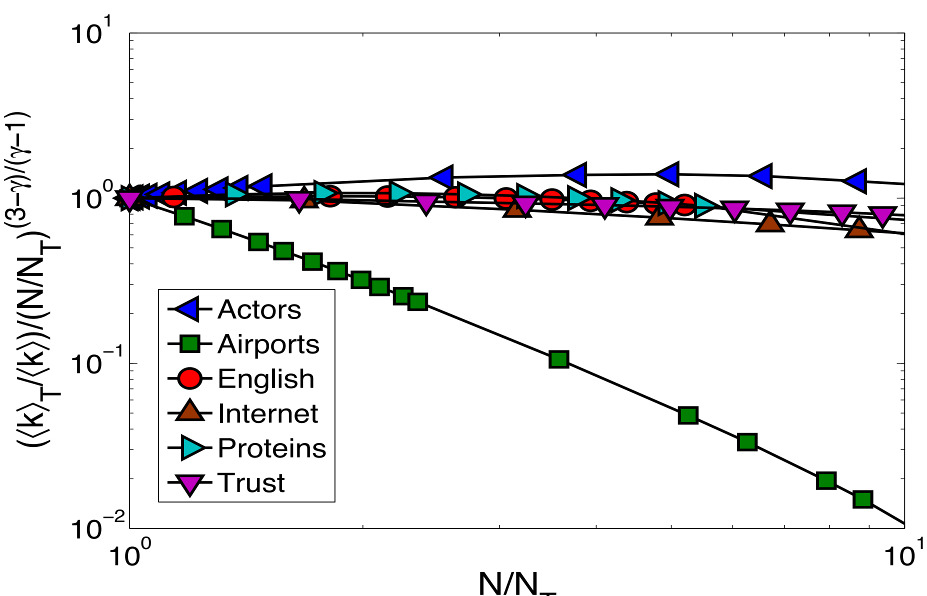
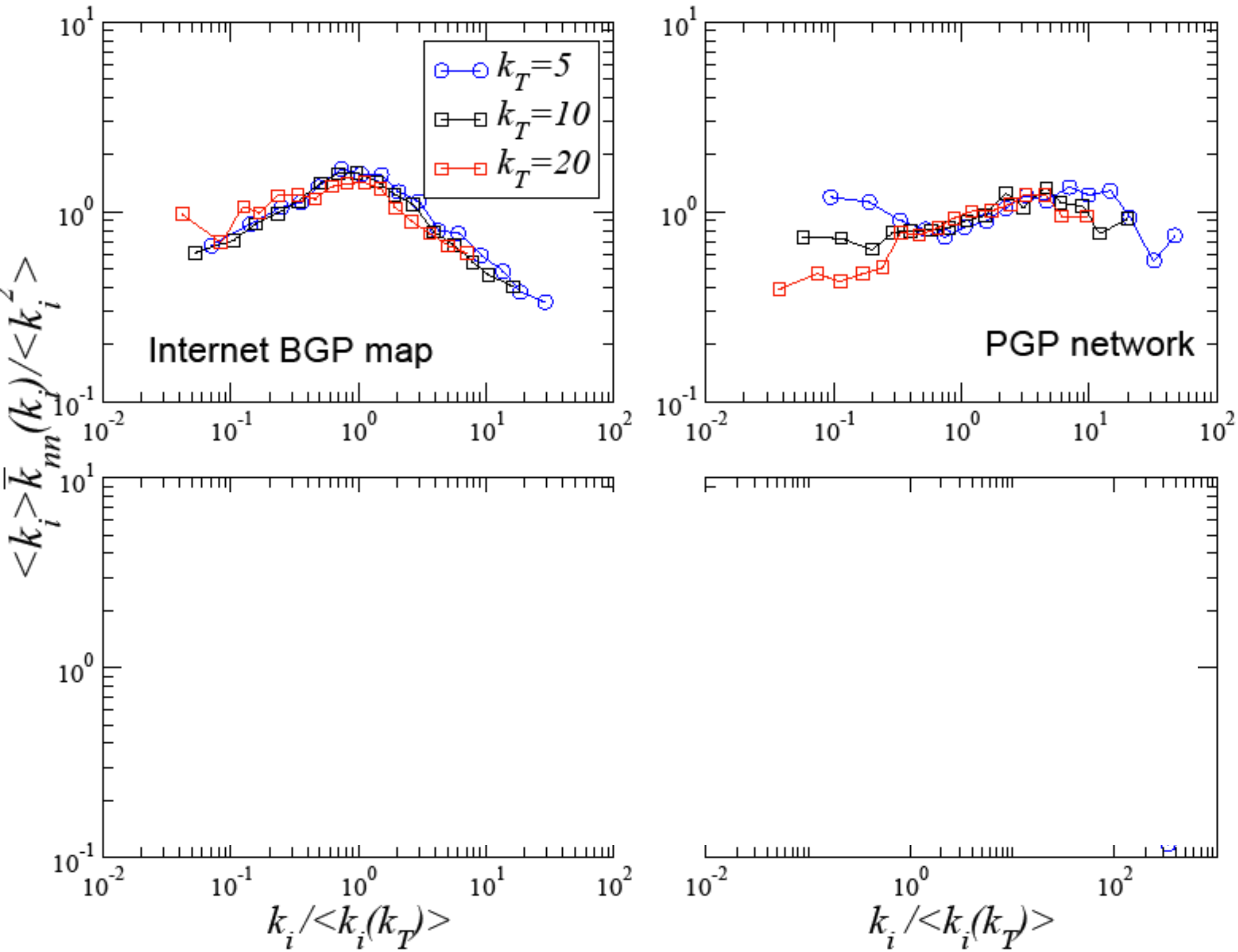


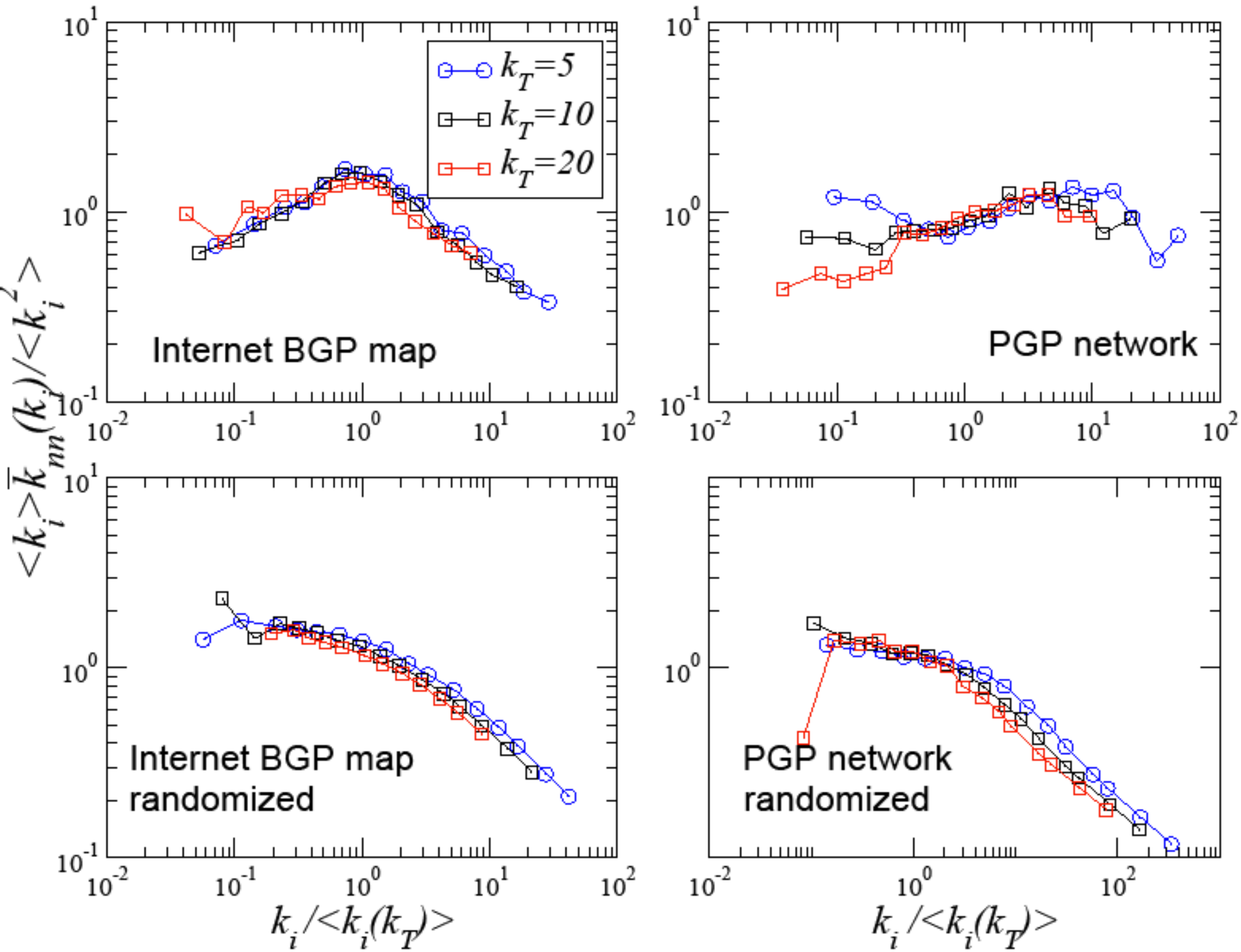
# Self-similarity of real networks

## degree-degree correlations



# Self-similarity of real networks

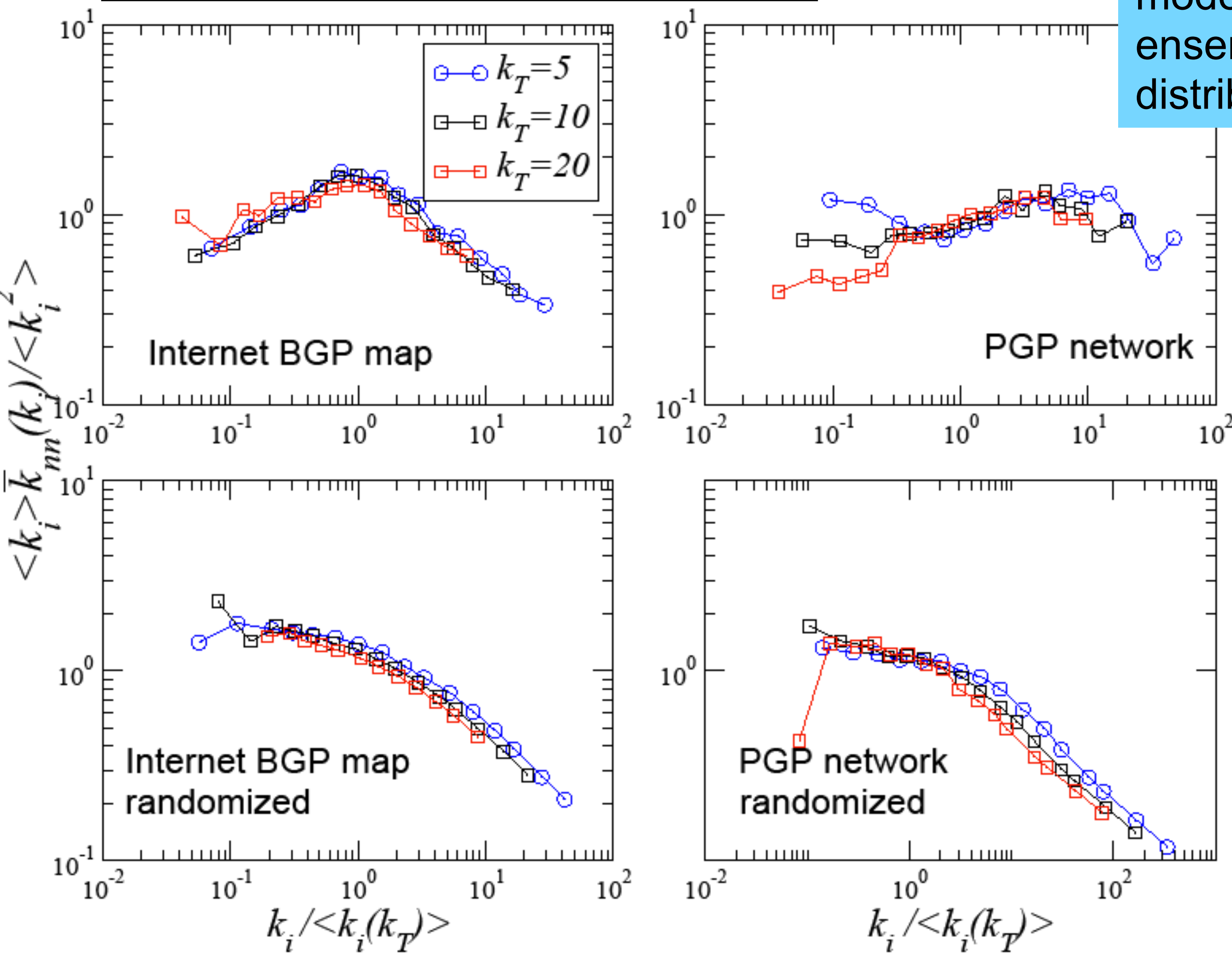
## degree-degree correlations



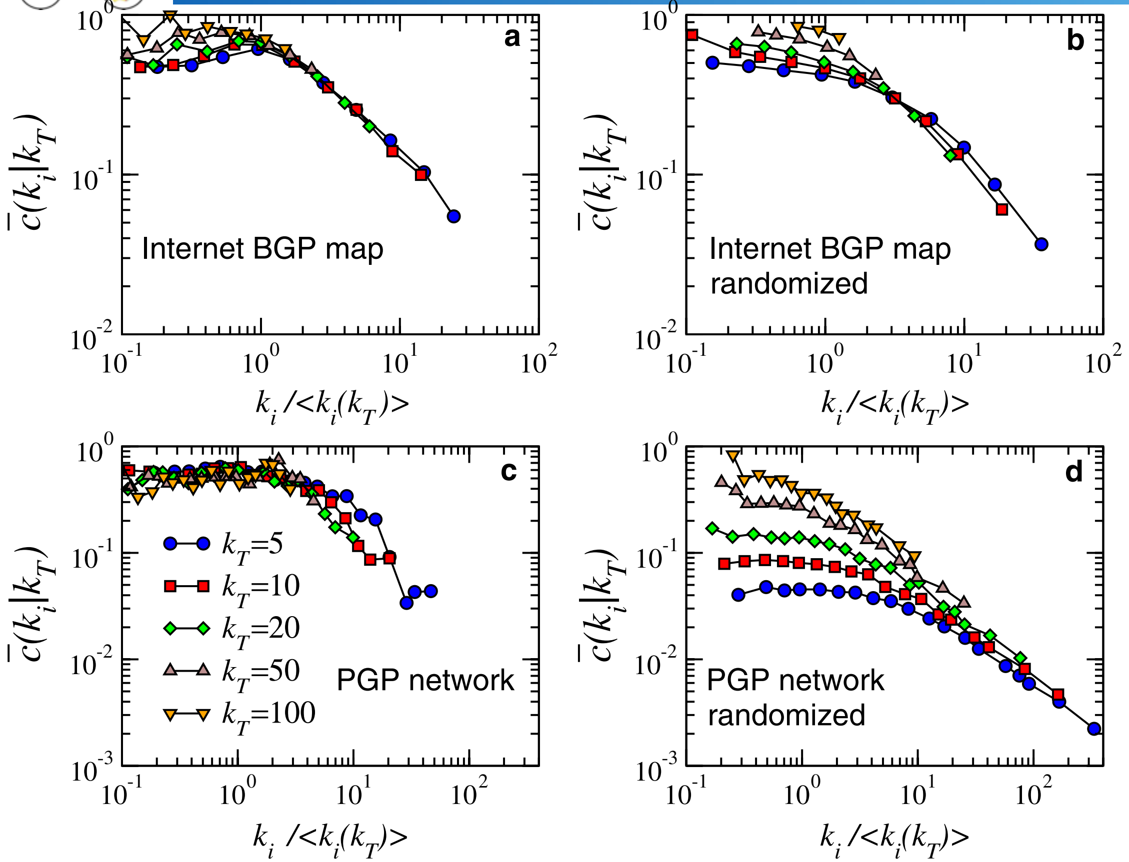
# Self-similarity of real networks

degree-degree correlations

The configuration model is a self-similar ensemble if the degree distribution is scale-free



# measuring clustering



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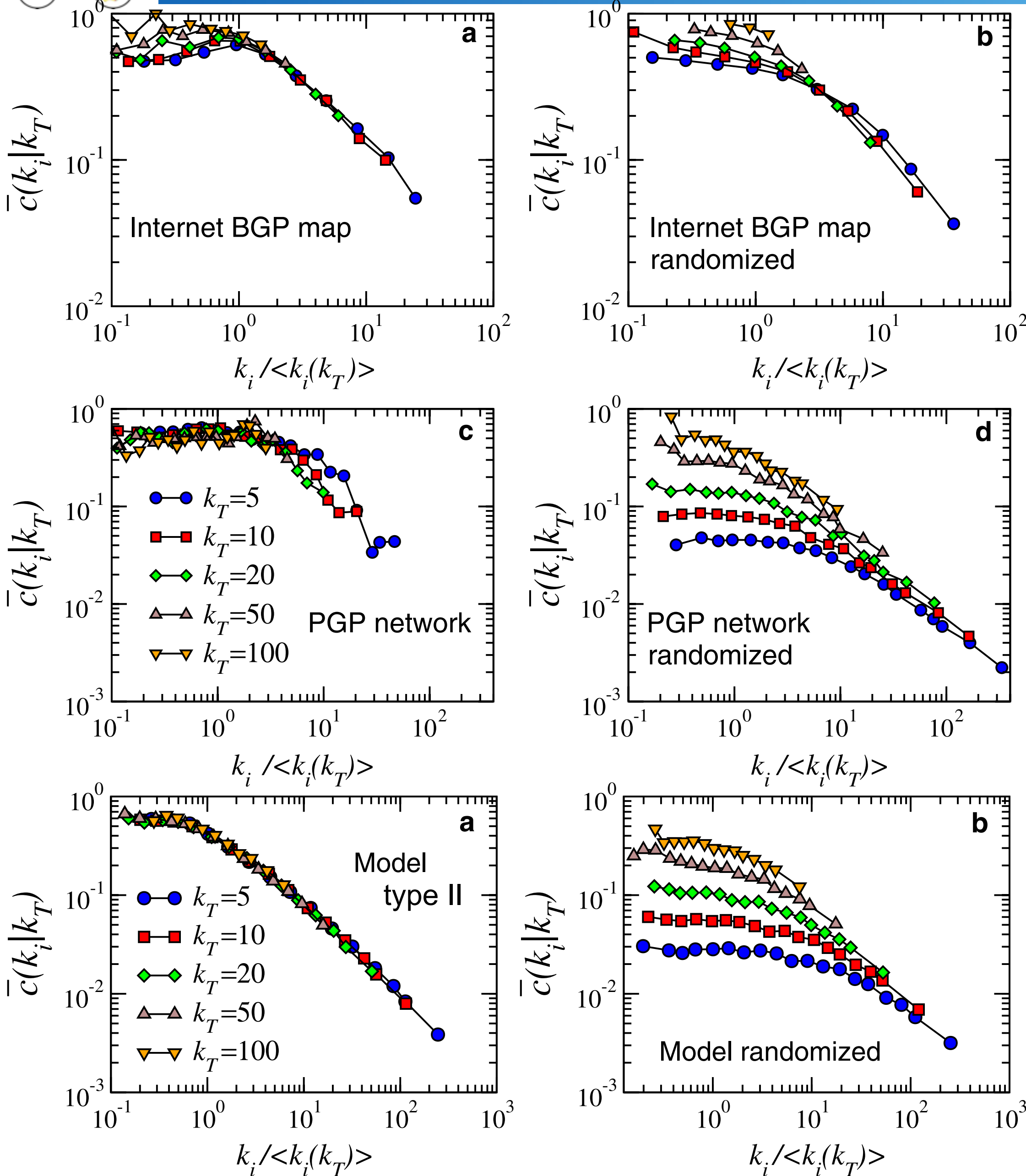

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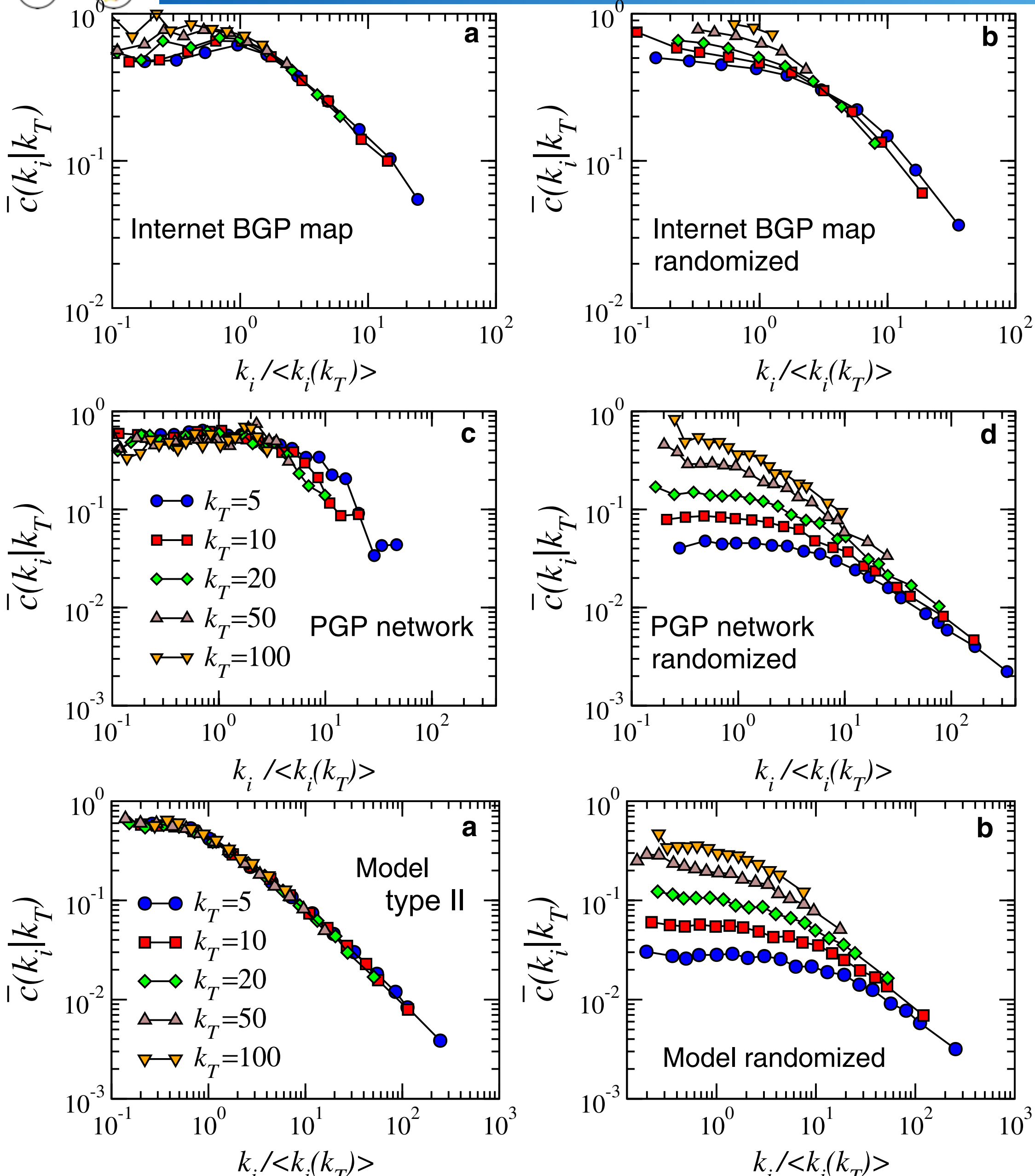
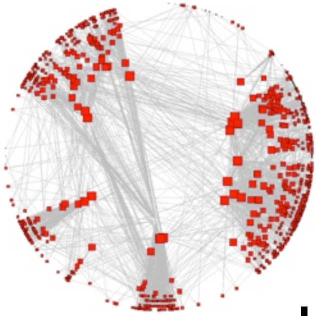



Real networks are close to type II

# Aplications: percolation threshold



$$\langle k \rangle \longrightarrow \langle k \rangle_T = \langle k \rangle \left( \frac{N}{N_T} \right)^{(3-\gamma)/(\gamma-1)}$$

If $2 < \gamma < 3$ the average degree in the subgraphs is bigger than in the original graph

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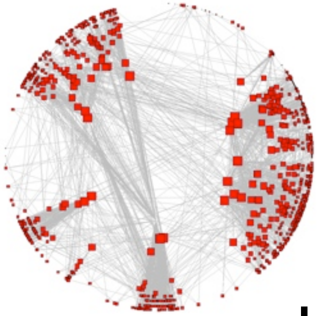

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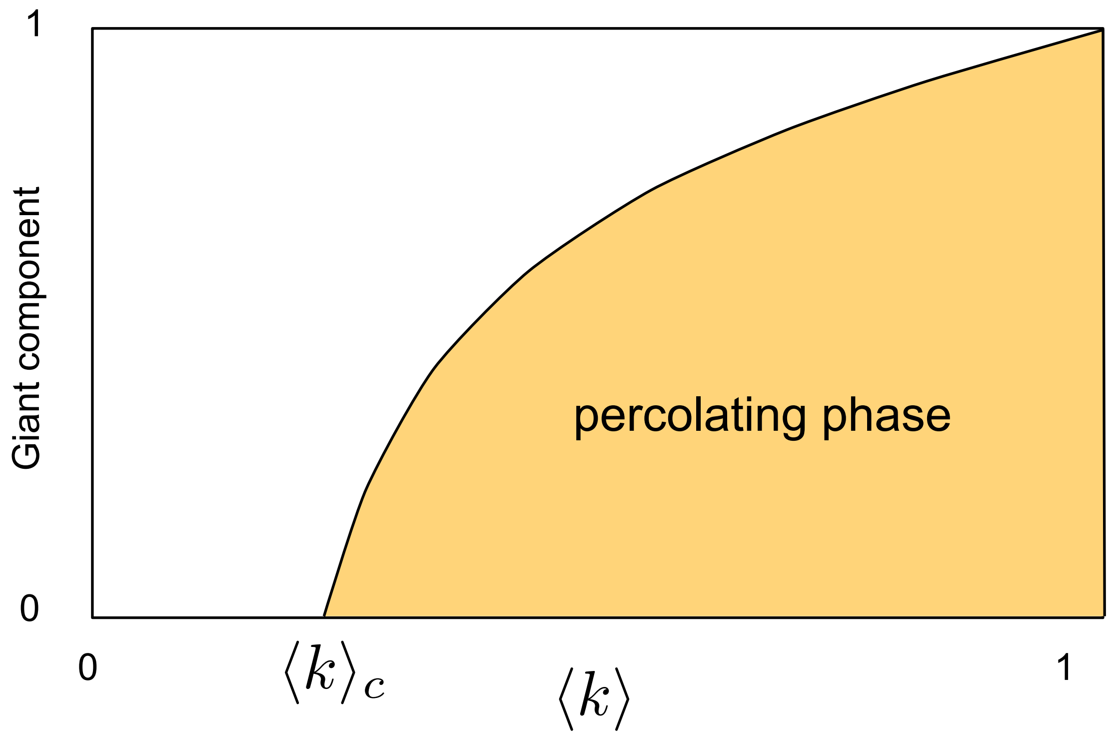

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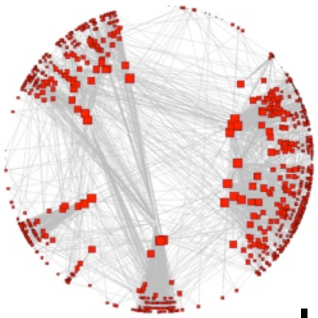

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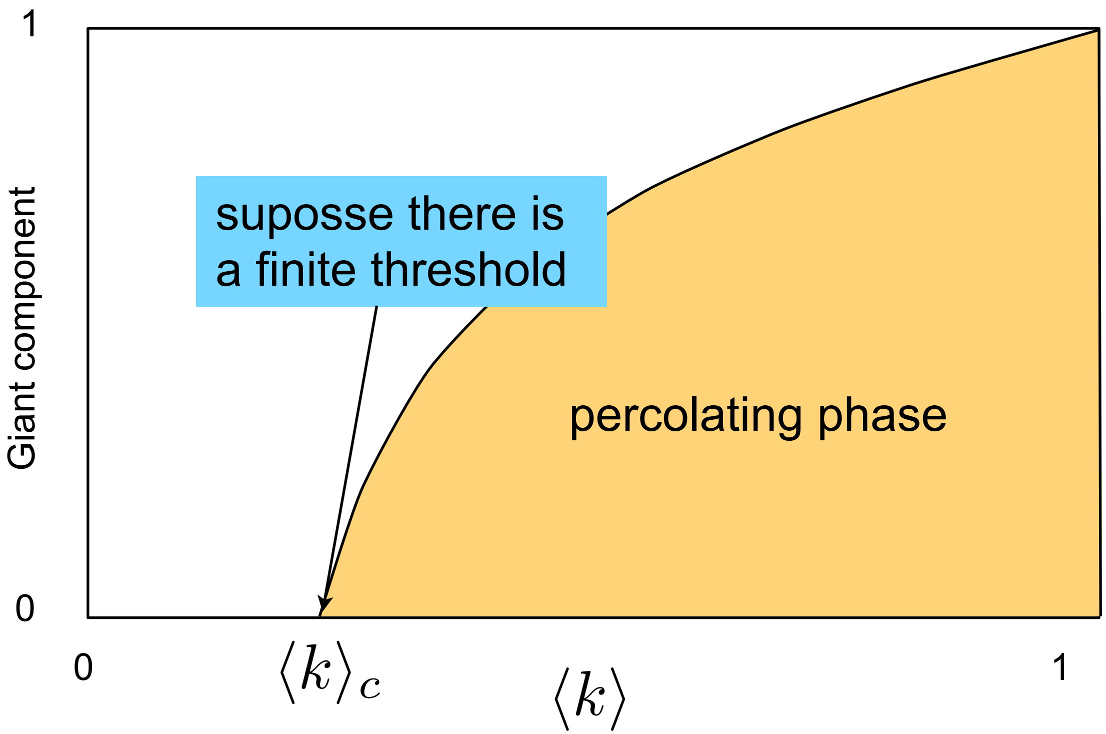

# Aplications: percolation threshold

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we have a graph here

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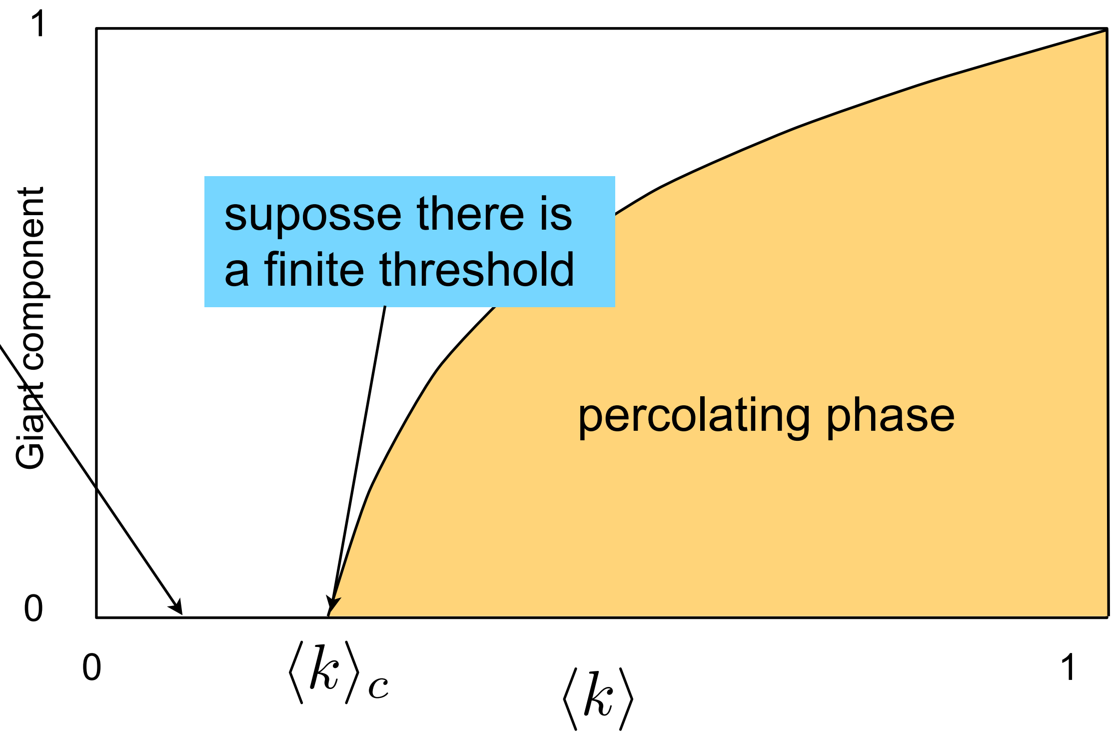

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If $2 < \gamma < 3$ the ave[...]
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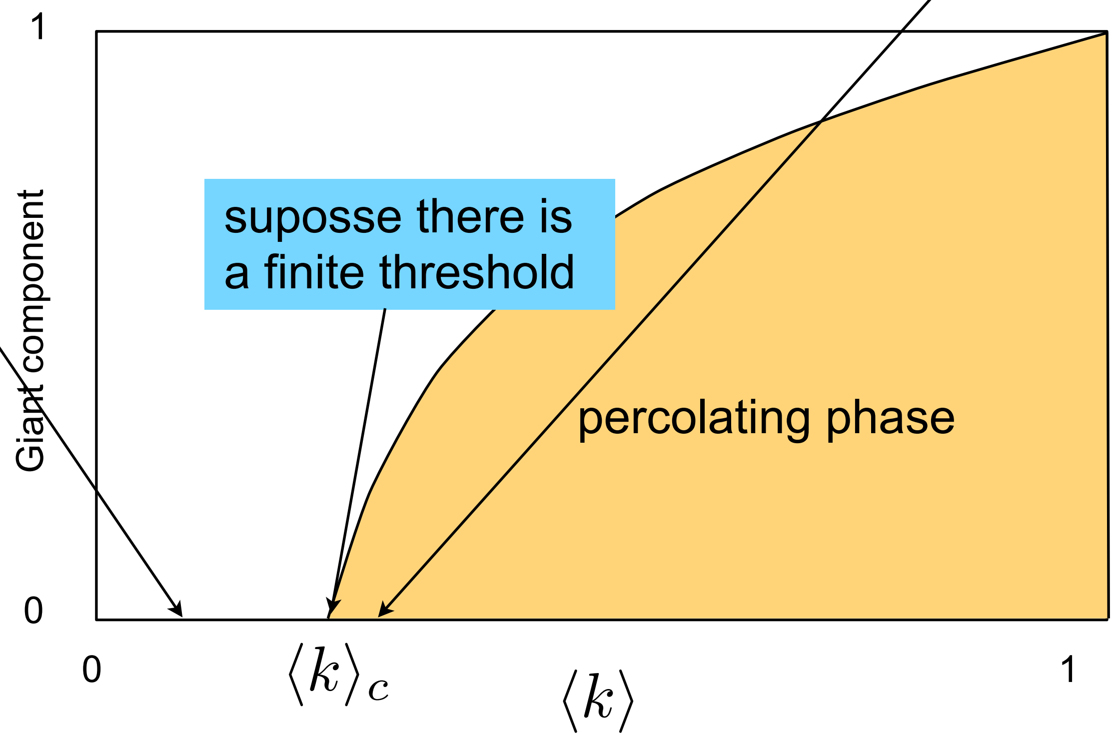
There exists a subgraph with this average degree, which is therefore percolated



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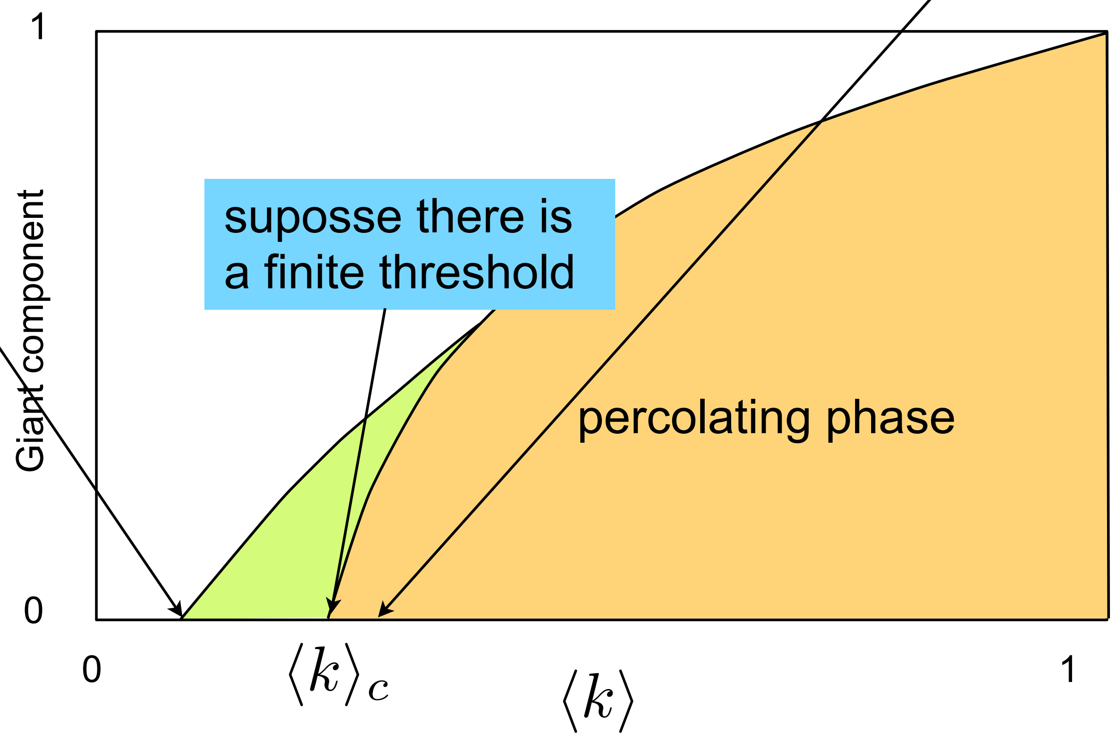

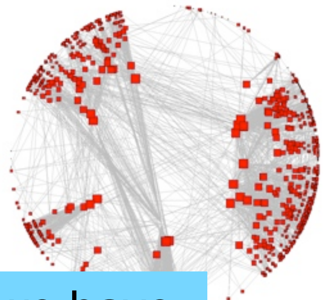

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we have a graph here

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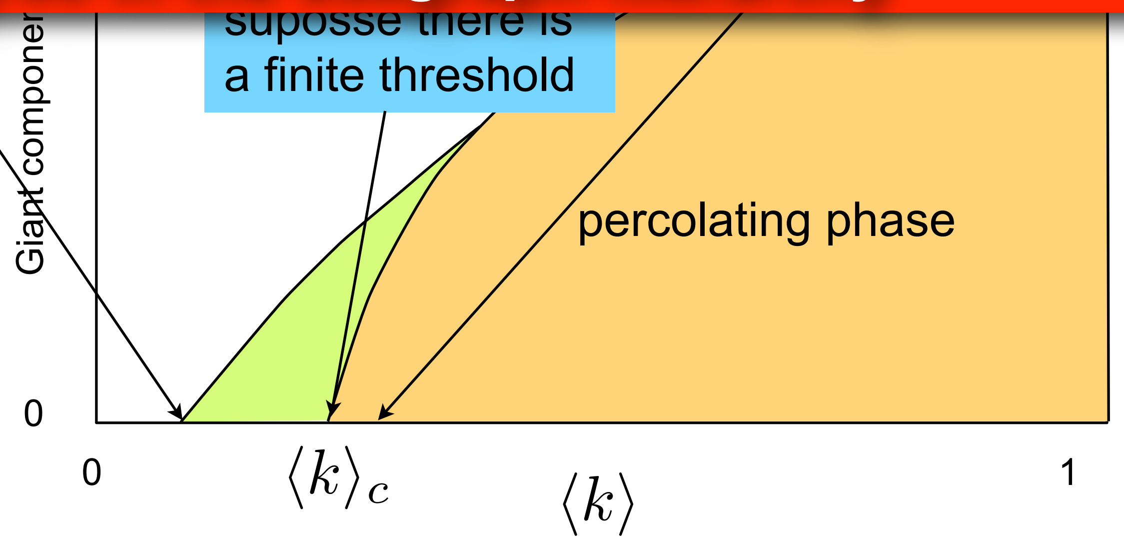
There exists a subgraph with this average degree, which is therefore percolated

subgraphs is bigger than in the original graph

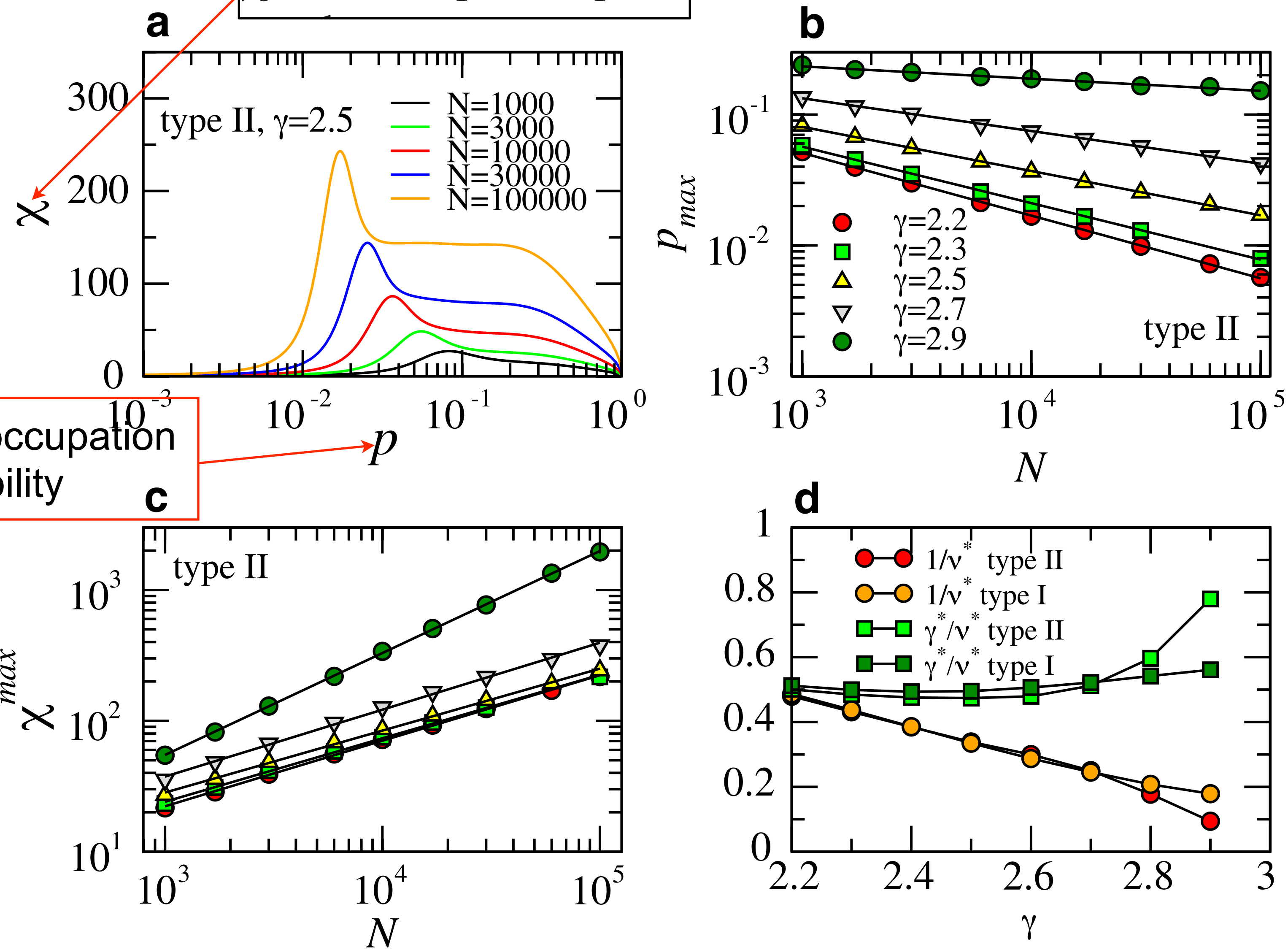
**The percolation threshold is zero in self-similar networks with increasing average degree in the subgraph hierarchy**



# Simulation results: bond percolation

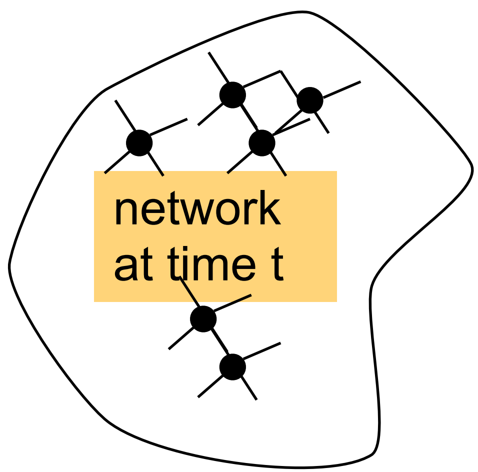
$$\chi = \sqrt{\langle (S_1 - \langle S_1 \rangle)^2 \rangle}$$

size of the giant component




bond occupation probability

Typical growing network models are self-similar by construction under a transformation that selects nodes older than a certain age



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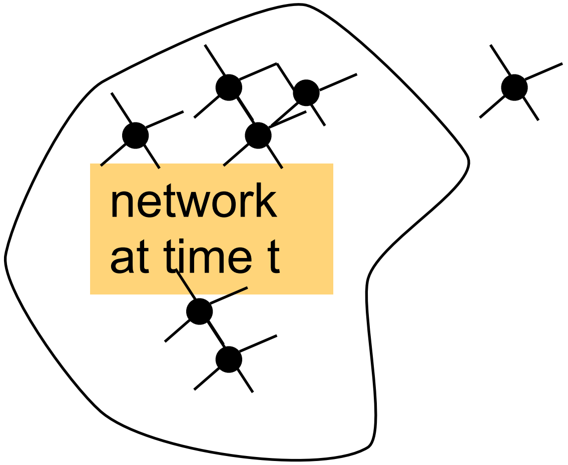

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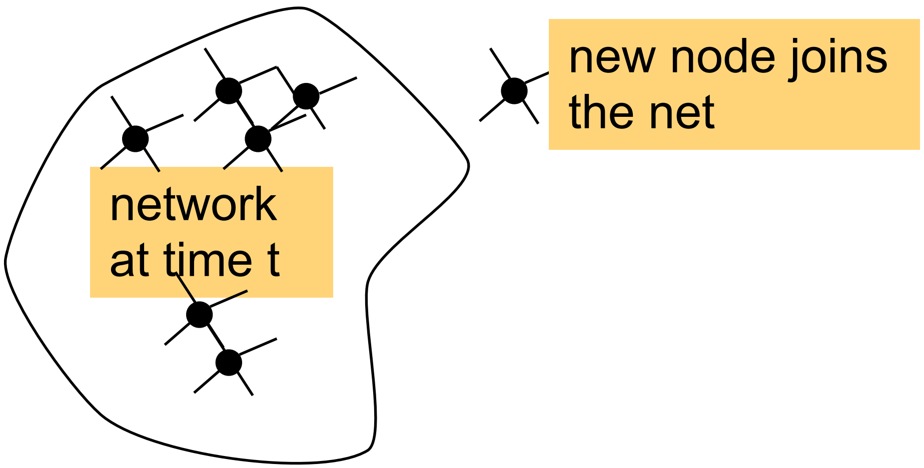

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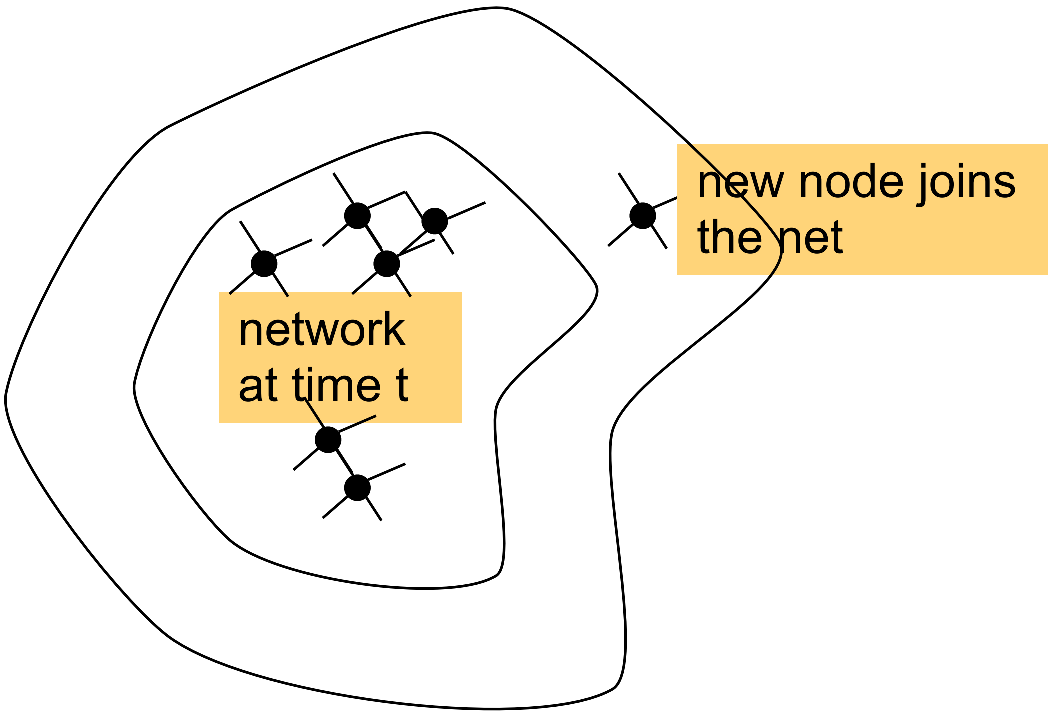

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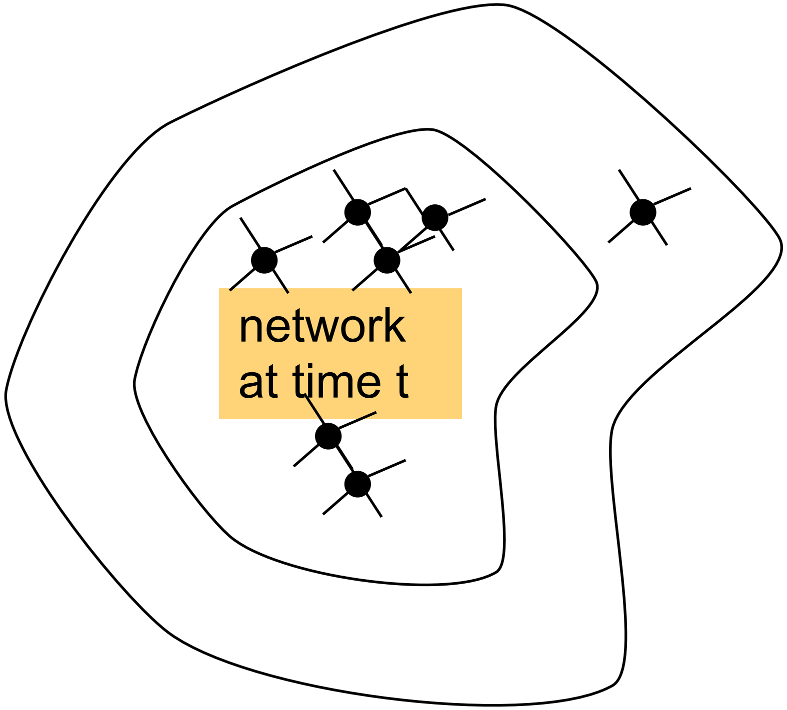

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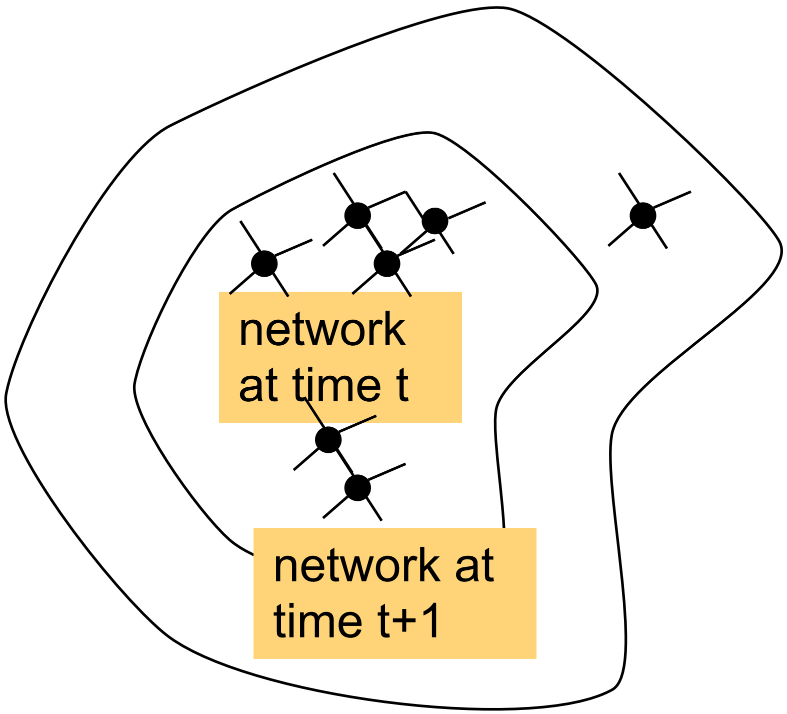

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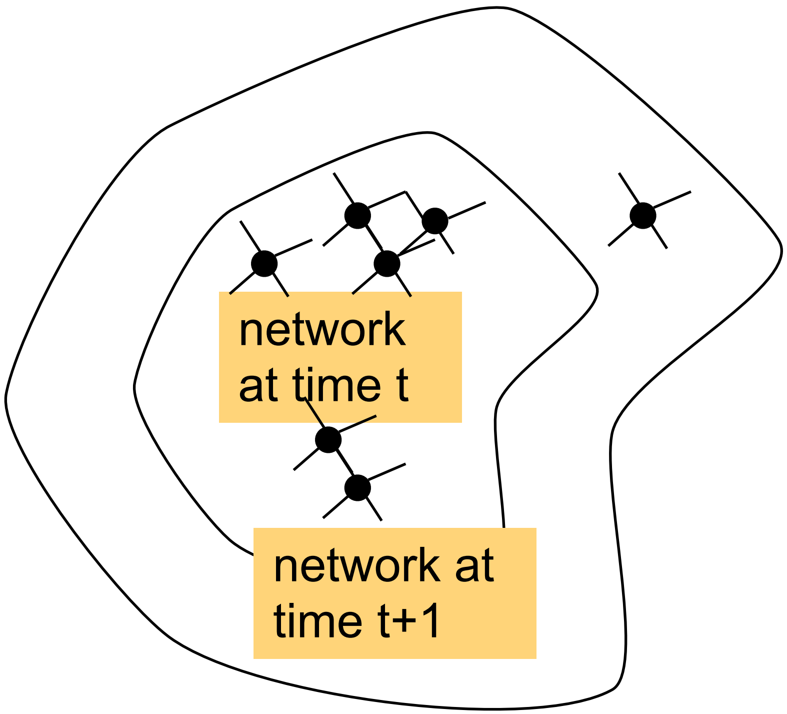
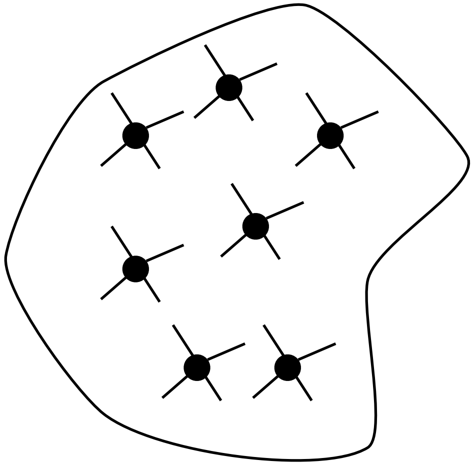



but are designed to have a constant average degree within self-similar subgraphs

# Models: type III



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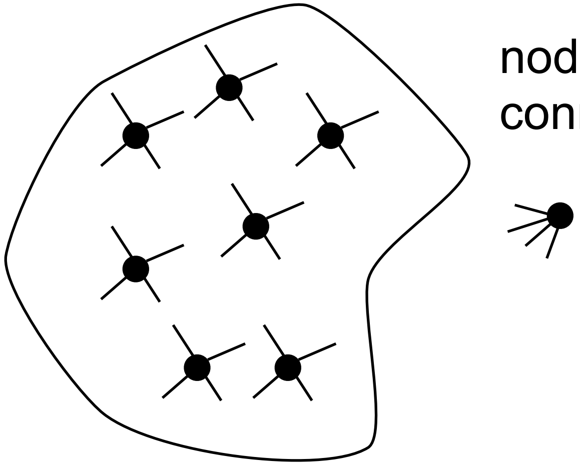
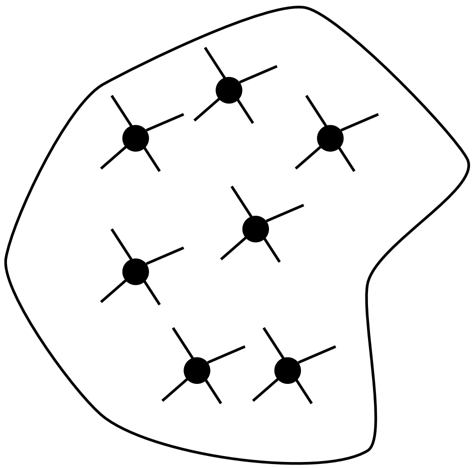


node *i* appearing at time *i* with $m_i = m_0 \left(\frac{N}{i}\right)^{\eta}$ new connections $(0 < \eta < 1)$

# Models: type III



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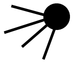

the node connects to randomly chosen existing nodes

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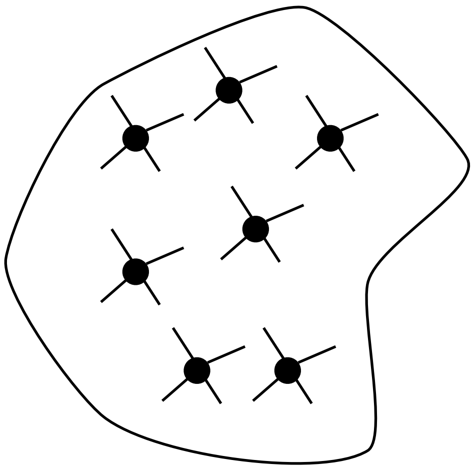

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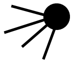

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if $\eta = 0$ the network has an exponential degree distribution

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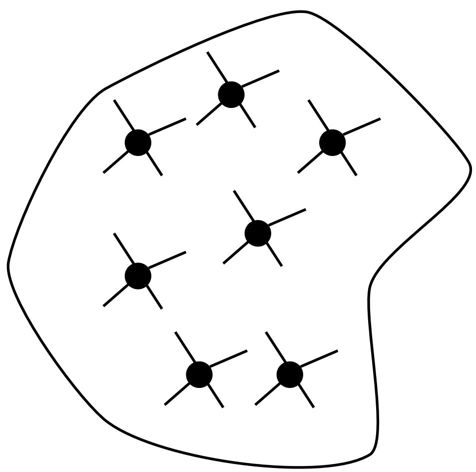

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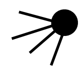

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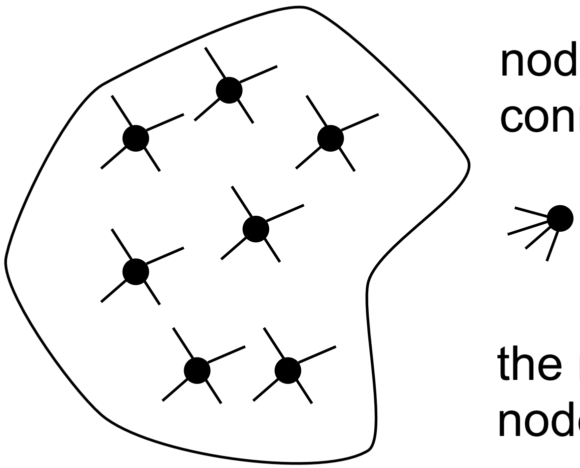
if $\eta > 0$ the degree distribution is power law with exponent $\gamma = 1 + 1/\eta$

# Models: type III



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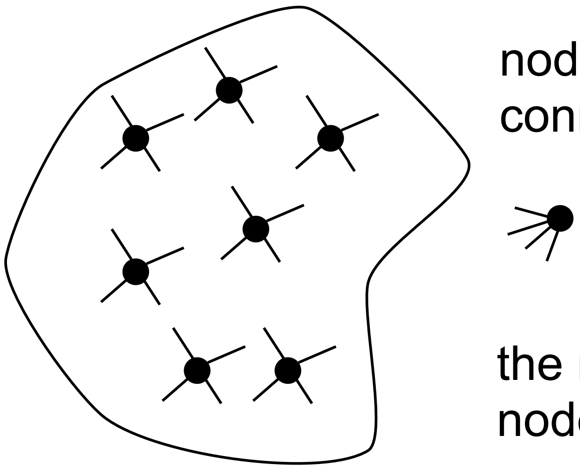

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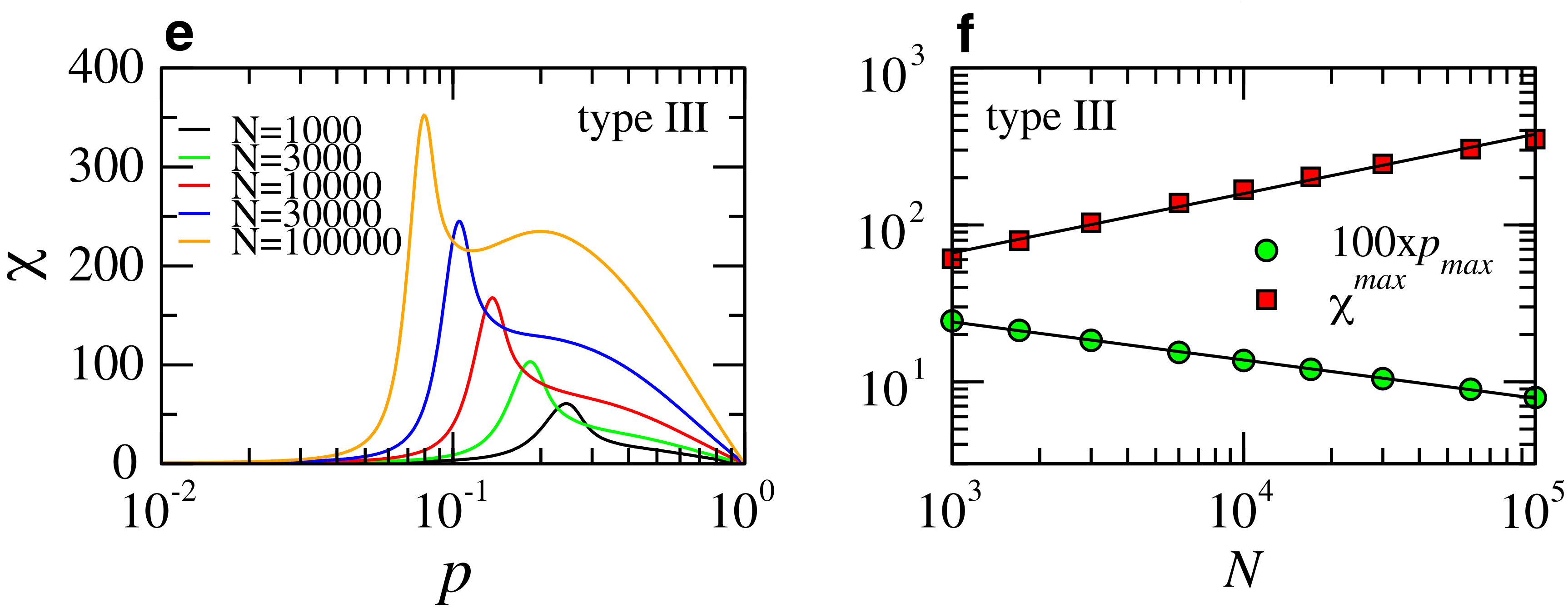
$$\langle k \rangle \rightarrow \langle k \rangle_T = \langle k \rangle \left(\frac{N}{N_T}\right)^{1/(\gamma-1)}$$

# Simulation results: type III

$$\eta = 1/4 \ \ (\gamma = 5)$$



# Summary

- Self-similarity is observed in many real networks and model ensembles
- It provides a minimalistic proof of the absence of percolation threshold in a very large number of network models
- The proof can be applied to any phase transition where the threshold is a monotonous function of the average degree

M. A. Serrano, D. Krioukov, and M. Boguñá, Phys. Rev. Lett. 100, 078701 (2008)

M. Boguñá, F. Papadopoulos, and D. Krioukov, Nat. Comm. 1, 62 (2010)

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