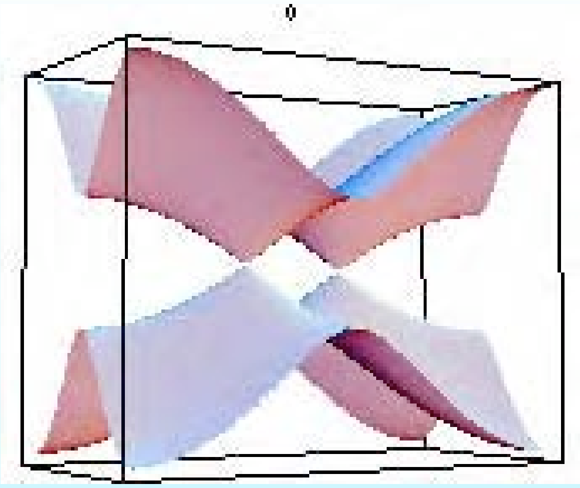
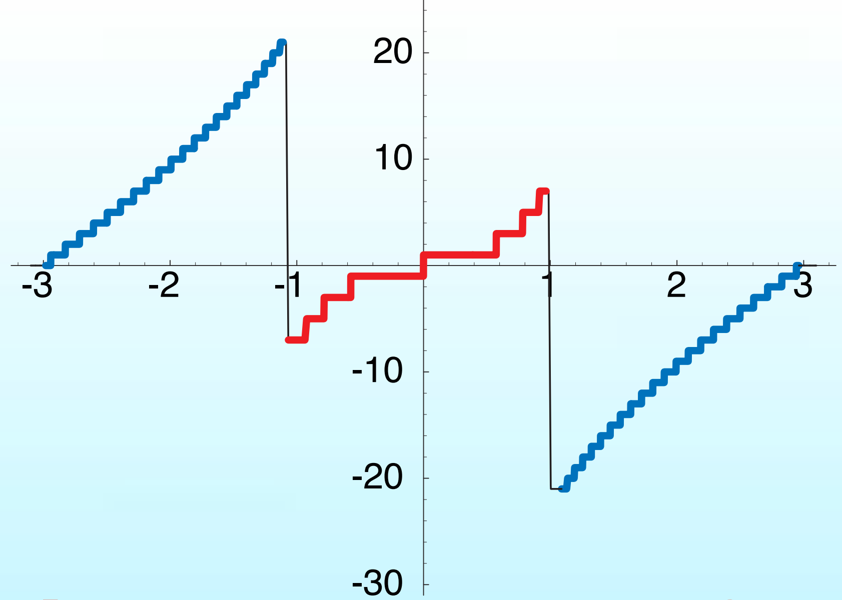
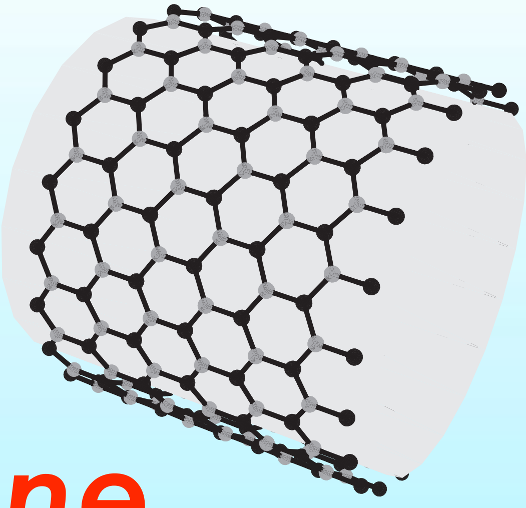






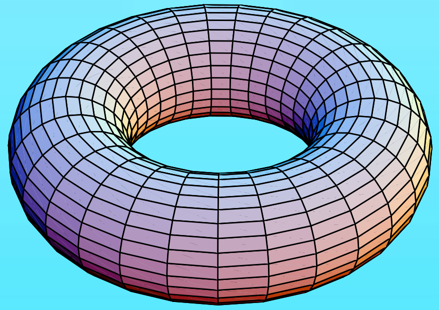
COQUSY06, Sep28, 2006

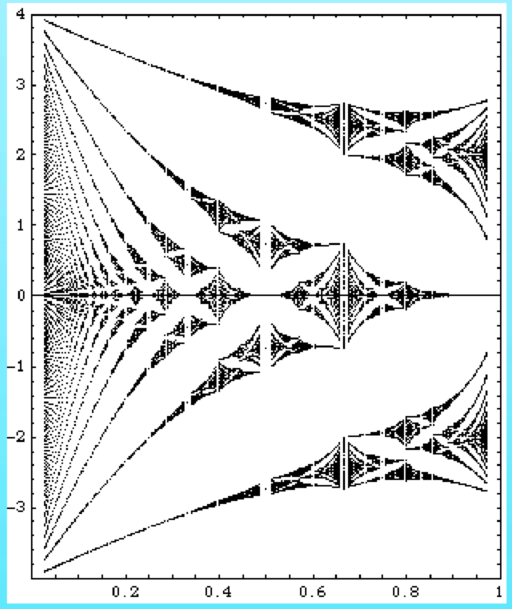
Topological Aspects of Graphene

Dirac Fermions and Bulk-Edge Correspondence in a Magnetic Field

Department of Applied Physics
University of Tokyo
Y. Hatsugai

with T. Fukui (Ibaragi U.)
H. Aoki (U. Tokyo)





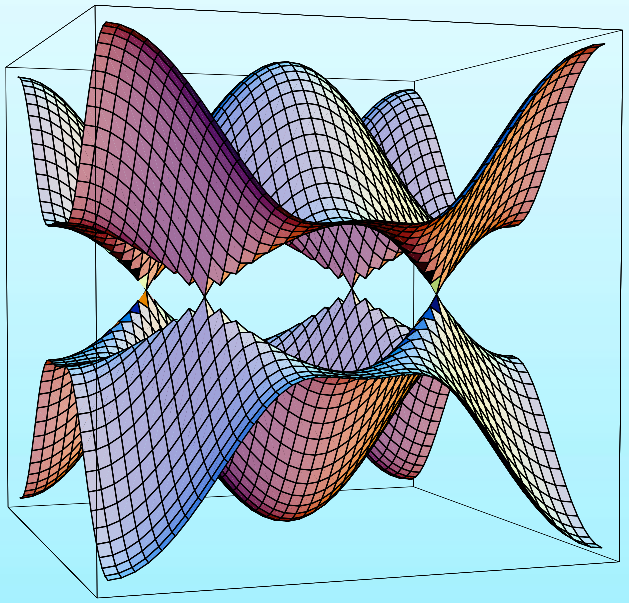
Ref. Y. Hatsugai, T. Fukui and H. Aoki,
to appear in Phys. Rev. B, cond-mat/0607669

Today's Talk

- Graphene as a basic platform of Dirac Fermions
  - Massless Dirac Fermions in Condensed Matter Physics
  - Anomalous Quantum Hall Effect (QHE) in Graphene
- Topological Aspects of Graphene (Bulk)
  - Topological Stability of the Dirac Fermions
  - Topological Stability of the Anomalous QHE
    - Adiabatic Principle and Topological Equivalence
    - Quantum phase Transition by chemical potential shift
    - Technical development for calculating Chern numbers (Lattice Gauge Theory)
- Topological Aspects of Graphene (Edge)
  - Without Magnetic field
    - Topological Origin of Zero Modes in Graphene (c.f. d-wave superconductors)
  - With Magnetic field
    - Edge States of Dirac Fermions
- Bulk – Edge Correspondence (Analytical & Numerical)
  - Edge States and Bloch States (complex energy structure)

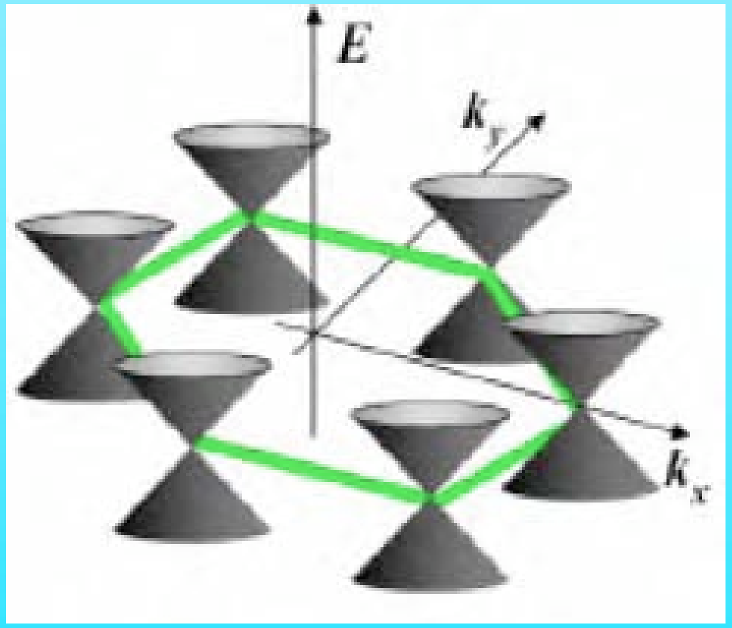
Massless Dirac Fermions in Condensed Matter

- Gapless Superconductor with point Nodes



d-wave superconductivity

- Graphene as a 2D Carbon sheet



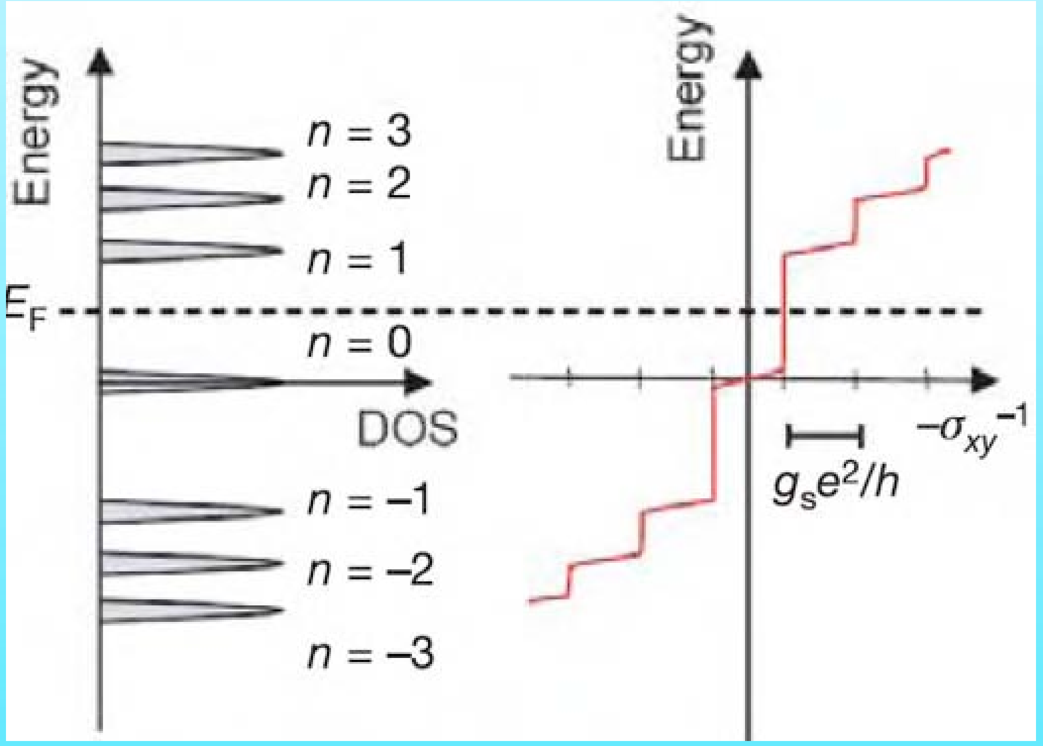
Wallace (1946)

Fig.Zhang et al. (2005)

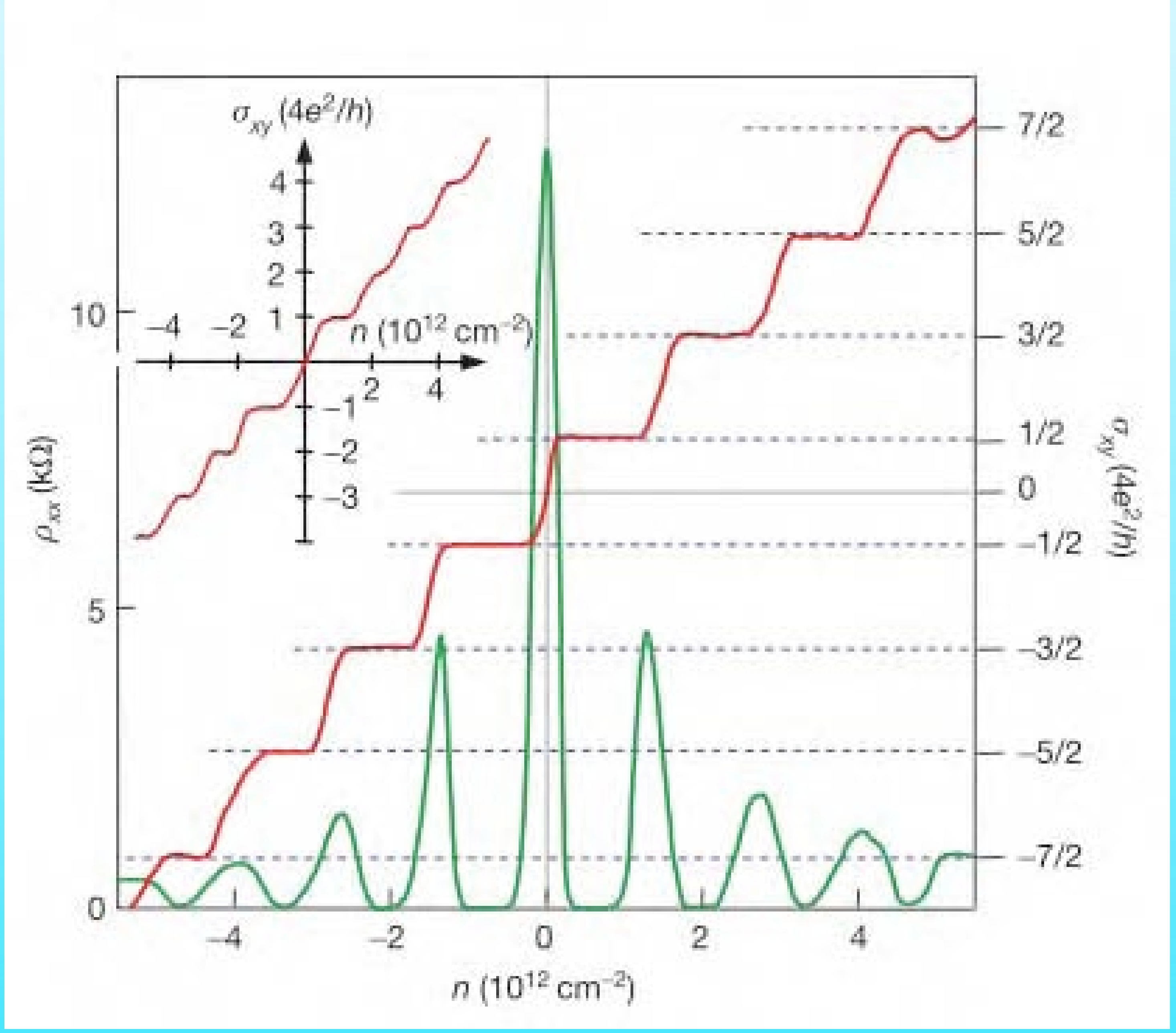
Observation of Anomalous QHE in Graphene

- Anomalous QHE of gapless Dirac Fermions

$$\sigma_{xy} = \frac{e^2}{h}(2n+1), \;\; n = 0, \pm 1, \pm 2, \cdots$$
$$= 2\frac{e^2}{h}(n + \frac{1}{2})$$



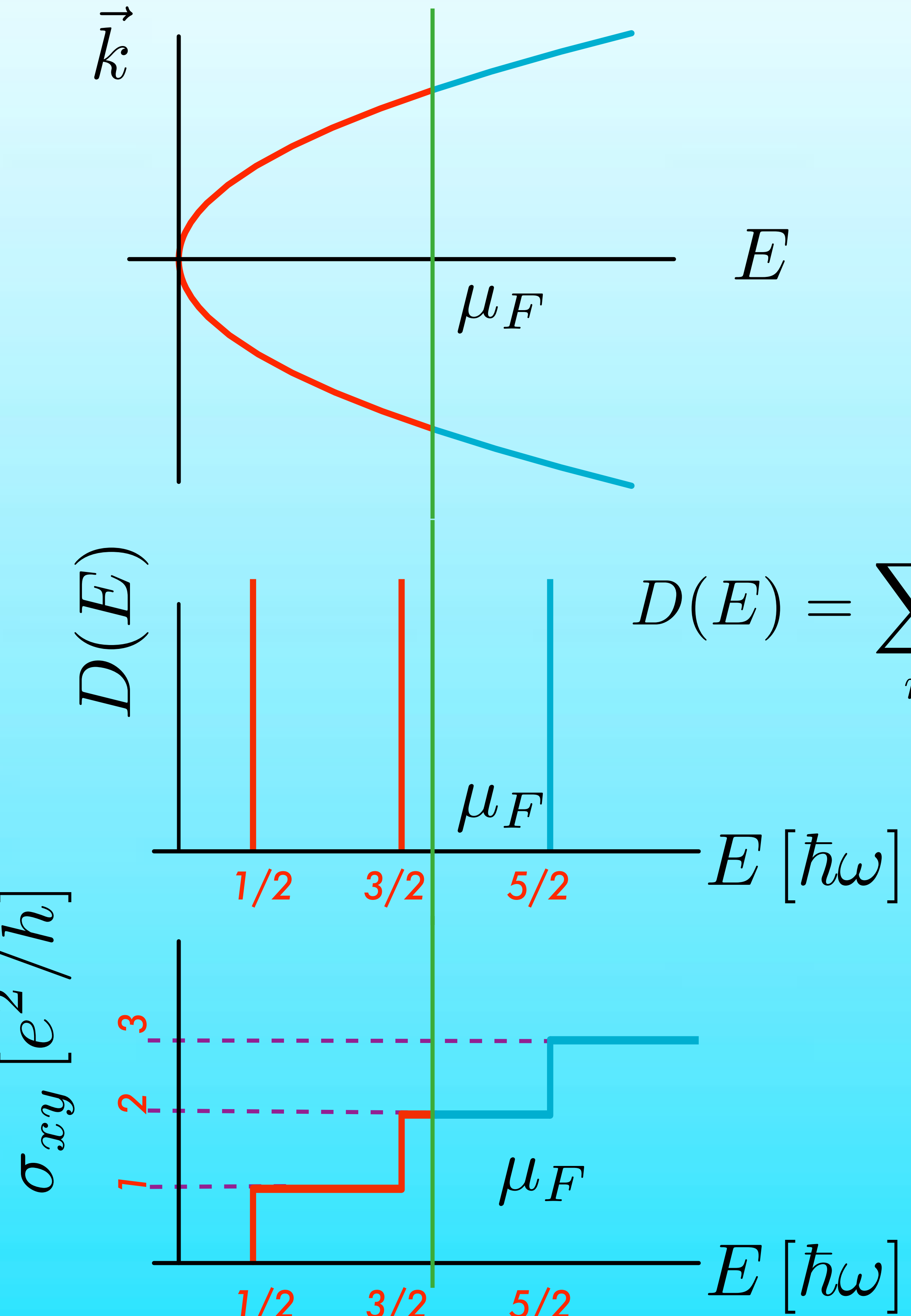
Zhang et al. Nature 2005



Novoselov et al. Nature 2005

Conventional QHE

☆ Landau Level and Integer QHE



$$E(B=0) = \frac{\hbar^2}{2m}k^2$$

$$D(E) = \sum_n \delta(E - \epsilon_n) \qquad \epsilon_n = \hbar\omega(n + \frac{1}{2})$$

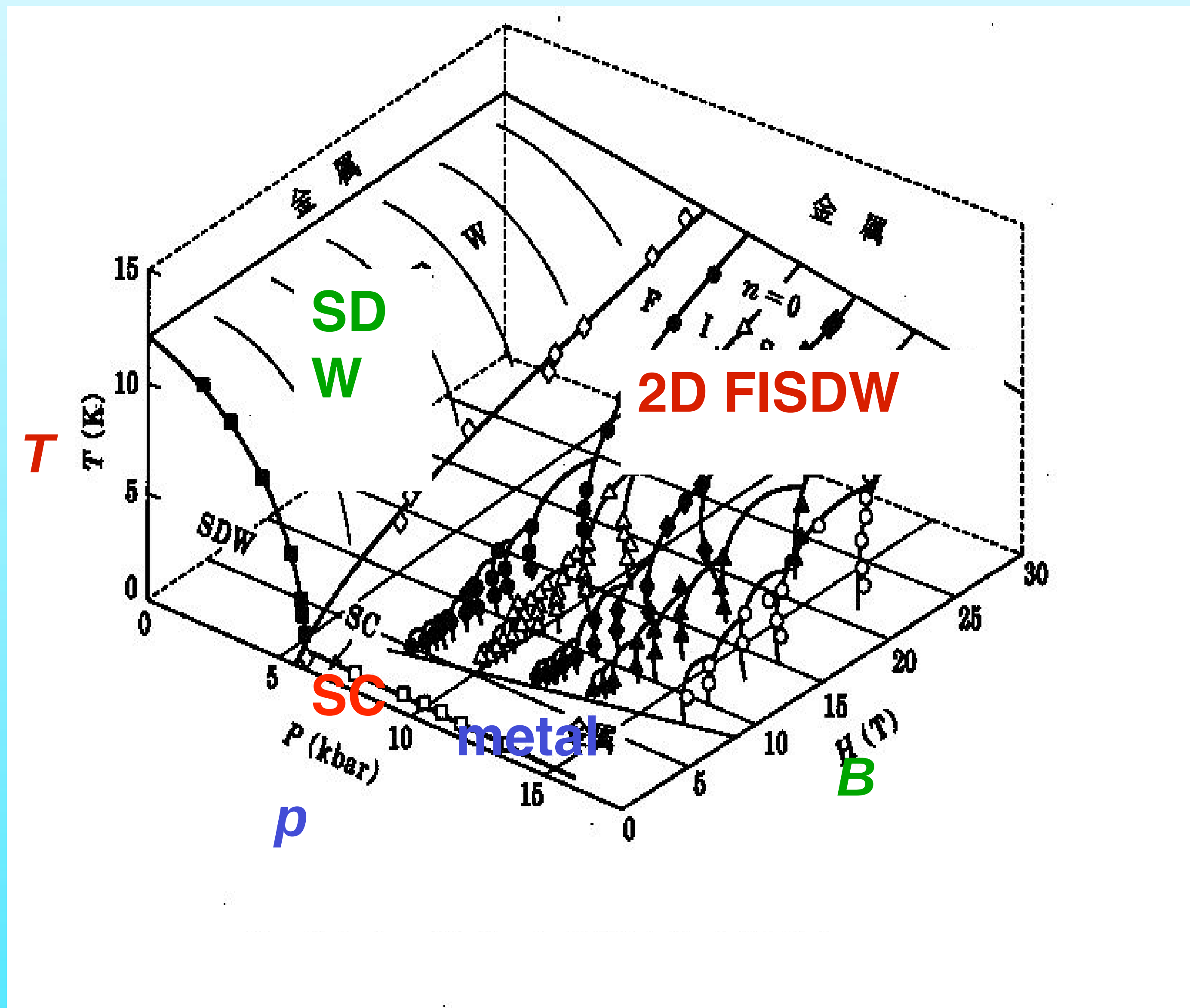
$$\sigma_{xy}(\mu_F) = \frac{e^2}{\hbar} n, \quad n = 1, 2, 3, \cdots$$

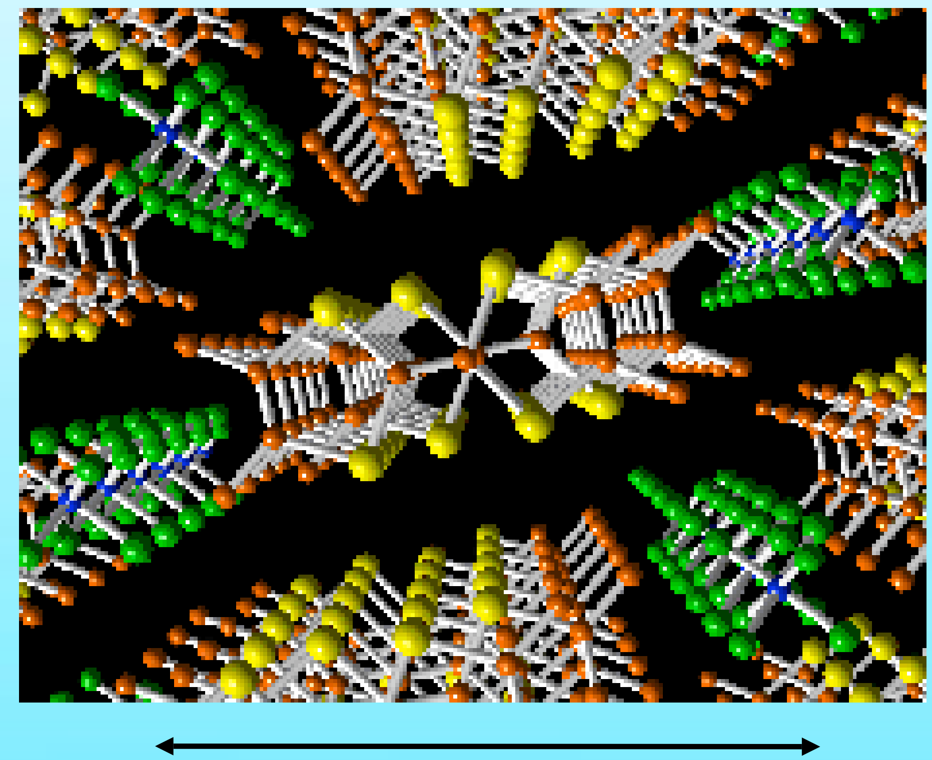
$$\epsilon_{n-1} < \mu_F < \epsilon_n$$

QHE and Band Structures

☆ QHE of electrons and holes

2D organic metal $(TMTSF)_2PF_6$ (Chaikin et al)

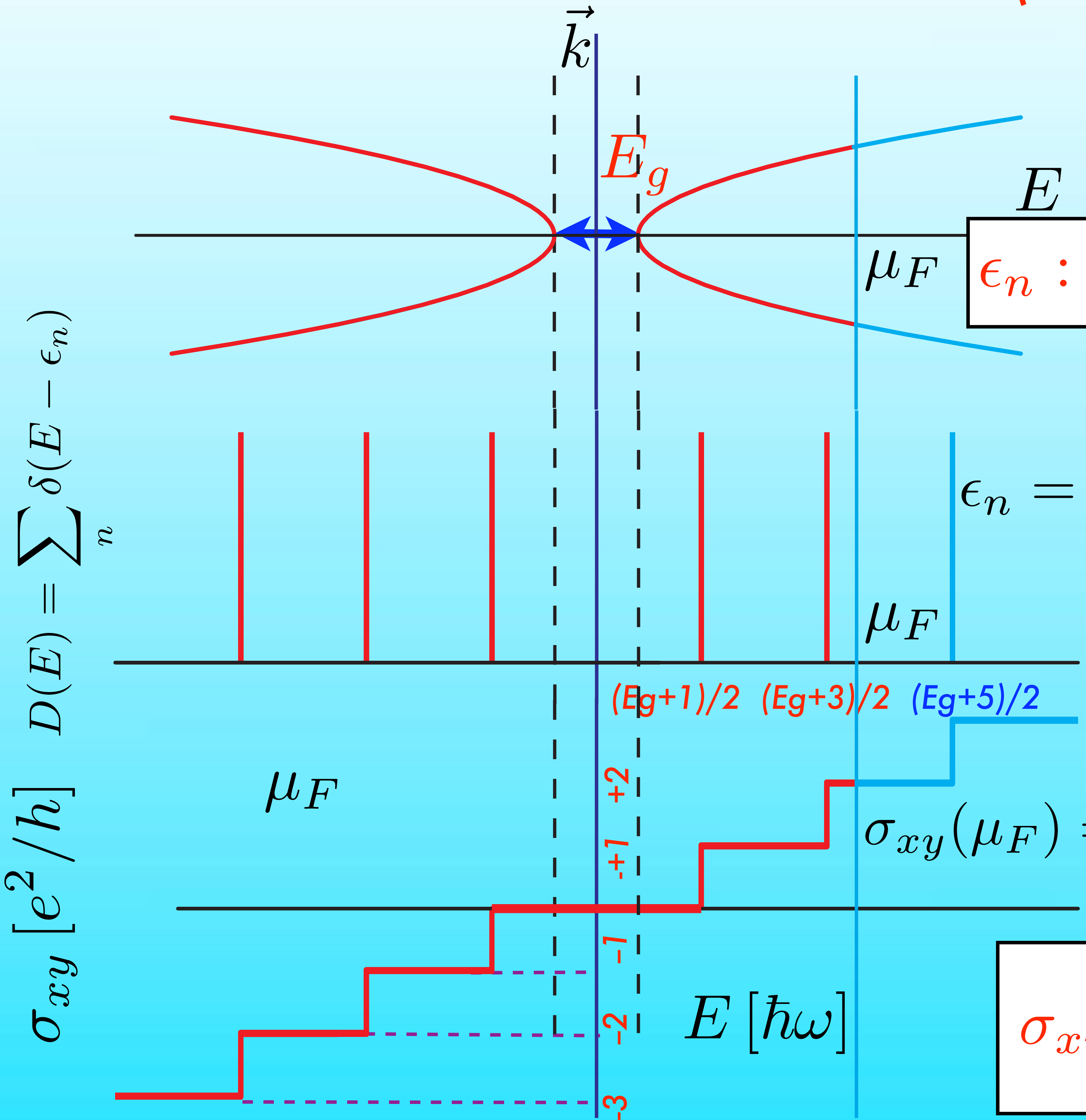




Lab. de Physique des Solides

QHE of Semiconductors

- Landau Level of Conduction band (Electrons)



$$E(B=0) = \pm \frac{\hbar^2}{2m} k^2$$

ϵ_n : zero point energy 1/2

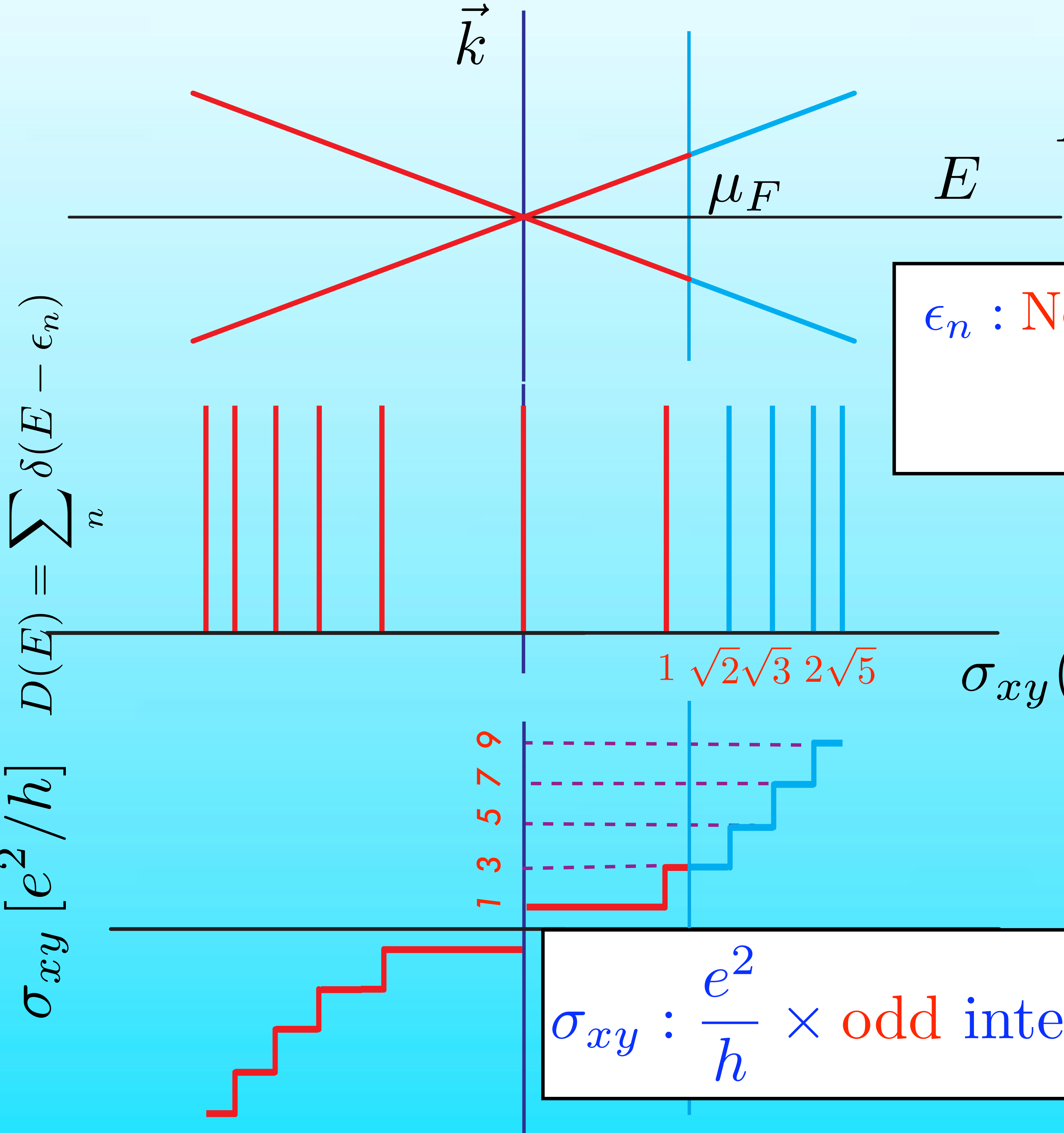
$$\epsilon_n = \begin{cases} E_g/2 + \hbar\omega(n + \frac{1}{2}) & n \geq 0 \\ -E_g/2 + \hbar\omega(n + \frac{1}{2}) & n < 0 \end{cases}$$

$$\sigma_{xy}(\mu_F) = \frac{e^2}{\hbar} \times \begin{cases} n & (n > 0) \\ n+1 & (n \leq 0) \end{cases}$$

$\sigma_{xy} : \frac{e^2}{h} \times$ integer

QHE of Graphene (Gapless Semiconductor)

☆ Landau Level of Doubled Dirac Fermions



$$E(B=0) = \pm \hbar c |\vec{k}|$$

ϵ_n : No zero point energy shift

$$\epsilon_n = \pm C\sqrt{Bn}$$

McClure, 1956

$$\sigma_{xy}(\mu_F) = \frac{e^2}{\hbar}(2n+1)$$

$$n = 0, 1, 2, 3, \cdots$$

$$\epsilon_{n-1} < \mu_F < \epsilon_n$$

Zheng-Ando 2002

Gusynin-Sharapov, 2005

Peres-Guinea-Neto 2006

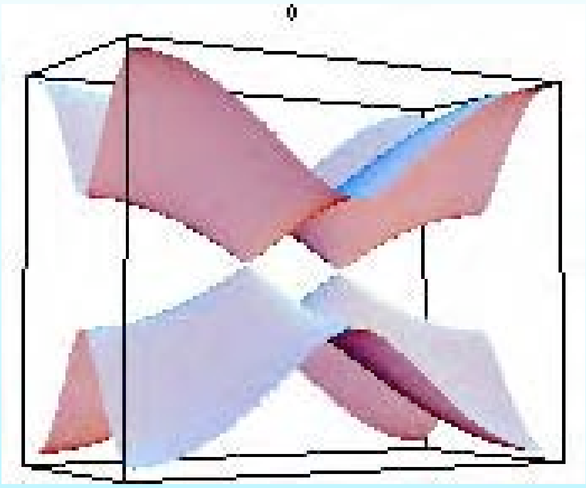
...

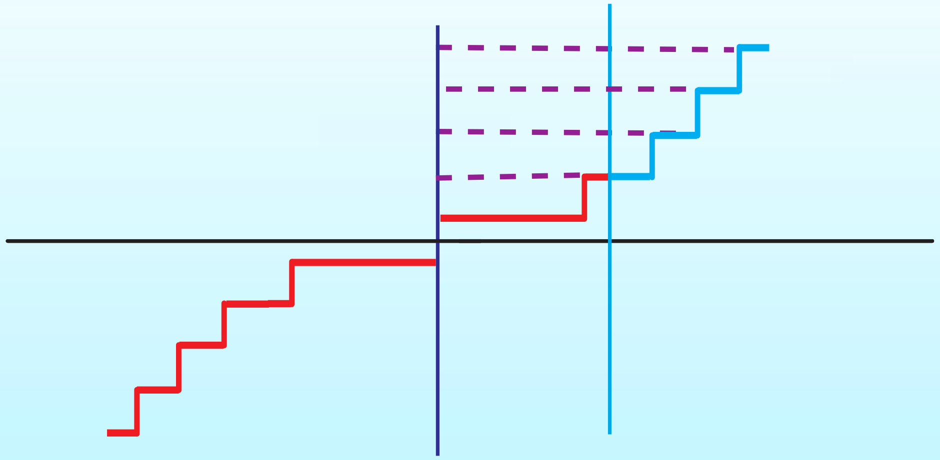
$\sigma_{xy} : \frac{e^2}{h} \times$ odd integer

Many Relevant Papers

- 89 Matches on Graphene in cond-mat in the past year
- Experiments and theories

Motivations Here





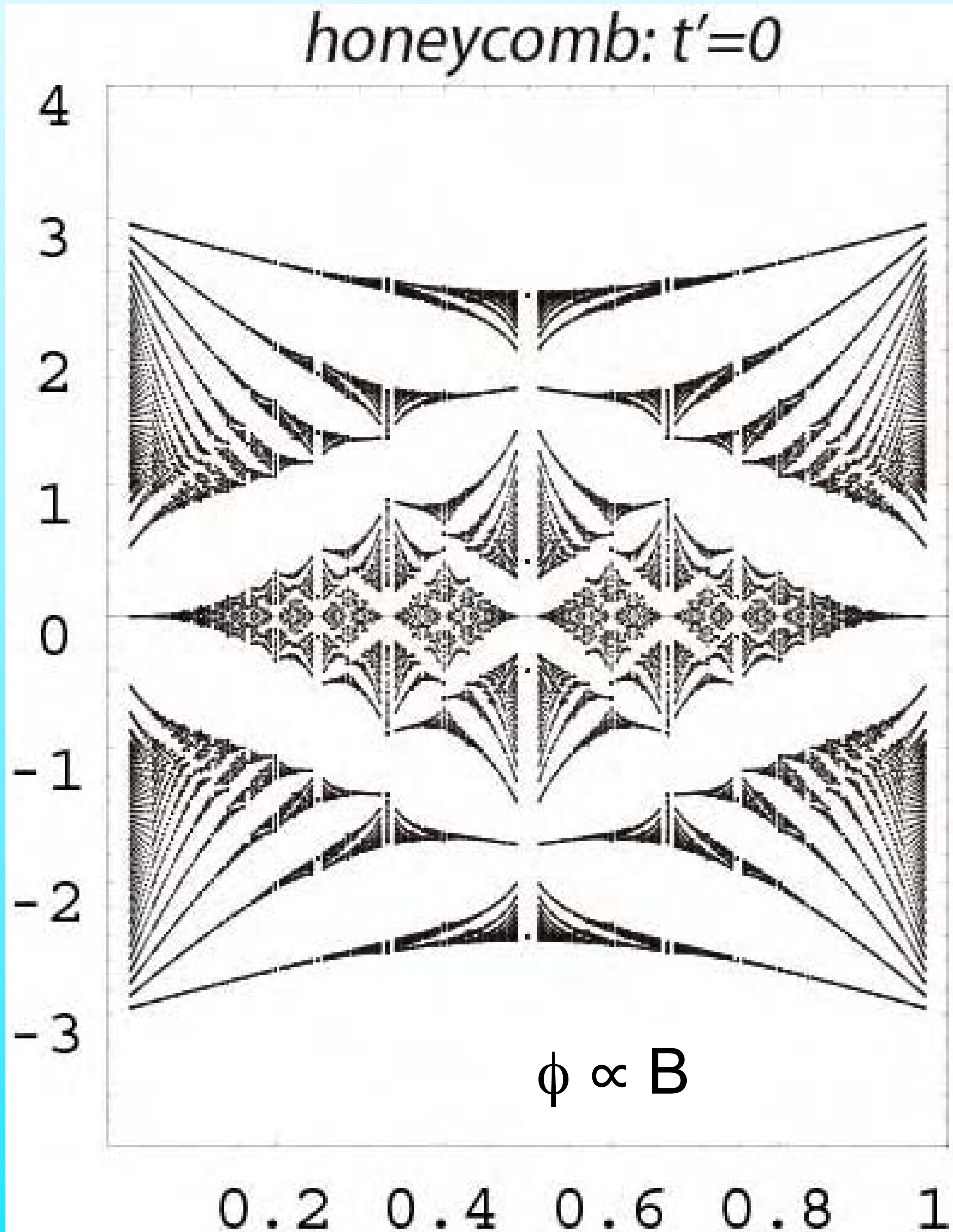
- *How does the Anomalous QHE persist for higher Energy?*
  - *Quantum Phase Transition?*
- *Is it specific to the honeycomb lattice?*
- *Edge State?*
  - *How do the edge states look like?*
  - *Edge States & Topological Numbers*
  - *How about the bulk-edge correspondence?*

Laughlin 81
Halperin 82

Hatsugai 1993

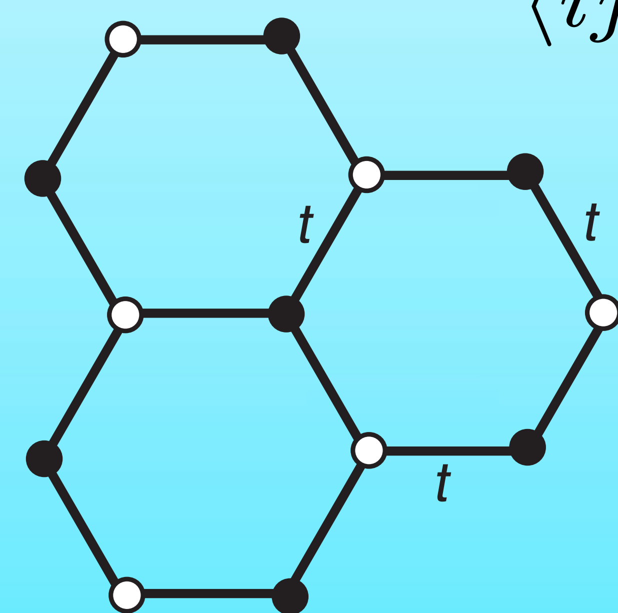
Hofstadter diagram for the honeycomb

- *Tight-binding model on a honeycomb lattice*



$$H = \sum_{\langle ij \rangle} t_{ij} e^{i\theta_{ij}}$$

$$2\pi\phi_P = \sum_{\langle ij \rangle \in P} \theta_{ij}$$



$$\phi_P = \phi = \frac{p}{q}$$

$$(p, q) = 1$$

Rammal 1985

E=0 Landau level :
outside Onsager's semiclassical quantisation scheme

Bulk σ_{xy} by the topological invariant

- Hall conductance by Chern number

Counting vortices in the band

$$\sigma_{xy}^j = \frac{e^2}{h}\sum_{\ell=1}^{j} C_\ell, \quad C_\ell = \frac{1}{2\pi i}\int_{BZ} dA_\ell, \quad A_\ell = \langle \psi_\ell | d\psi_\ell \rangle$$

$\epsilon_\ell(k) < \mu_F, \ \ell = 1, \cdots, j$

Thouless-Kohmoto-Nightingale-den Nijs 1982
with randomness Aoki-Ando 1986

- Integration of the NonAbelian Berry Connection of the "Fermi Sea"

$$\sigma_{xy} = \frac{e^2}{h}\frac{1}{2\pi i}\int_{BZ} \mathrm{Tr}_j \, dA_{\mathrm{FS}}$$

Fermi Sea of j filled bands

$$A_{\mathrm{FS}} = \Psi^\dagger d\Psi, \ \Psi = (\psi_1, \cdots, \psi_j)$$

Hatsugai 2004

- Tolological Invariant on Discretized Lattice

Lattice in k space

Technical Advantage for large Chern Numbers

$$\sigma_{xy} = \frac{e^2}{h}\frac{1}{2\pi i}\sum F_{1234}$$

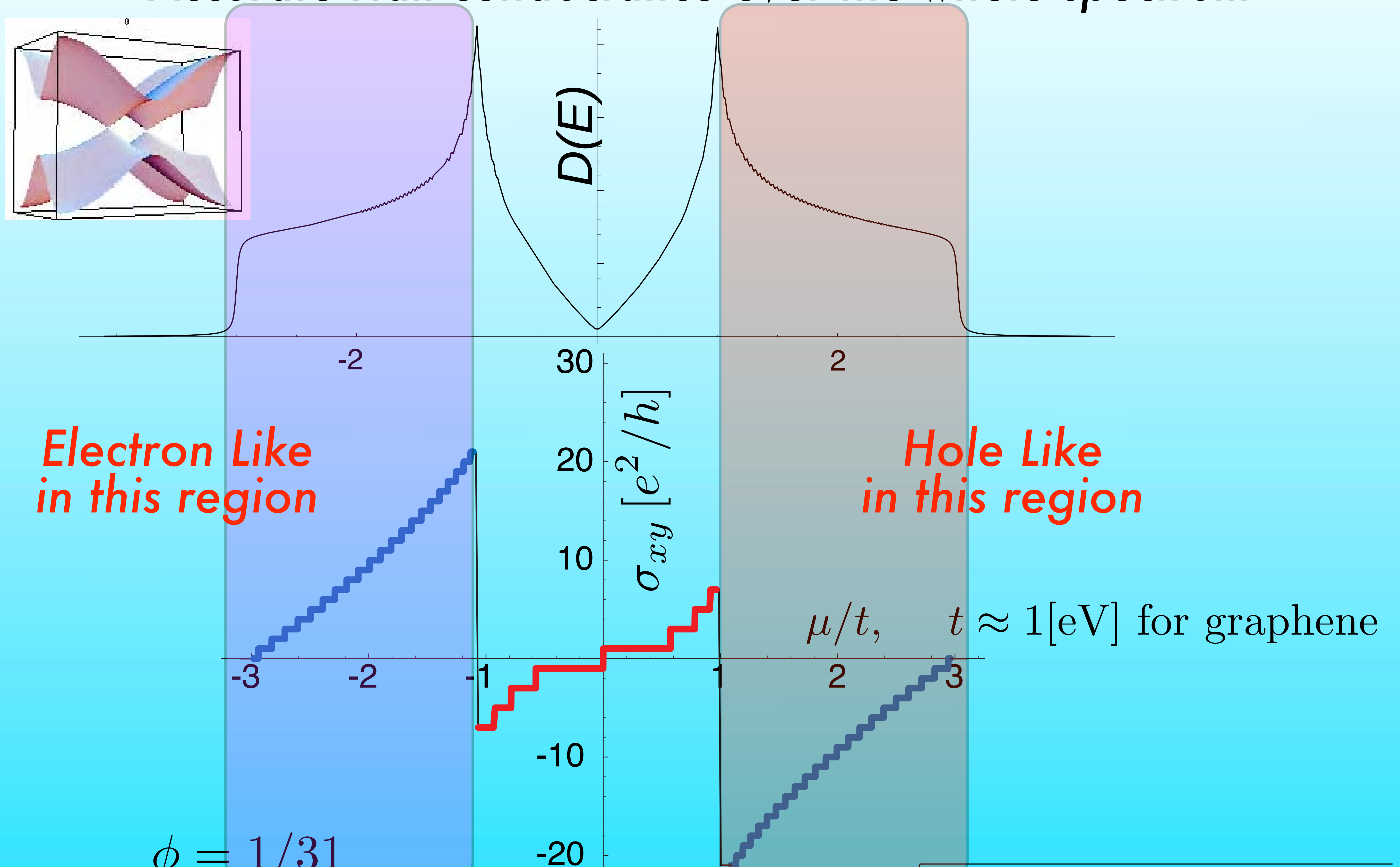
$$F_{1234} = \mathrm{Im}\,\log U_{12}U_{23}U_{34}U_{41}$$

Fukui-Hatsugai-Suzuki 2005

$$U_{mn} = \det_j \Psi_m^\dagger \Psi_n, \quad \Psi_n = (\psi_1(k_n), \cdots, \psi_j(k_n))$$

Hall Conductace vs chemical potential

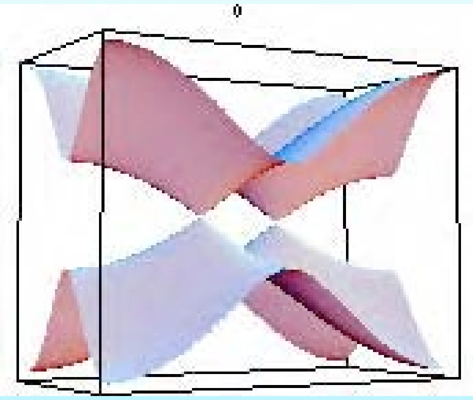
- ☆ Accurate Hall conductance over the whole spectrum

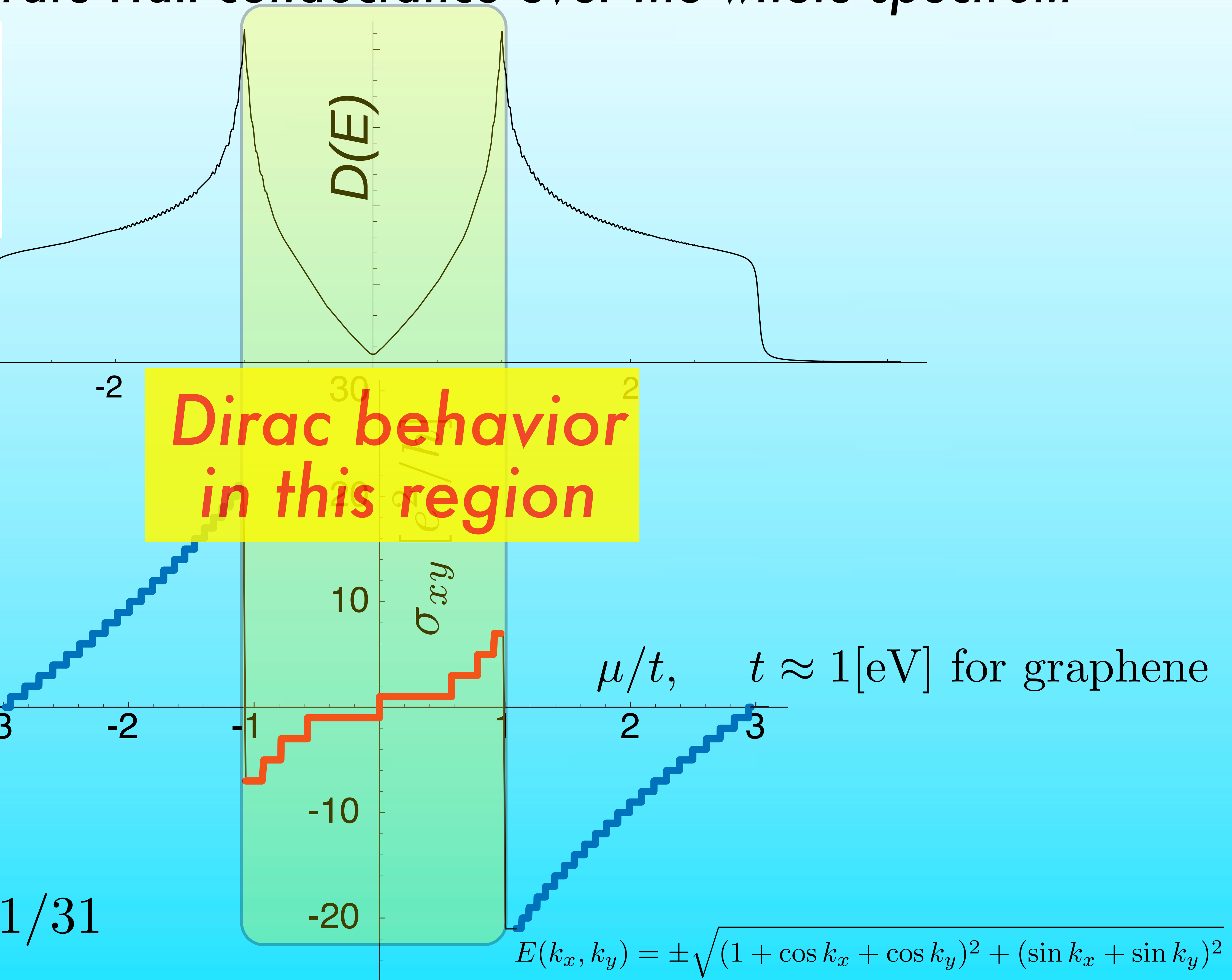


$$E(k_x, k_y) = \pm\sqrt{(1+\cos k_x + \cos k_y)^2 + (\sin k_x + \sin k_y)^2}$$

Hall Conductace vs chemical potential

- Accurate Hall conductance over the whole spectrum





Hall Conductace vs chemical potential

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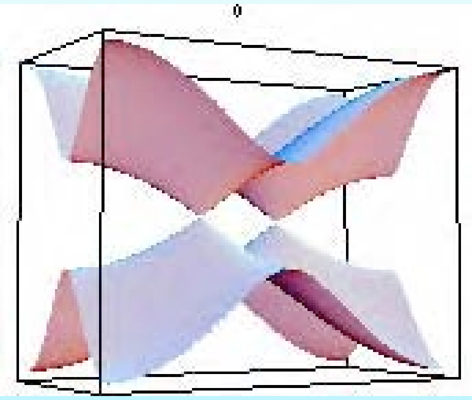
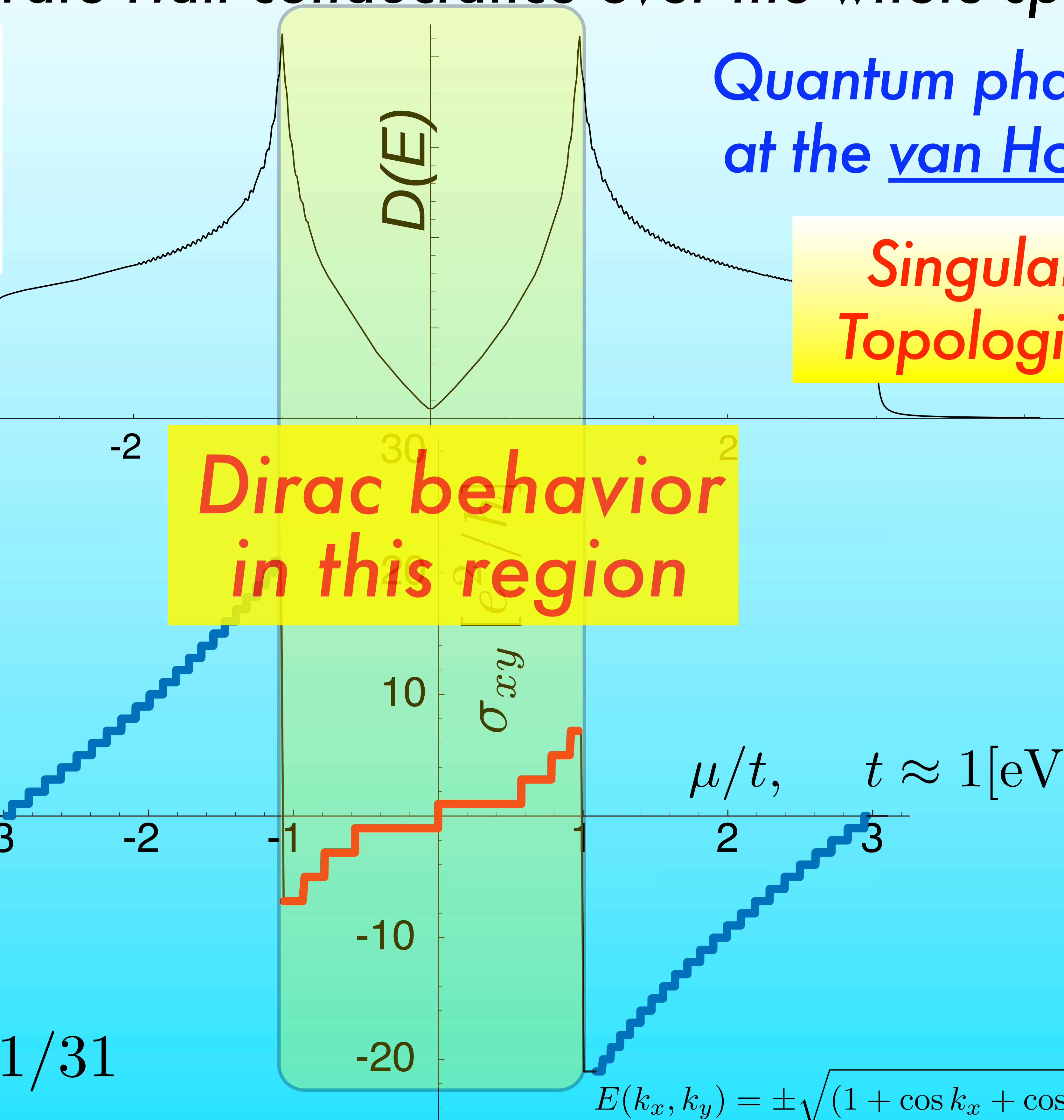




Quantum phase transition at the van Hove Energies

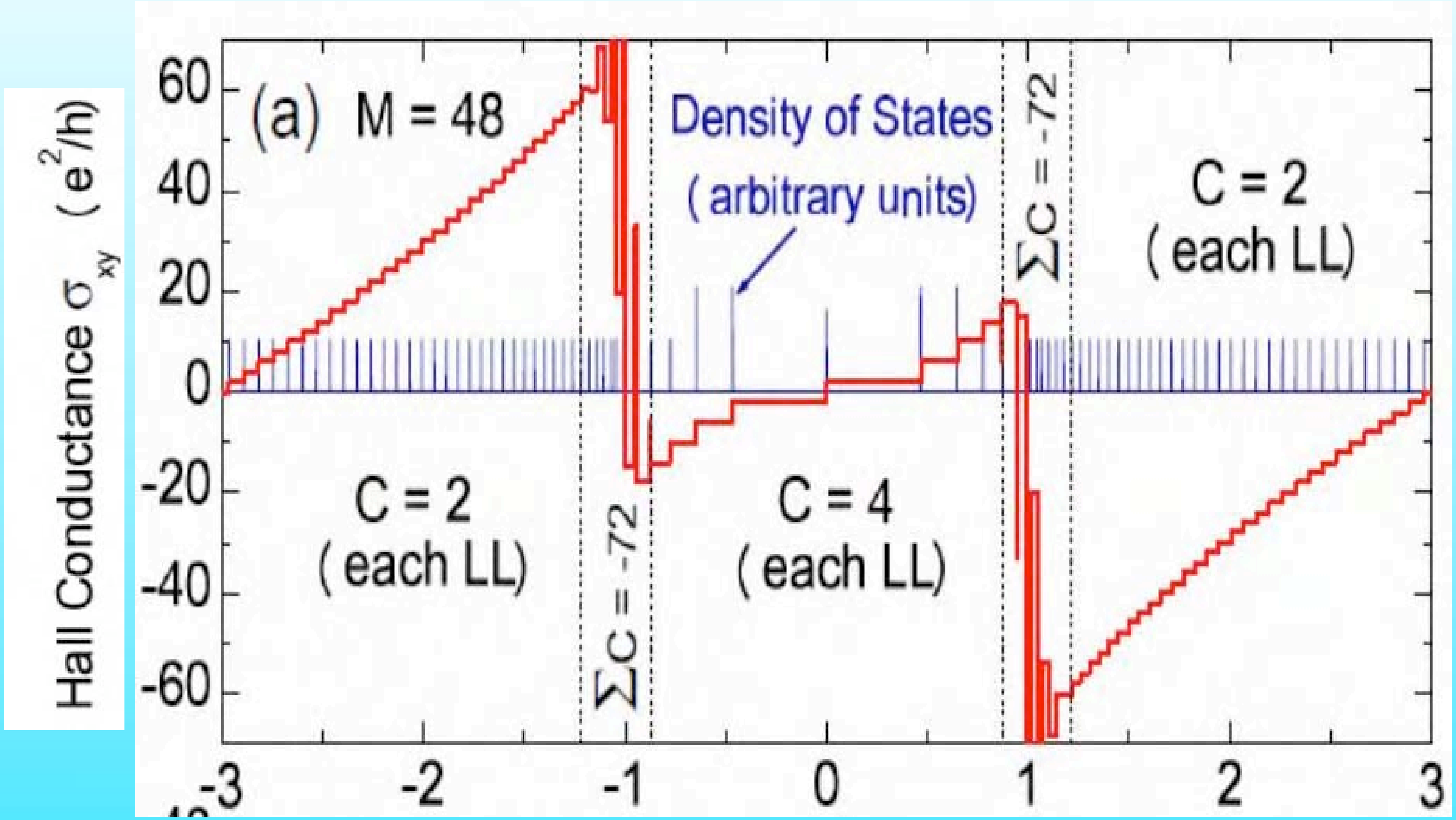
Singularity breaks Topological Stability

$\mu/t, \quad t \approx 1[\mathrm{eV}]$ for graphene

$\phi = 1/31$

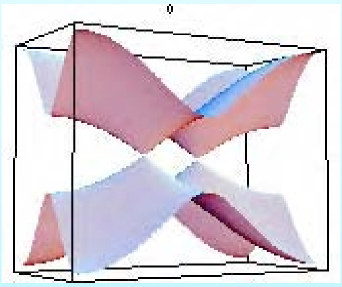
$E(k_x, k_y) = \pm\sqrt{(1 + \cos k_x + \cos k_y)^2 + (\sin k_x + \sin k_y)^2}$

(Sheng et al, cond-mat/0602190)



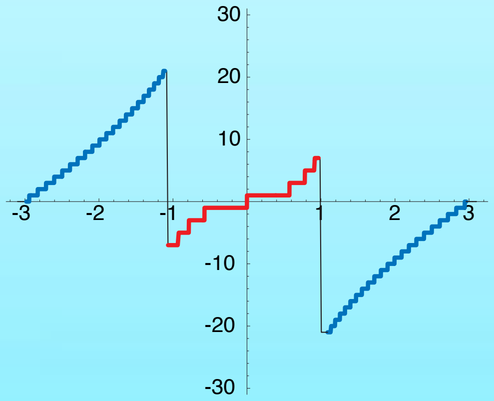
E

Is this anomalous behavior specific to the honeycomb lattice ?



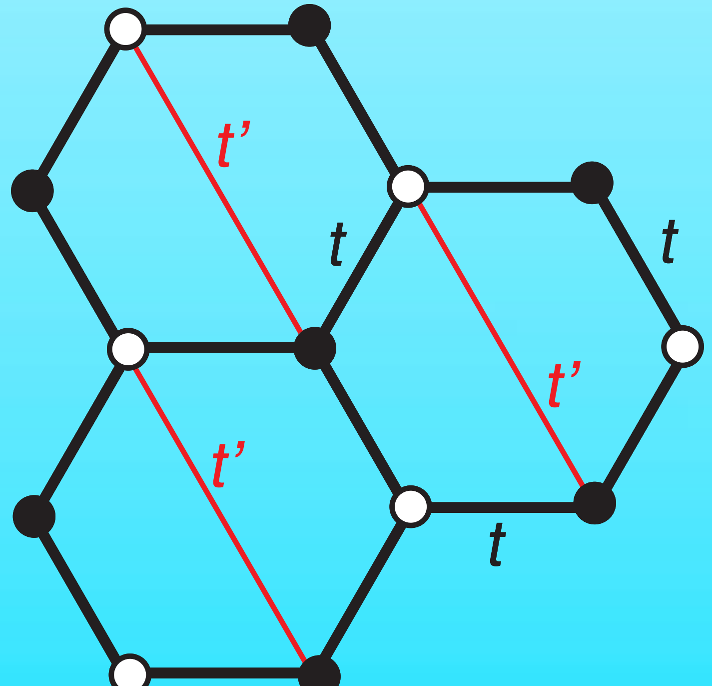
No! : It has topological Stability

- ☆ Vanishing DOS of the Dirac Fermions'
- ☆ Anomalous behavior of the Hall conductance



t'/t = 1 : Square Lattice
t'/t =0 :Honeycomb Lattice
t'/t=-1 : π Flux State

Chiral Symmetry (Bipartite Structure)



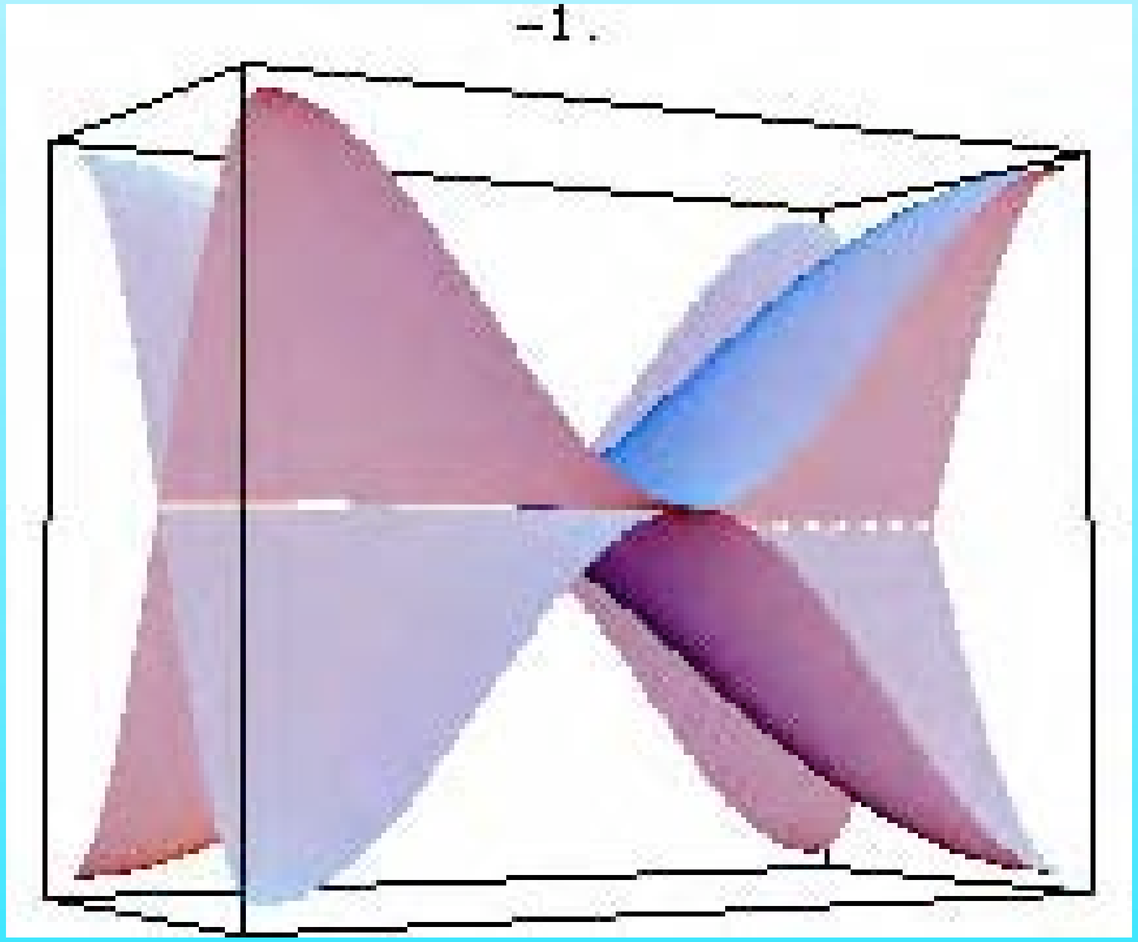
To demonstrate

introduce 2nd nearest neighbor hopping

Dirac Cones are Stable!

- The Dirac Cornes are not accidental
- It has topological stability

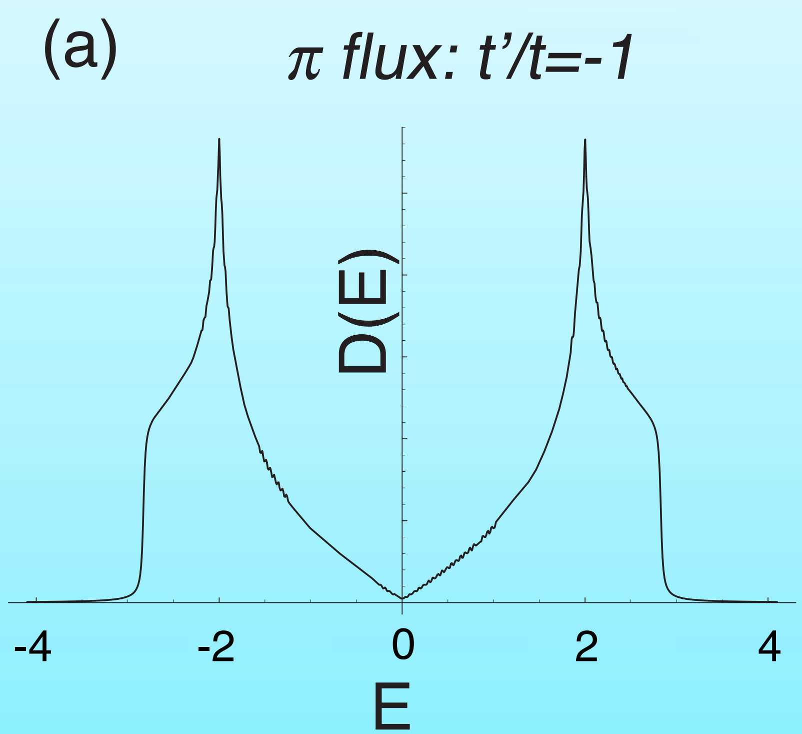
$$-3 < \frac{t'}{t} < 1 \rightarrow \text{Doubled Dirac Cones}$$

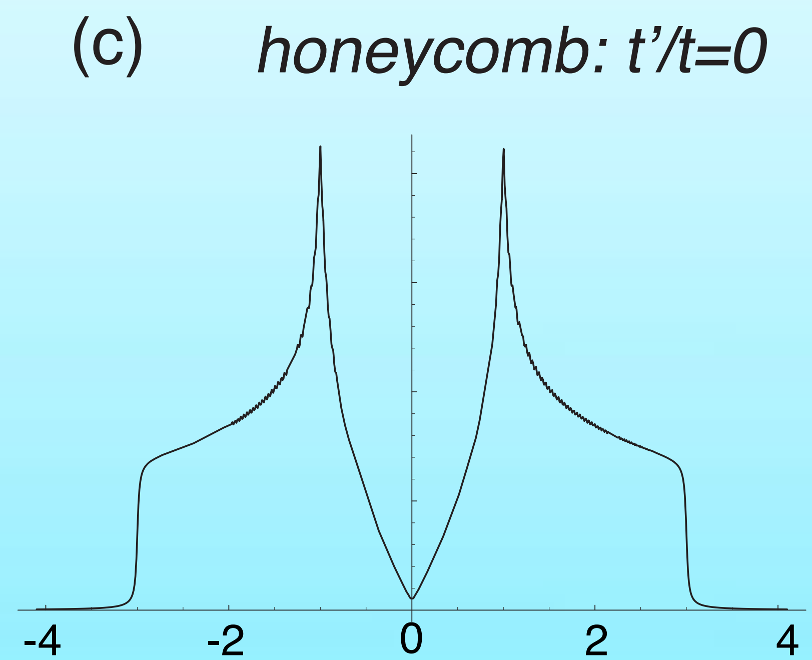


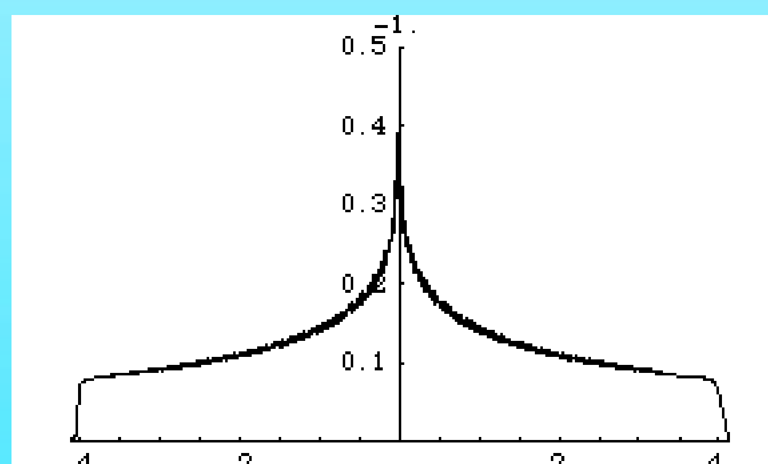
t′/t = 1 : Square Lattice
t′/t =0 :Honeycomb Lattice
t′/t=-1 : π Flux State

Density of States

- ☆ Vanishing DOS near the zero energy







$t'/t = 1$: Square Lattice

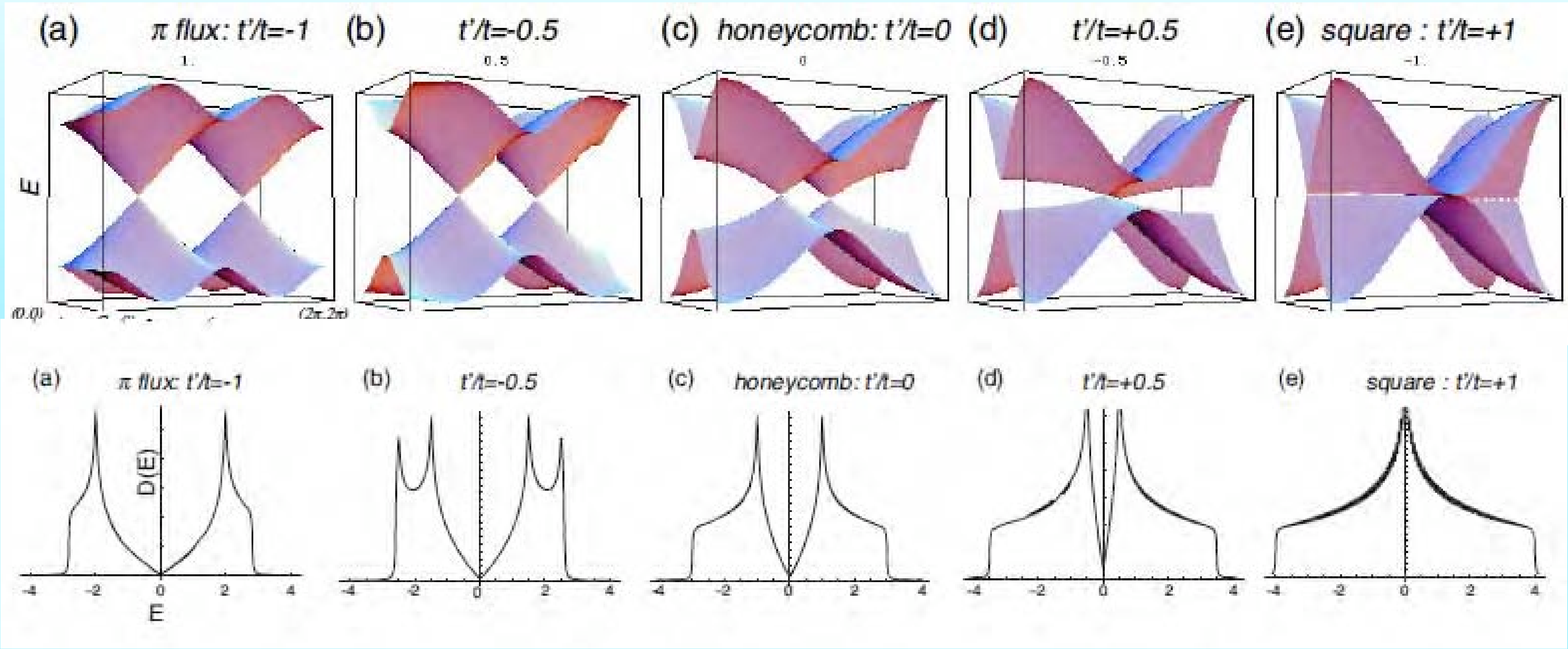
$t'/t = 0$: Honeycomb Lattice

$t'/t = -1$: π Flux State

Stability of the Dirac Cornes!

Dirac Cornes

☆ Adiabatic Equivalence



t′/t = 1 : Square Lattice

t′/t =0 :Honeycomb Lattice

t′/t=-1 : π Flux State

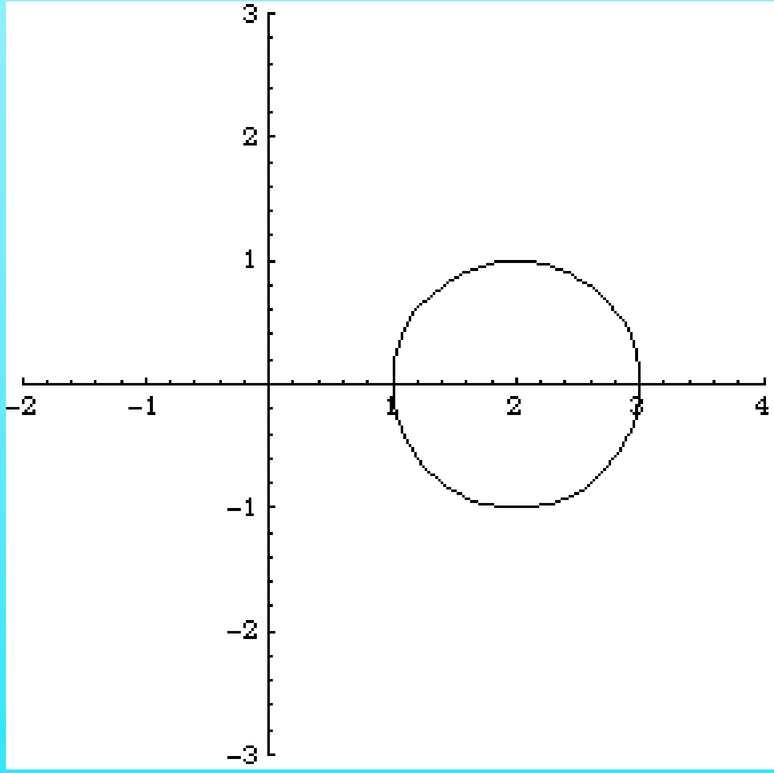
Topological Stability of the Dirac Cornes

$$H(k_x, k_y) = \begin{pmatrix} 0 & \Delta \\ \Delta^* & 0 \end{pmatrix}$$

$$\Delta = -t(1 + e^{ik_y} + e^{ik_x}(1 + re^{-ik_y}), \quad r = t'/t$$

$$\begin{aligned} E(k_x, k_y) &= \pm|\Delta| \\ &= \pm\sqrt{(1 + \cos k_x + \cos k_y)^2 + (\sin k_x + \sin k_y)^2} \end{aligned}$$

☆ General zeros of $\Delta(k_x, k_y)$ ⟶ Dirac Cones



$$\Delta(k_x, k_y), \ k_x : 0 \to 2\pi : \ \text{loop } C(k_y) \text{ in } \mathbb{C}$$

$$\text{loop } C(k_y) \text{ moves } : k_y : 0 \to 2\pi$$

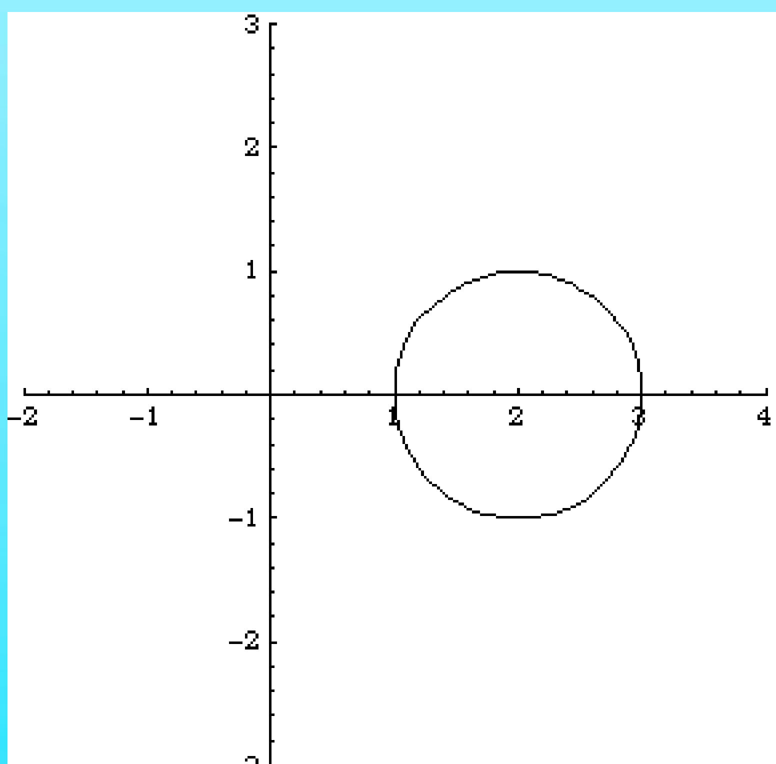
Topological Stability of the Dirac Cornes

$$H(k_x, k_y) = \begin{pmatrix} 0 & \Delta \\ \Delta^* & 0 \end{pmatrix}$$

$$\Delta = -t(1 + e^{ik_y} + e^{ik_x}(1 + re^{-ik_y}), \quad r = t'/t$$

$$\begin{aligned} E(k_x, k_y) &= \pm|\Delta| \\ &= \pm\sqrt{(1 + \cos k_x + \cos k_y)^2 + (\sin k_x + \sin k_y)^2} \end{aligned}$$

☆ General zeros of $\Delta(k_x, k_y)$ ⟶ Dirac Cones



$\Delta(k_x, k_y)$, $k_x : 0 \to 2\pi$: loop $C(k_y)$ in $\mathbb{C}$

loop $C(k_y)$ moves : $k_y : 0 \to 2\pi$

The loop cut the origin ⟶ Dirac Cones

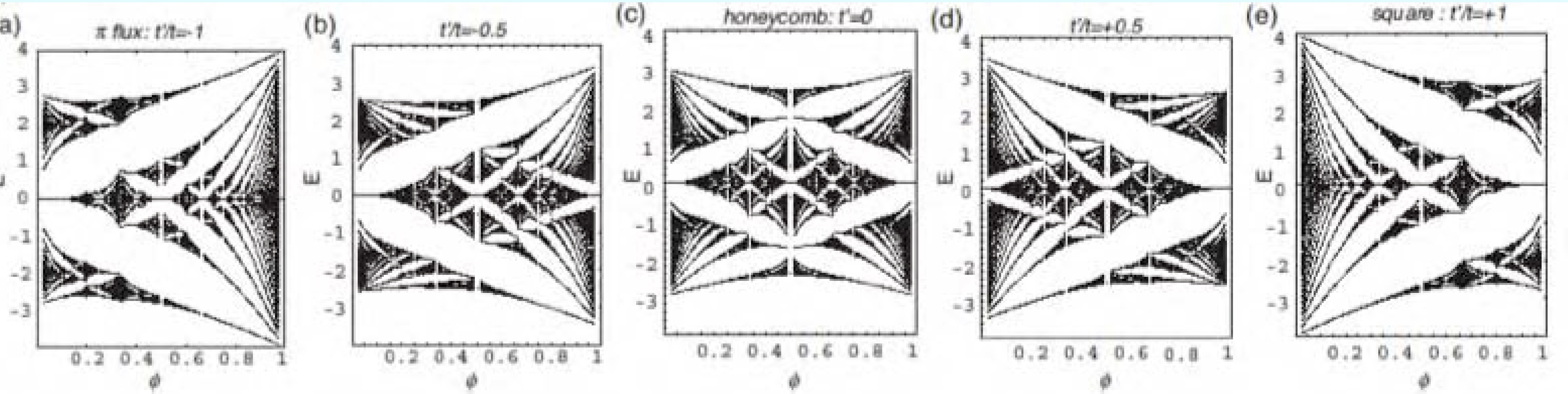
Topological Stability

of

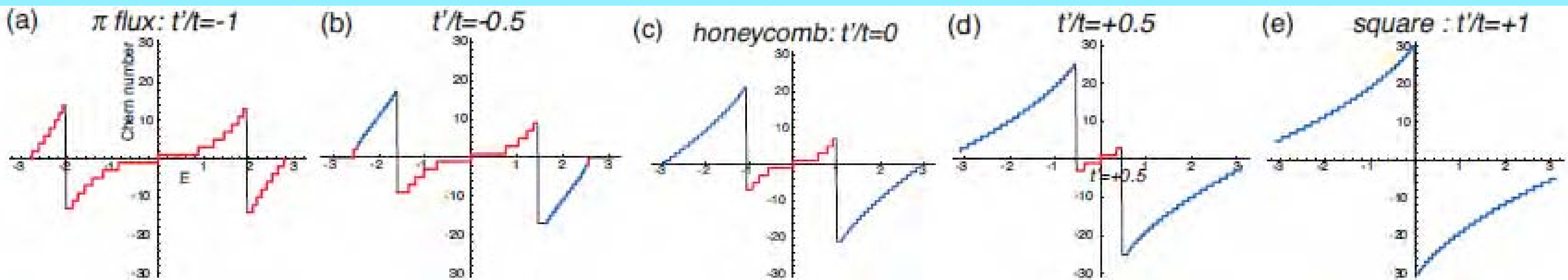
the doubled Dirac Cones

Hofstadter Diagrams

☆ *Adiabatic Equivalence & Duality*



☆ *Hall Conductance v.s. chemical potential*

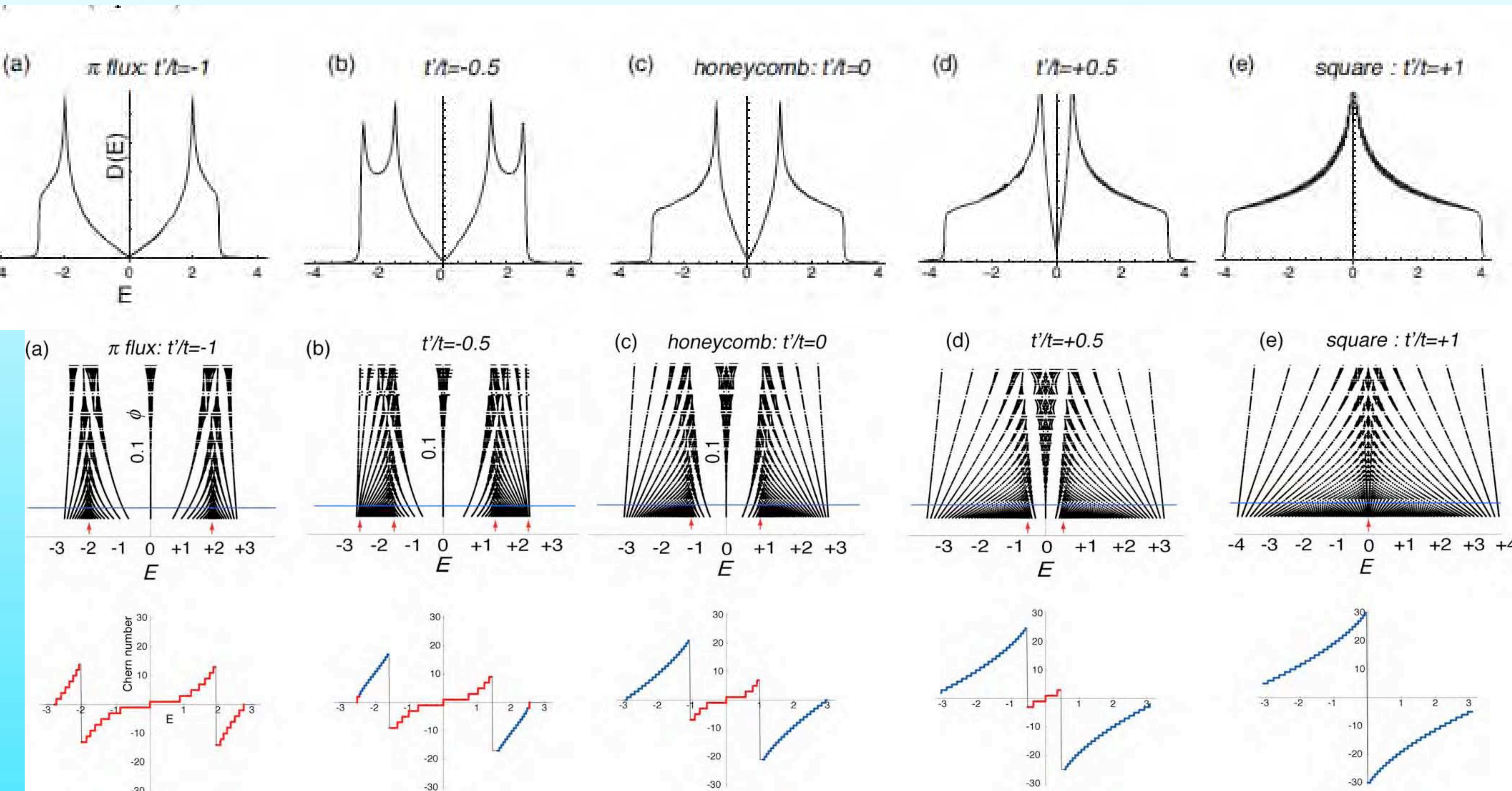


t'/t = 1 : Square Lattice

t'/t =0 :Honeycomb Lattice

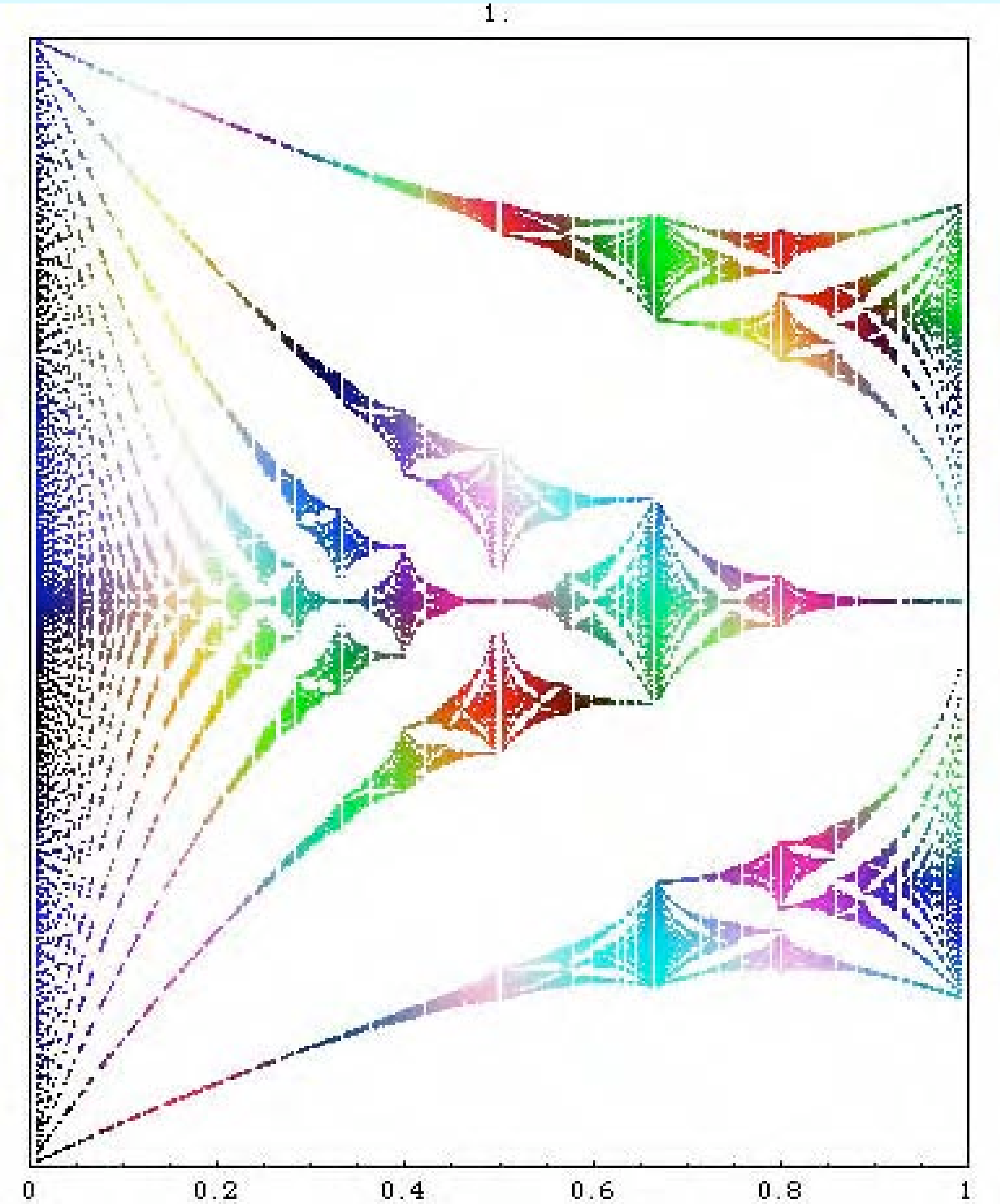
t'/t=-1 : π Flux State

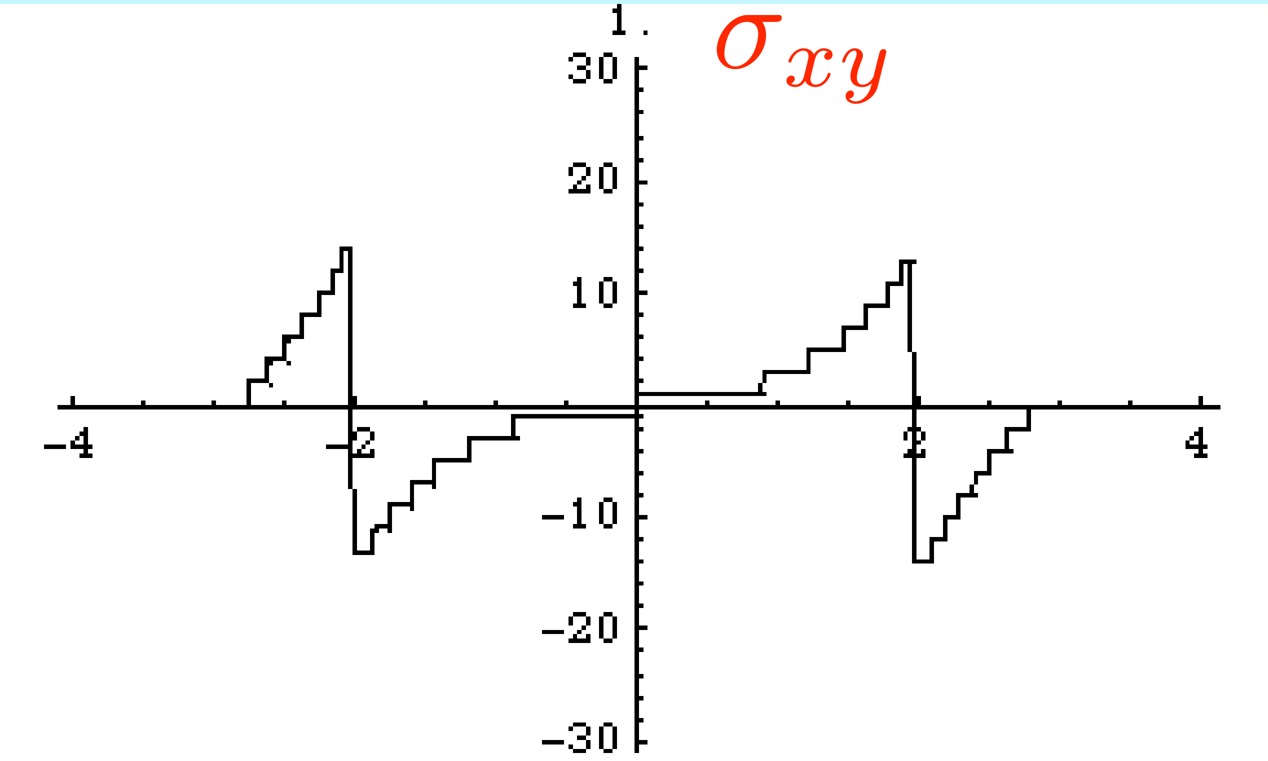
van Hove singularity & Hall Conductance



σ_{xy} by Adiabatic Principle

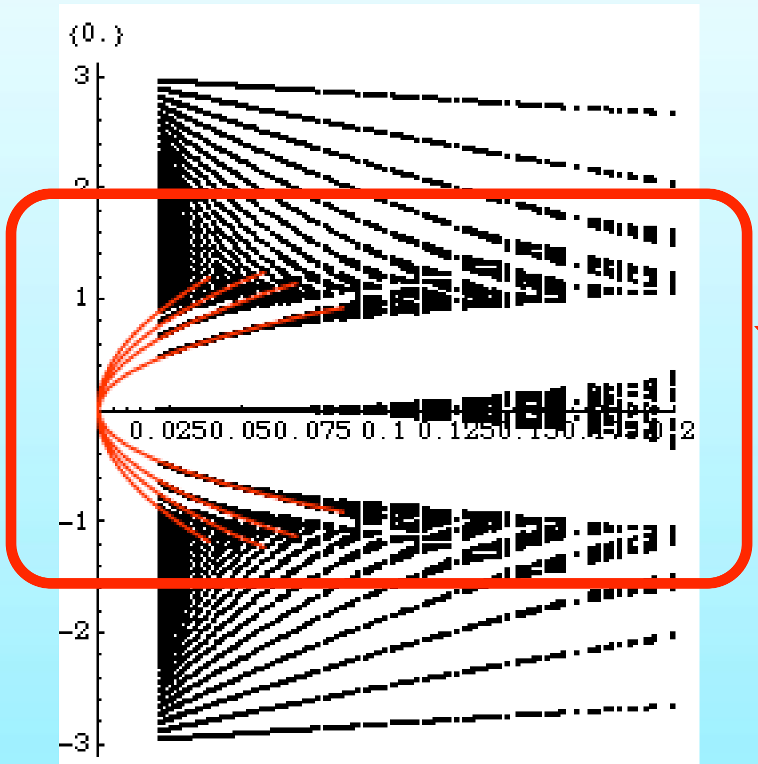
☆ Hofstadter Diagram and σ_{xy} with t′/t





t′/t = 1 : Square Lattice
t′/t =0 :Honeycomb Lattice
t′/t=-1 : π Flux State

Adiabatic Connections Near zero Field

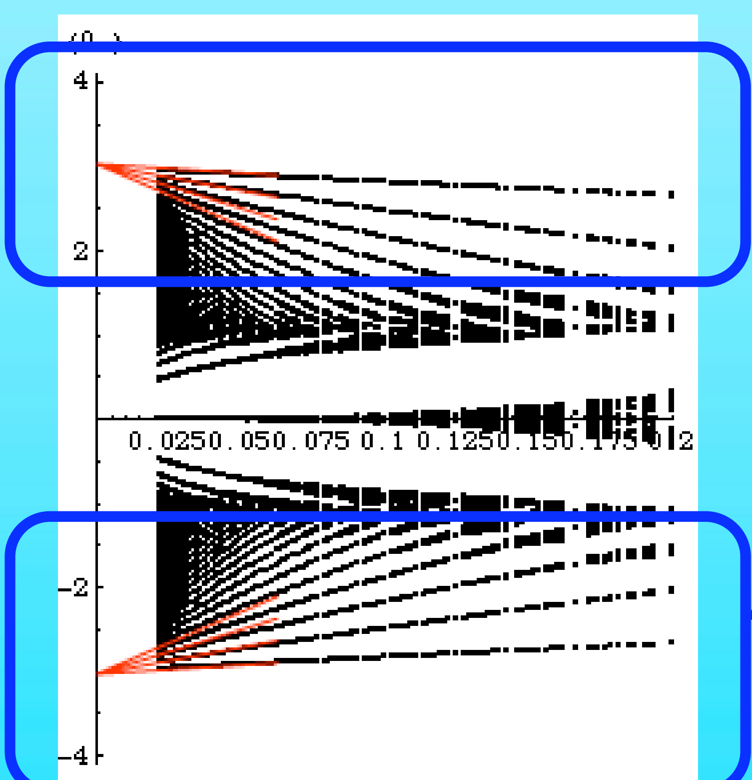


☆ Honeycomb Lattice ⟷ π flux

Main Gaps Preserve

Near E=0

Honeycomb System is
Topologically Equivalent to
π flux System Near E=0



☆ Honeycomb Lattice ⟷ Square Lattice

Main Gaps Preserve

Near Band Edges

Honeycomb System is
Topologically Equivalent to
Square System Near the Band Edges

Honeycomb σ_{xy} From Diophantine Equations

- As for the π flux system and square system, σ_{xy} is determined by a Diophantine equation
- By the Adiabatic Equivalence, σ_{xy} of the honeycomb is determined algebraically.

Master Eq.

$$\Phi = \frac{P}{Q}, \quad J \equiv P c_J, \quad (\mathrm{mod}\, Q), \quad |c_J| < Q/2$$

TKNN1982

- By the Adiabatic Equivalence

$$\Phi = \frac{P}{Q} = \frac{1}{2} + \frac{\phi}{2} = \frac{q+1}{2q}, \quad \phi = \frac{1}{q}$$

$$P = q+1, \; Q = 2q$$

$$J = q - 1 + 2(N+1) = q + 2N + 1$$

Dirac Fermion Type Quantization: Algebraically on Honeycomb Lattice

$$c_J = 2N + 1, \; N = 0, 1, 2, \cdots$$

Edge states of Graphene

- Without magnetic field

YH
with S. Ryu (now KITP)

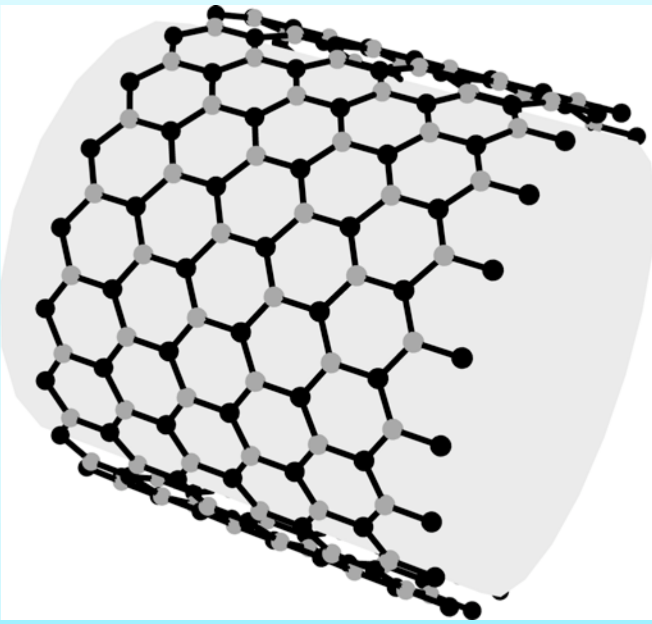
Ref.[1] Phys. Rev. B65, 212510 (2002)
[2] Phys. Rev.Lett. 89, 077002(2002)
[3] Physics C 388-389, 78 (2003), ibid 90 (2003)
[4] Phys. Rev. B67, 165410 (2003)
[5] Physica E 22, 679 (2004)

- With magnetic field

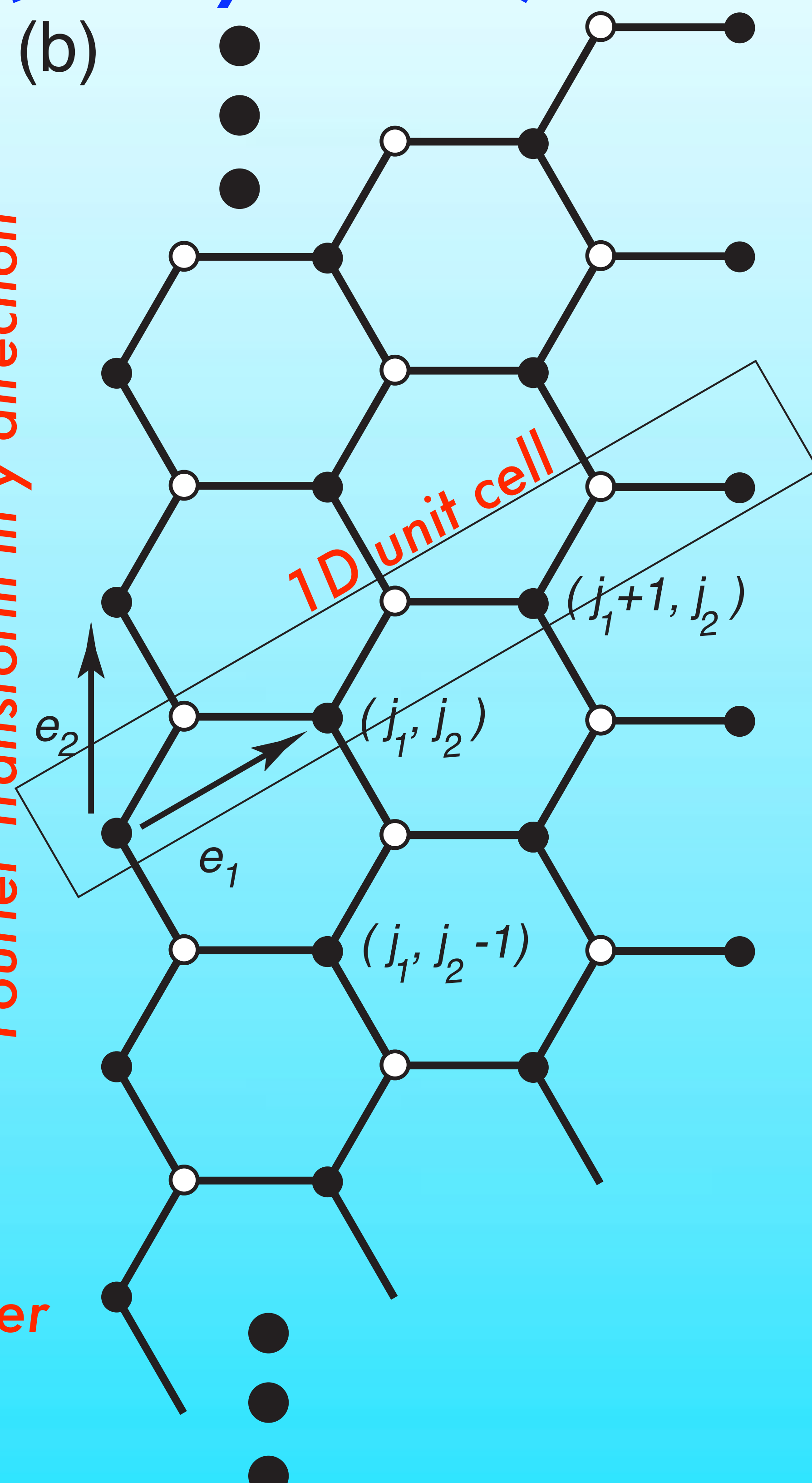
Recent works

Graphene on a Cylinder

☆ Zigzag Edges (on Cylinder)



Fourier Transform in y direction

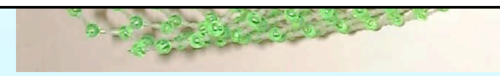


$$H_{\rm total} = \sum_{k_y} H(k_y)$$

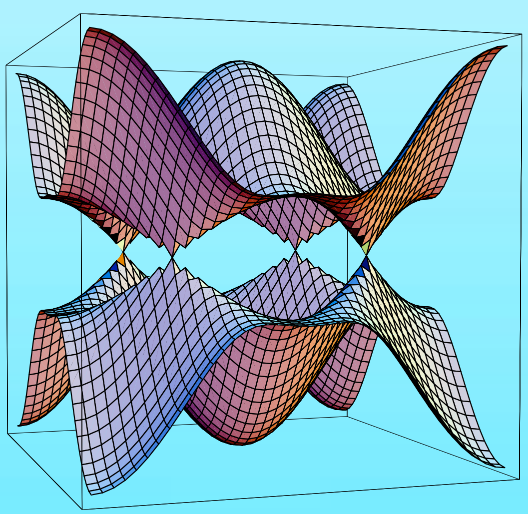
Total System as a sum of 1D system parametrized by k_y

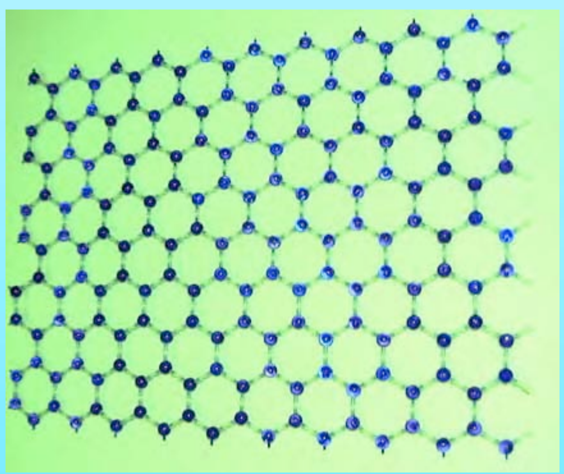
1D system with parameter k_y

Let me remind old works
without magnetic field for a while



Topological Equivalence between Anisotropic Superconductors and Carbon 2D Systems





From Topological Orders

Y. Hatsugai
with S.Ryu
Department of Applied Physics
University of Tokyo

Ref.[1] Phys. Rev. B65, 212510 (2002)
[2] Phys. Rev.Lett. 89, 077002(2002)
[3] Physics C 388-389, 78 (2003), ibid 90 (2003)
[4] Phys. Rev. B67, 165410 (2003)
[5] Physica E 22, 679 (2004)

Localized Boundary State in Carbon Sheet (1)

now called as Graphene

Tight-binding Model Calculation

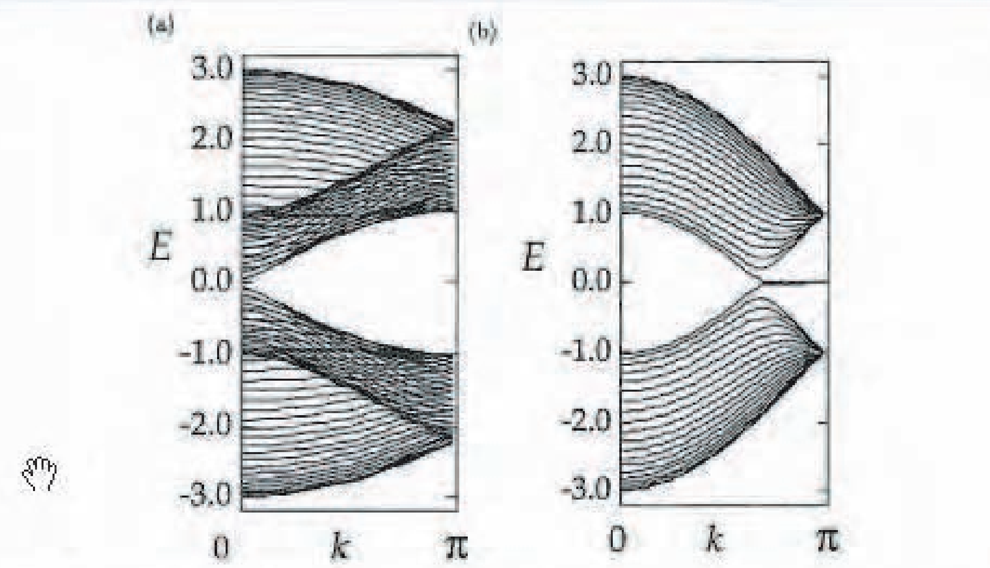


Fig. 2. Band structure of (a) armchair and (b) zigzag ribbons with width $N=20$.

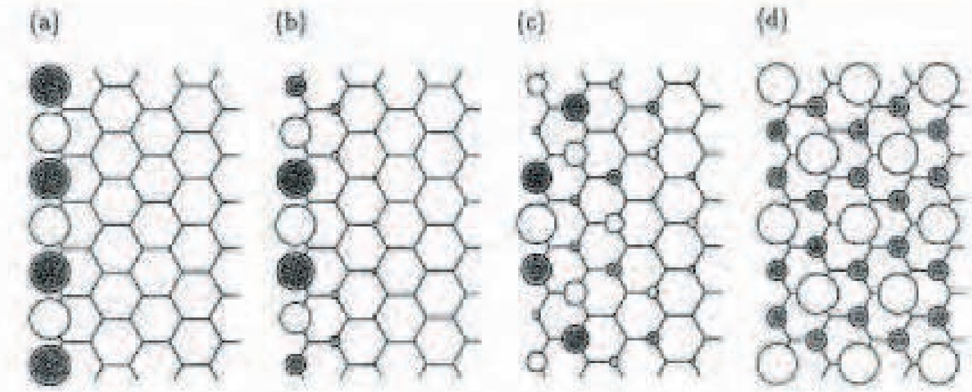


Fig. 3. Schematic figures of the real part of analytic solutions for the edge state in semi-infinite graphite, when (a) $k=\pi$, (b) $8\pi/9$, (c) $7\pi/9$ and (d) $2\pi/3$.

“ Peculiar Localized State at Zigzag Graphite Edge “ M. Fujita, K. Wakabayashi, K. Nakada and K. Kusakabe, JPSJ 65, 1920 (1996)

Localized Boundary State in Carbon Sheet (2)

Local Spin Density Functional Appr. Calculation

VOLUME 87, NUMBER 14 PHYSICAL REVIEW LETTERS 1 OCTOBER 2001

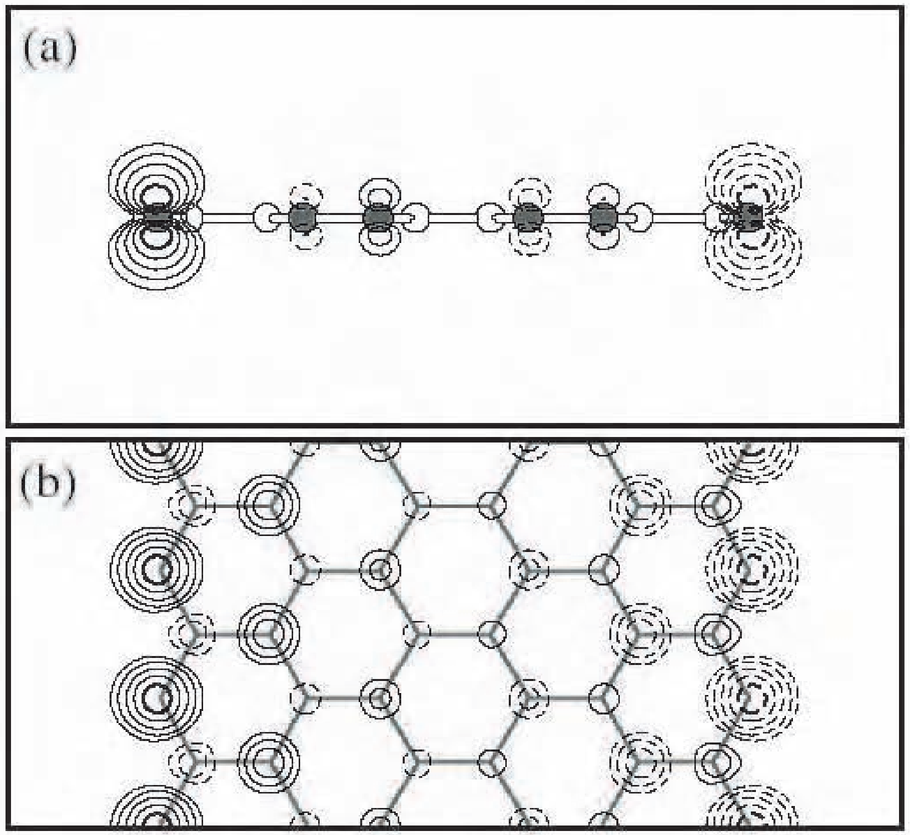


FIG. 1. Contour plots of spin density $n_\uparrow(r) - n_\downarrow(r)$ (a) on a plane perpendicular to a graphite flake with zigzag edges and (b) on a plane including the graphite flake. In (a) the edges are perpendicular to the plane and C atoms on the plane are depicted by shaded circles. Positive and negative values of the spin density are shown by solid and dashed lines, respectively. Each contour represents twice (or half) the density of the adjacent contour lines.

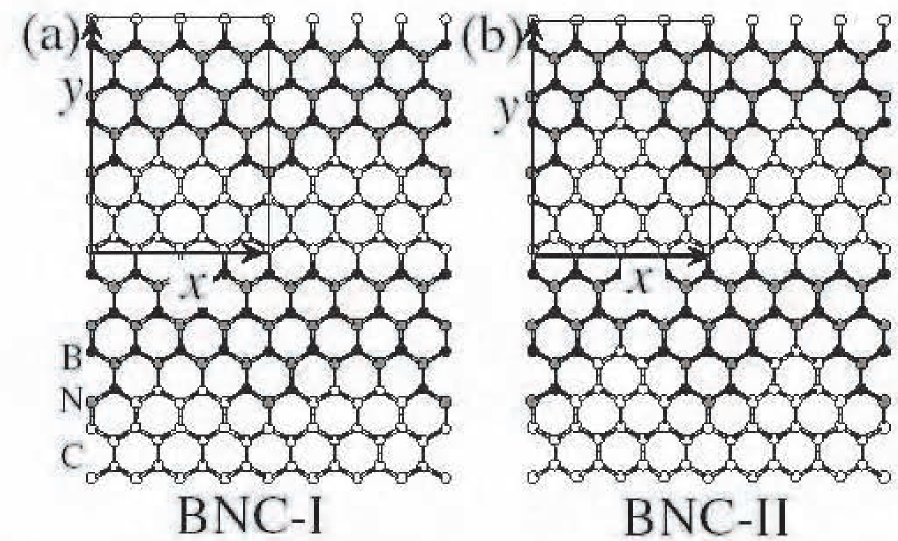


FIG. 2. Top views of fully optimized BNC heterosheets. (a) BNC-I and (b) BNC-II. White, shaded, and black circles denote C, B, and N atoms, respectively. The rectangle in each figure denotes the unit cell.

B, N, and C atoms have been observed indeed [8–10]. Second, the phase separation of graphite and BN regions leading to the striped structures above is energetically favorable. In fact, we have performed the total-energy calculations for graphite, BN, BC, and NC heterosheets by DFT. The calculated bond energies of B-C and N-C are smaller than that of graphite by 1.52 and 0.81 eV, respectively. On the other hand, the bond energy of B-N is smaller than that of graphite only by 0.31 eV. Third, undulation

“Magnetic Ordering in Hexagonally Bonded Sheets with First-Row Elements”,

Okada, Oshiyama, Phys. Rev. Lett. 87, 146803 (2001)

Zero Bias Conductance Peak in Anisotropic Superconductivity

d-wave superconductivity

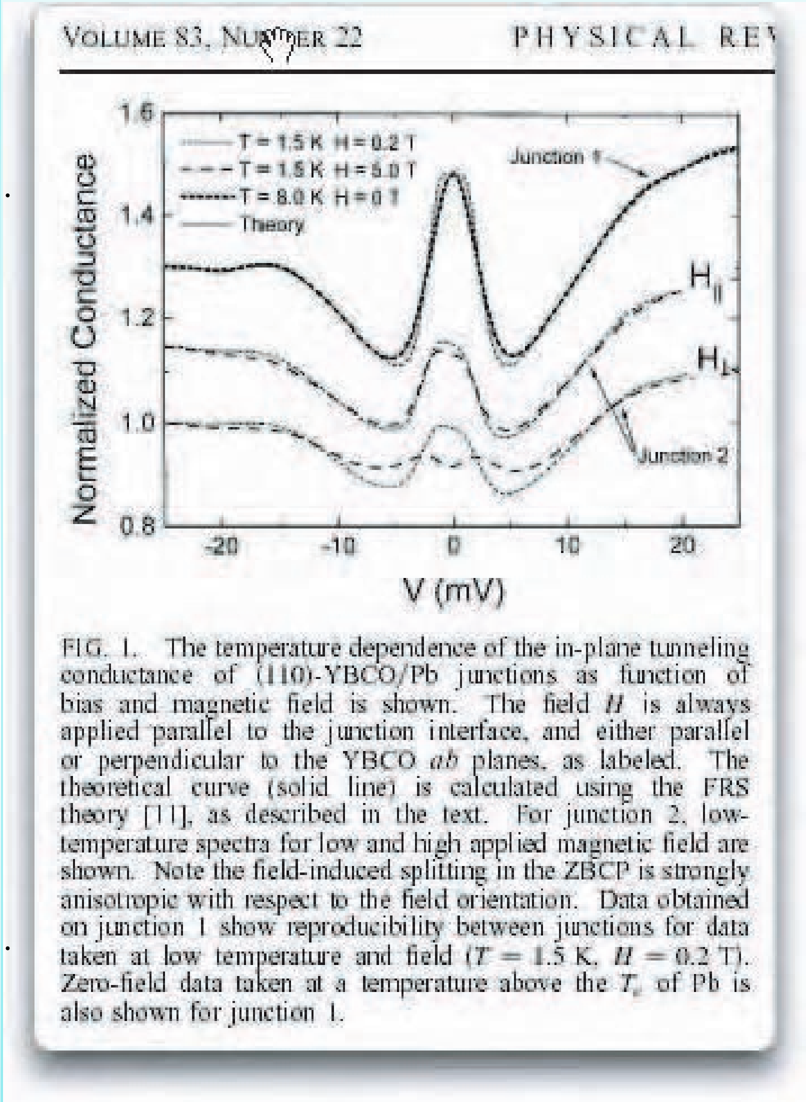

VOLUME 83, NUMBER 22 PHYSICAL REV

FIG. 1. The temperature dependence of the in-plane tunneling conductance of (110)-YBCO/Pb junctions as function of bias and magnetic field is shown. The field H is always applied parallel to the junction interface, and either parallel or perpendicular to the YBCO ab planes, as labeled. The theoretical curve (solid line) is calculated using the FRS theory [11], as described in the text. For junction 2, low-temperature spectra for low and high applied magnetic field are shown. Note the field-induced splitting in the ZBCP is strongly anisotropic with respect to the field orientation. Data obtained on junction 1 show reproducibility between junctions for data taken at low temperature and field ($T = 1.5$ K, $H = 0.2$ T). Zero-field data taken at a temperature above the T_c of Pb is also shown for junction 1.

Zero Energy Boundary States of Anisotropic Superconductivity

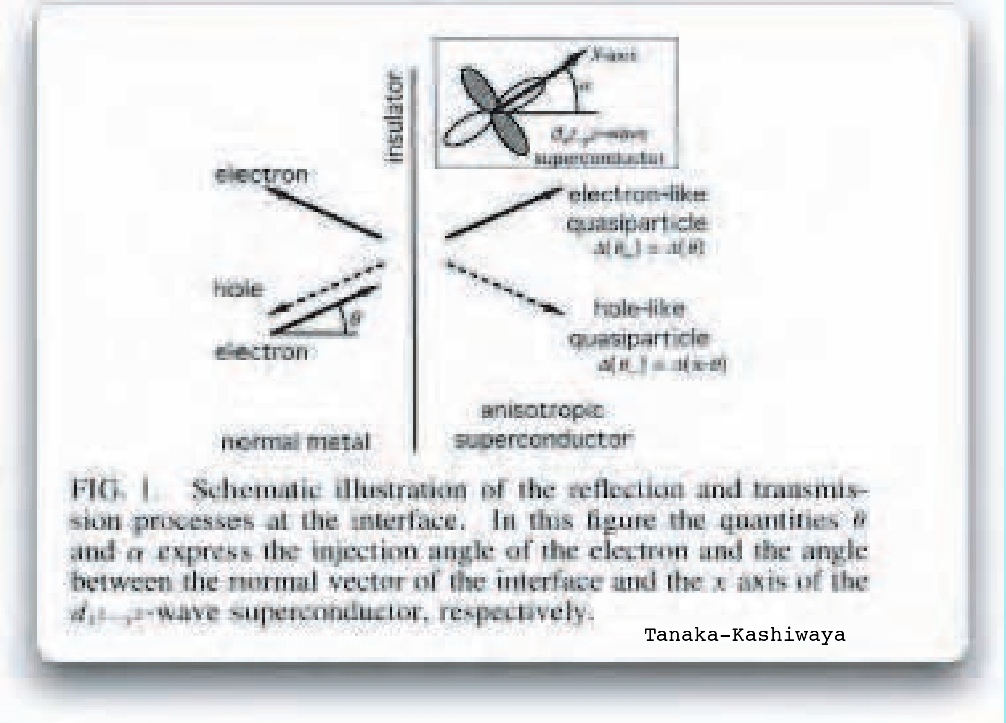


FIG. 1. Schematic illustration of the reflection and transmission processes at the interface. In this figure the quantities θ and α express the injection angle of the electron and the angle between the normal vector of the interface and the x axis of the $d_{x^2-y^2}$-wave superconductor, respectively.

Tanaka-Kashiwaya

L. J. Buchholtz,G. Zwicknagl, Phys. Rev. B 23, 5788 (1981) (p wave)

C.-R. Hu, Phys. Rev. Lett. 72, 1526 (1994) (d wave)

S. Kashiwaya, Y. Tanaka, Phys. Rev. Lett. 72, 1526 (1994)

M. Matsumoto and H. Shiba, JPSJ, 1703 (1995)

(fig.) M. Aprili, E. Badica, and L. H. Greene,Phys. Rev. Lett. 83, 4630 (1999)

Edge State and Zero Modes

1. Zero Bias Conductance Peak

2. Boundary Magnetism of the Carbon Nanotubes

These 2 systems are topologically equivalent with each other

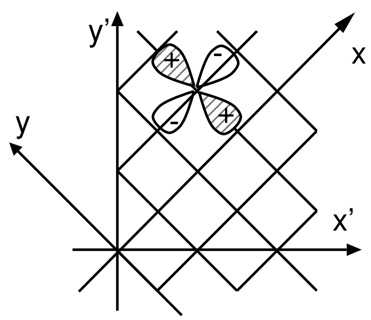
Localized zero modes of topological ordered states

cf. Witten's SUSY QM

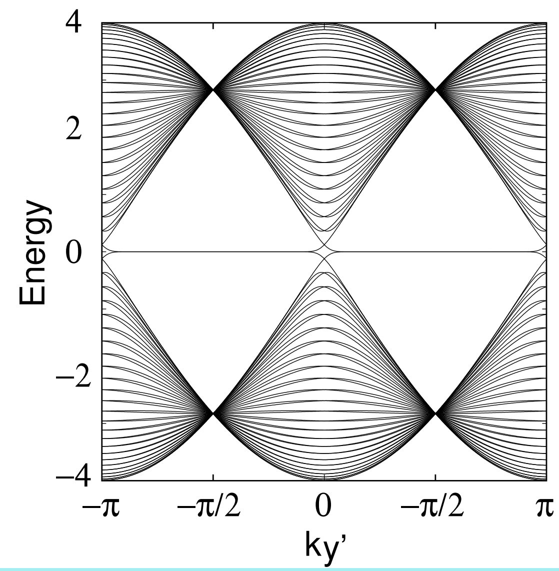
Zero Energy Edge States in Various Physical Systems

- Anisotropic Superconductivity ($d_{x^2-y^2}$-wave)

(a)

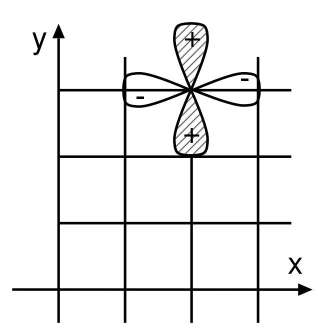


$(1,1,0)$ surface

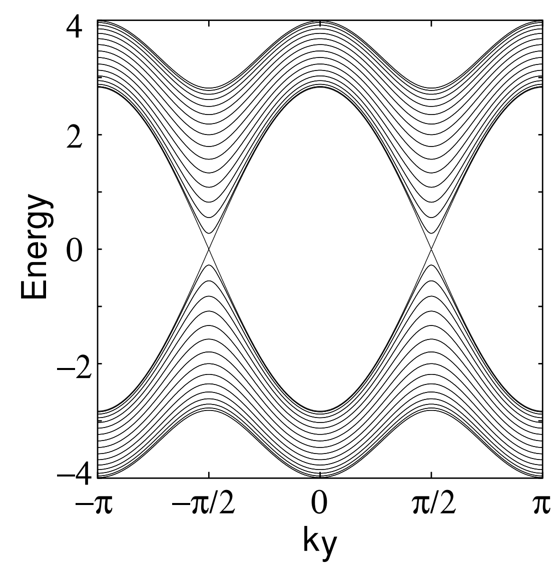


Zero Energy Edge States !

(b)



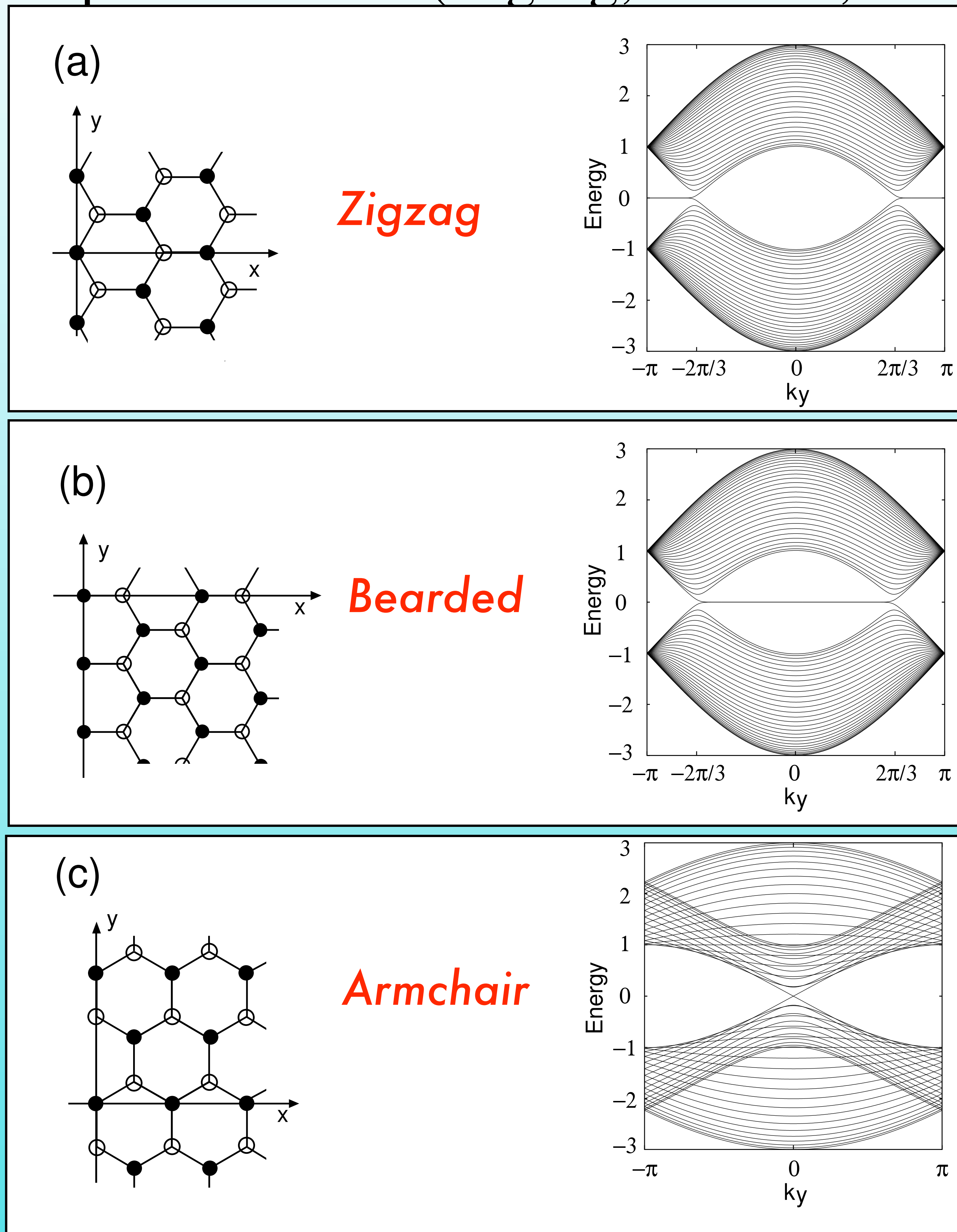
$(1,0,0)$ surface



No Edge States !

Zero Energy Edge States : cont.

- Graphite Ribbons (zigzag, bearded, and armchair edges)



Zero Energy Edge States !

Zero Energy Edge States !

No Edge States !

When and Why the Zero Energy Edge States Appear ?

Accidental ?

No !!

$\Downarrow$

Topological Origin !

- Bulk-Edge Correspondence
- Particle Hole Symmetry
- Topological Stability

S. Ryu and Y. Hatsugai, Phys. Rev. Lett. 89, 077002-1-4 (2002)

Berry's parametrization

As for a1D system parametrized by ky

$$h_k = \begin{pmatrix} \xi_k & \Delta_k \\ \Delta_k^* & -\xi_k \end{pmatrix} = \boldsymbol{R}(\boldsymbol{k}) \cdot \boldsymbol{\sigma}$$

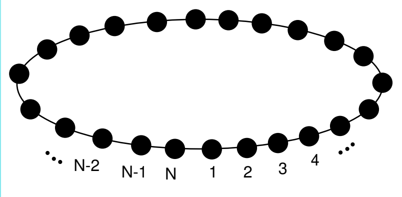
$\boldsymbol{\sigma}$: Pauli matrices

$$\boldsymbol{R}(\boldsymbol{k}) = (\mathrm{Re}\,\Delta_k, -\mathrm{Im}\,\Delta_k, \xi_k)$$

- ♦ Map from $\boldsymbol{k}$ to $\boldsymbol{R}$ as $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{k})$.
- ♦ In 1D, $k \in S^1$ ($k : 0 \to 2\pi$), so $\boldsymbol{R}$ forms a loop ℓ
  - This map is one to one

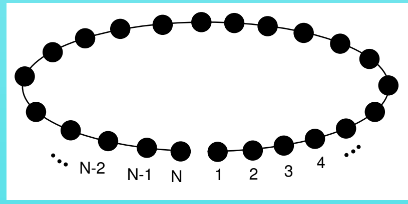
A loop in R space characterize the hamiltonian

$$H^{bulk}[\ell]$$



♦ The system with edges is also constructed by cutting all the matrix elements between the sites 1 and N in real space.

$$H^{edge}[\ell]$$



When the Zero Mode Edge States Exist ?

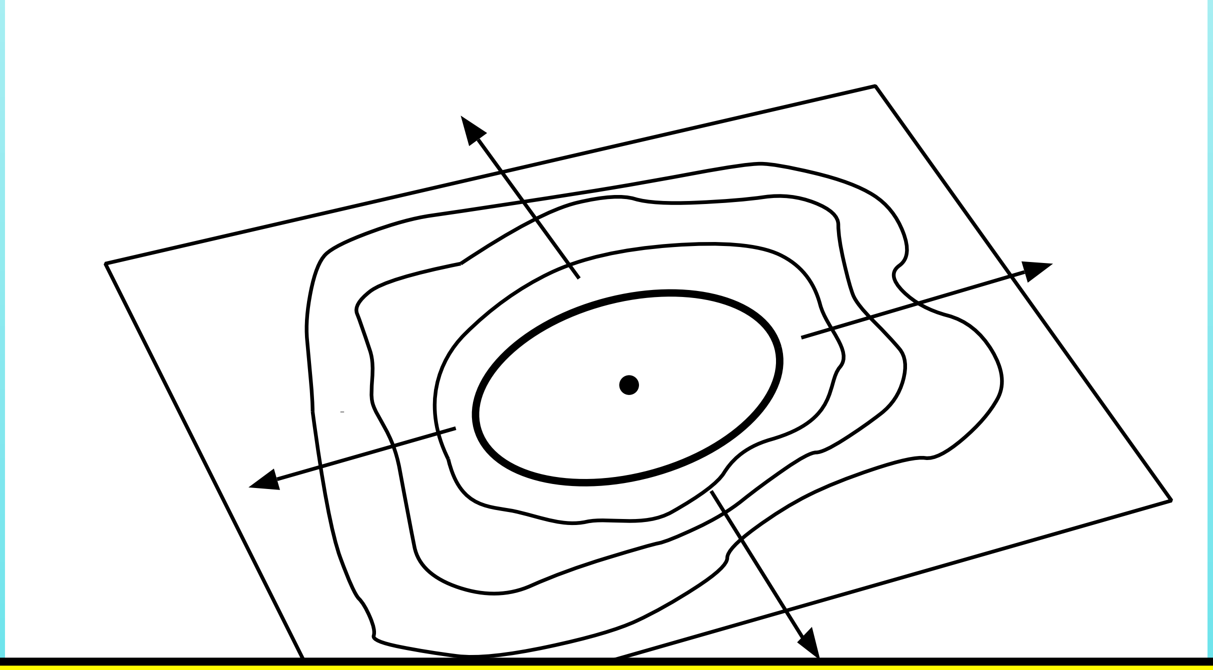
(Sufficient Condition)

S.Ryu & Y.Hatsugai, Phys. Rev. Lett. 89, 077002 (2002)

I. The loop ℓ is on the plane cutting the origin $\mathcal{O}$.

II. The loop ℓ is continuously deformed to the circle whose

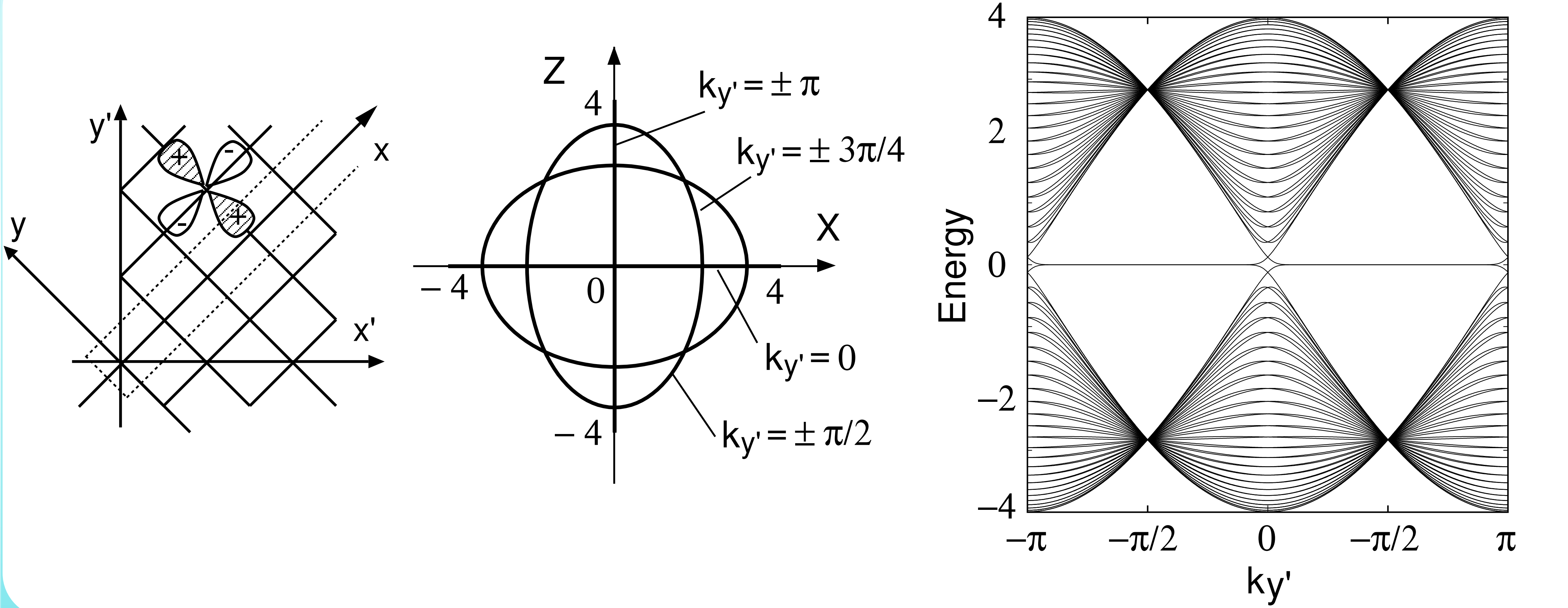
... origin is at $\mathcal{O}$ without passing through $\mathcal{O}$



Zero energy localized states EXIST

Check for the Anisotropic Superconductivity ($d_{x^2-y^2}$-wave)

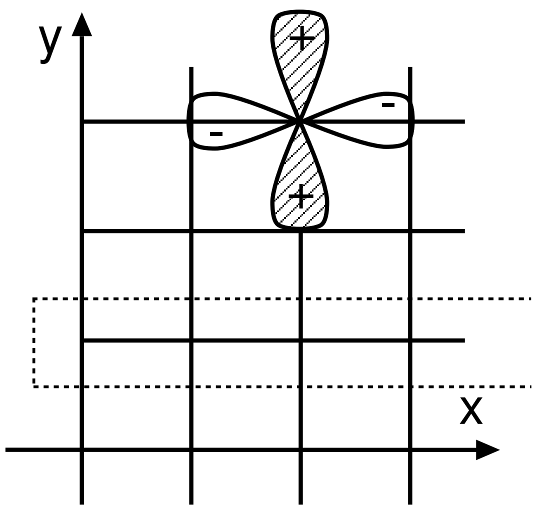
- (110) surface: the unit cells, loops, and the dispersion

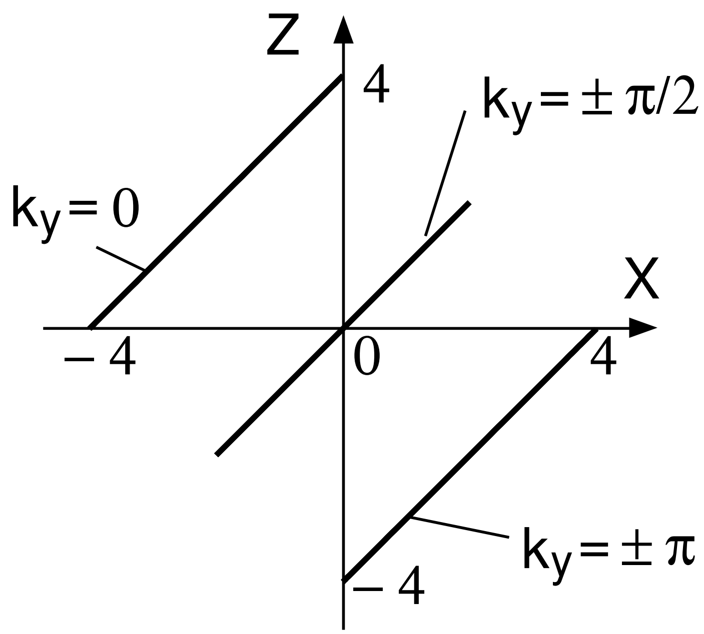


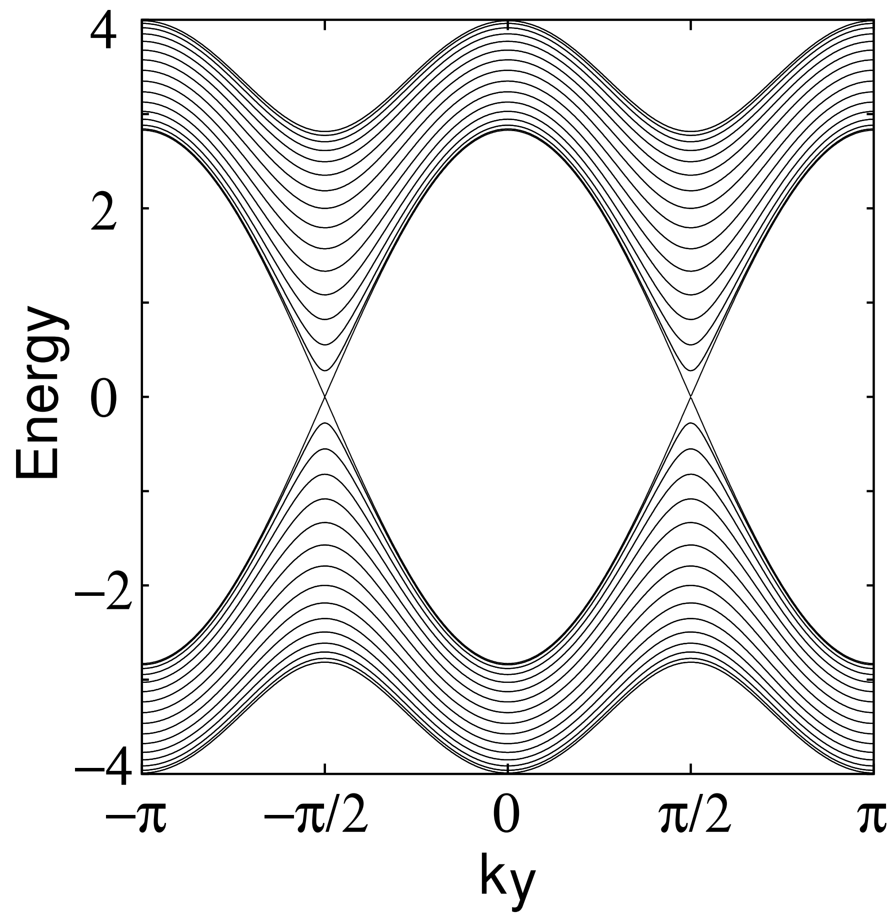
The origin $\mathcal{O}$ is always inside the loop.

Check for the Anisotropic Superconductivity : cont.

♦ (100) surface: the unit cells, loops, and the dispersion



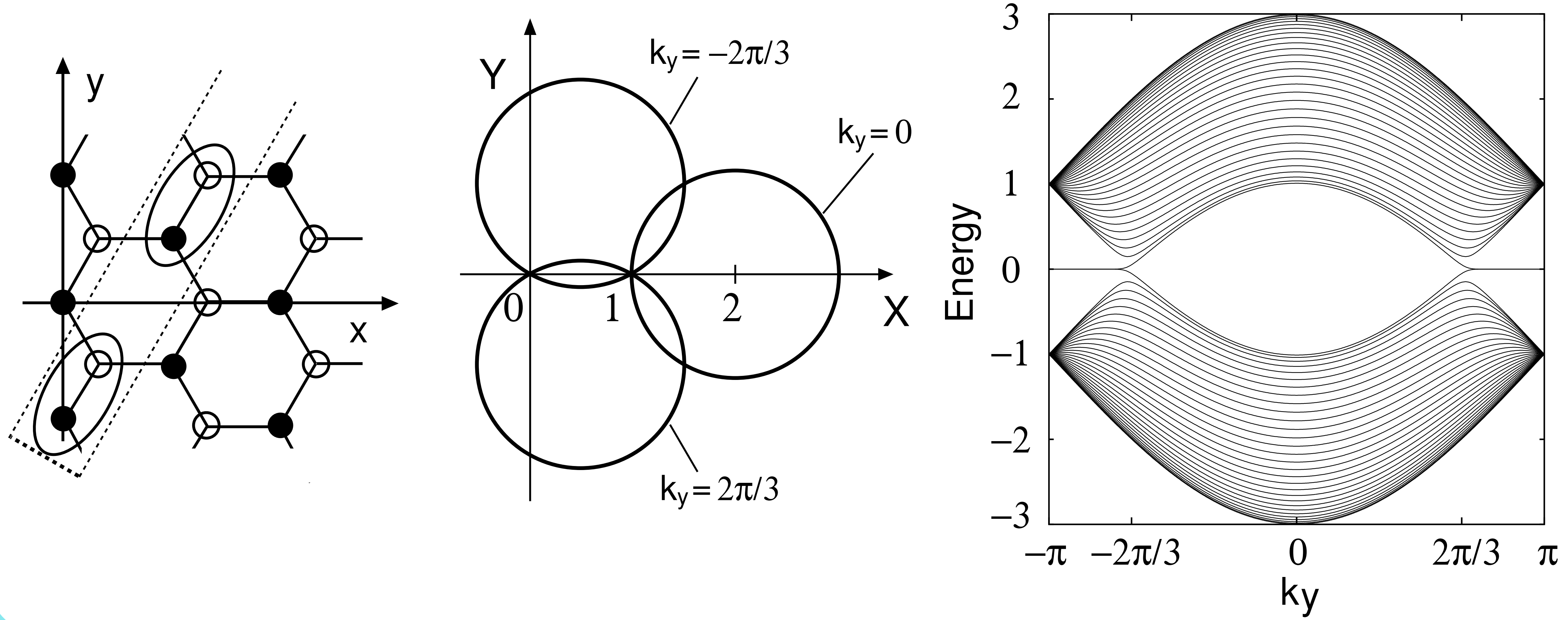




The origin $\mathcal{O}$ is never inside the loop except at $k_y = \pm\pi$.

Check for the Graphite Ribbons

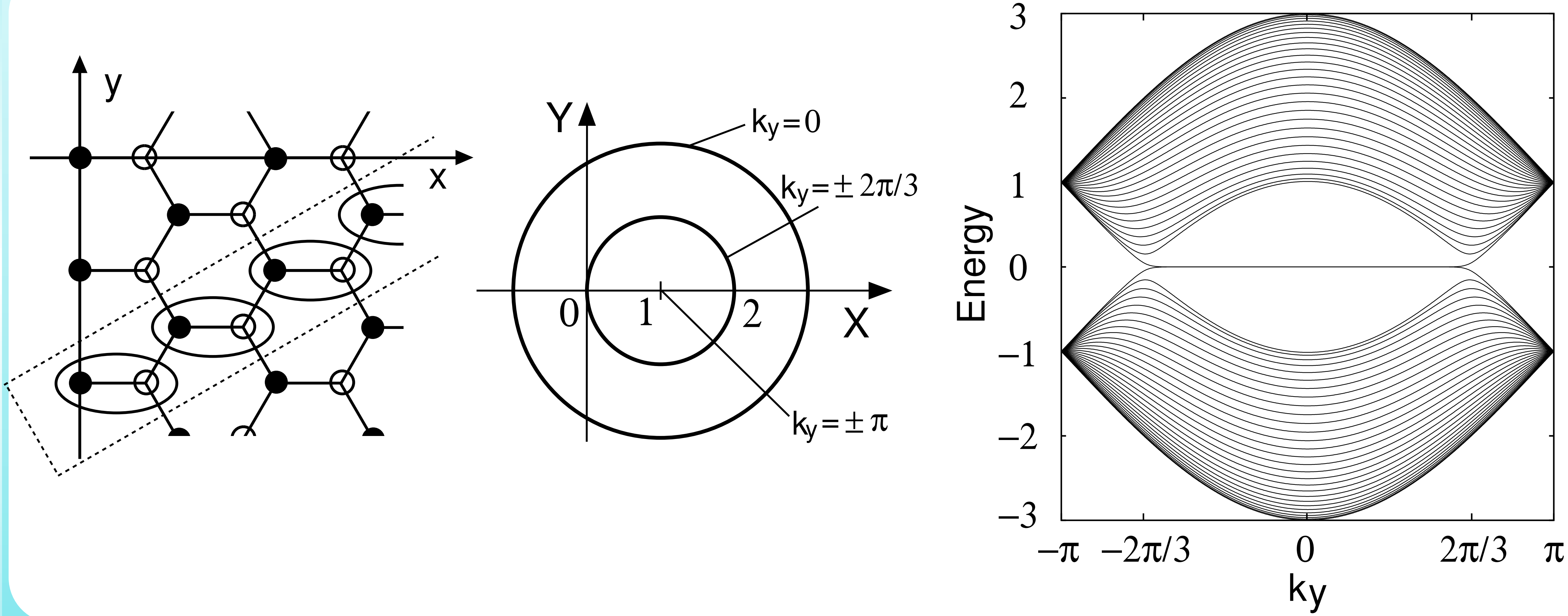
♦ Zigzag edge : the unit cells, loops, and the dispersion



The origin $\mathcal{O}$ is inside the loop when $|k_y| > 2\pi/3$.

Check for the Graphite Ribbons : cont.

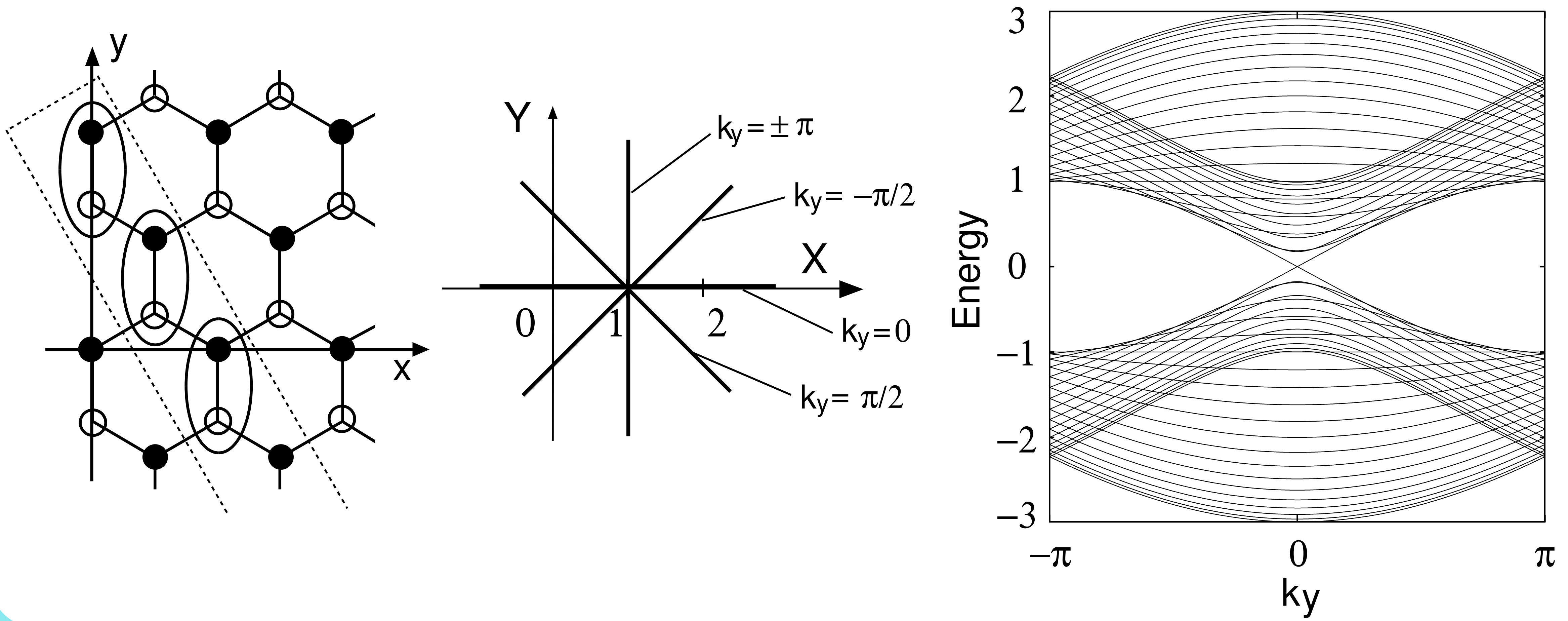
♦ Bearded edge : the unit cells, loops, and the dispersion



The origin $\mathcal{O}$ is inside the loop when $|k_y| < 2\pi/3$.

Check for the Graphite Ribbons : cont.

- ♦ Armchair edges : the unit cells, loops, and the dispersion

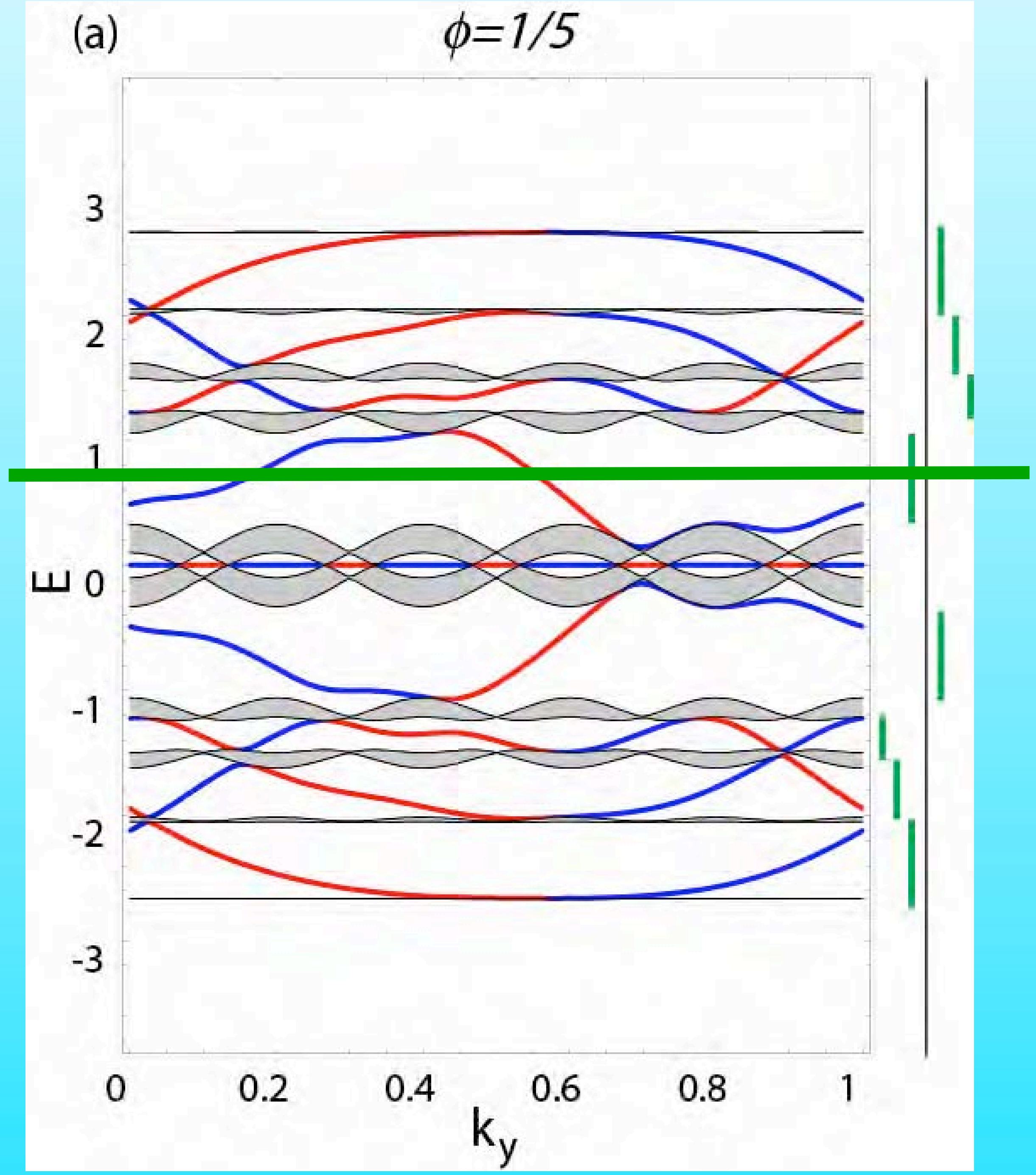


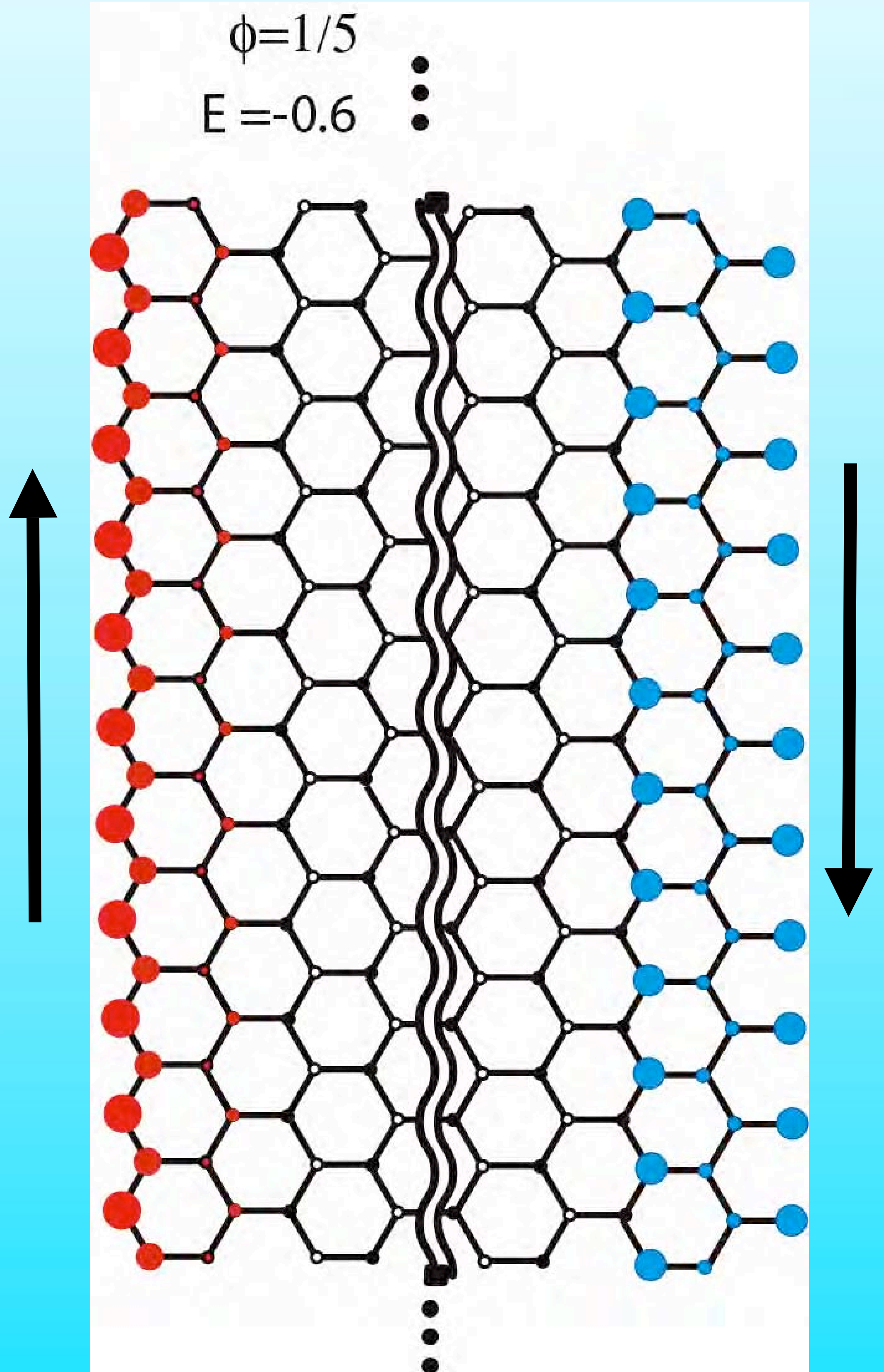
The origin $\mathcal{O}$ is always outside the loop.

Now go back to the present work

Edge States of Graphene with magnetic field

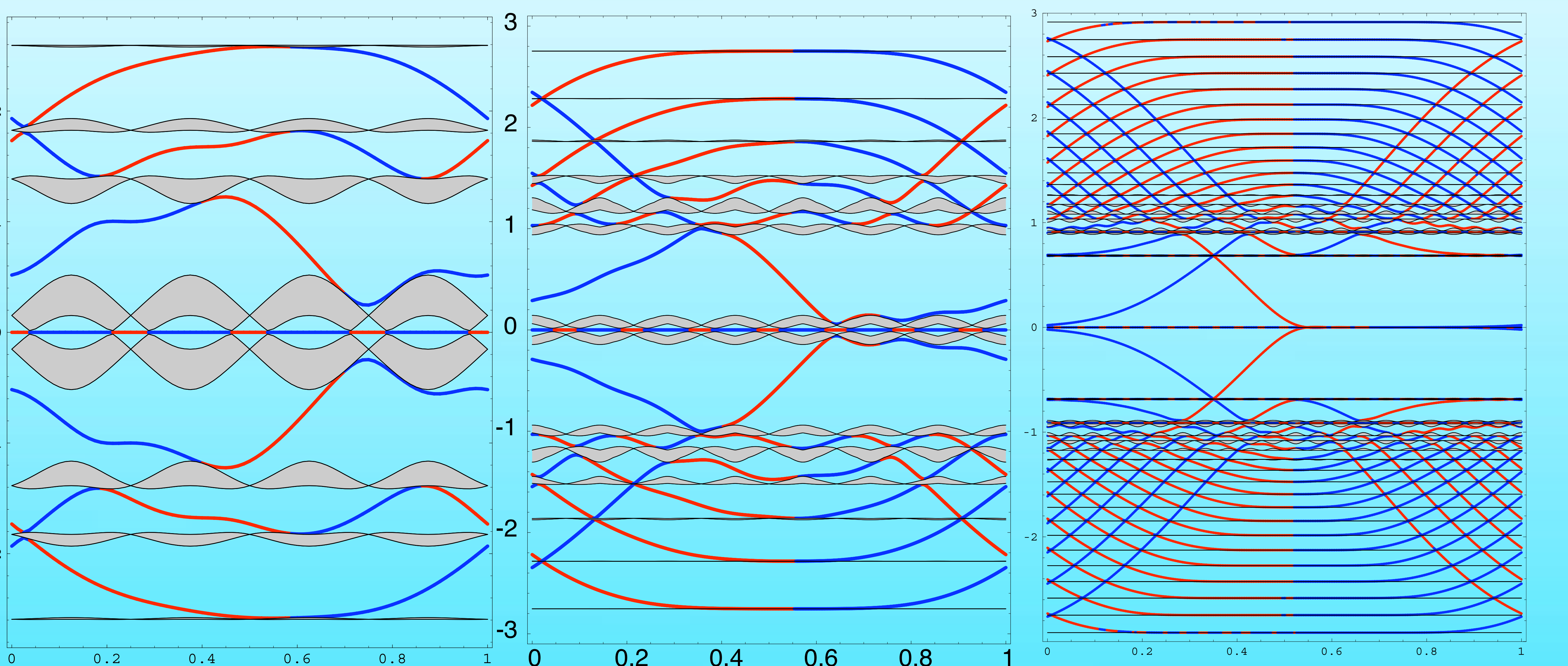
- Edge States and their local charges (Zigzag edges)





How the Edge states look like ?

☆ *Field dependence*



$\phi = 1/4$ $\phi = 1/7$ $\phi = 1/21$

strong ⟷ weak

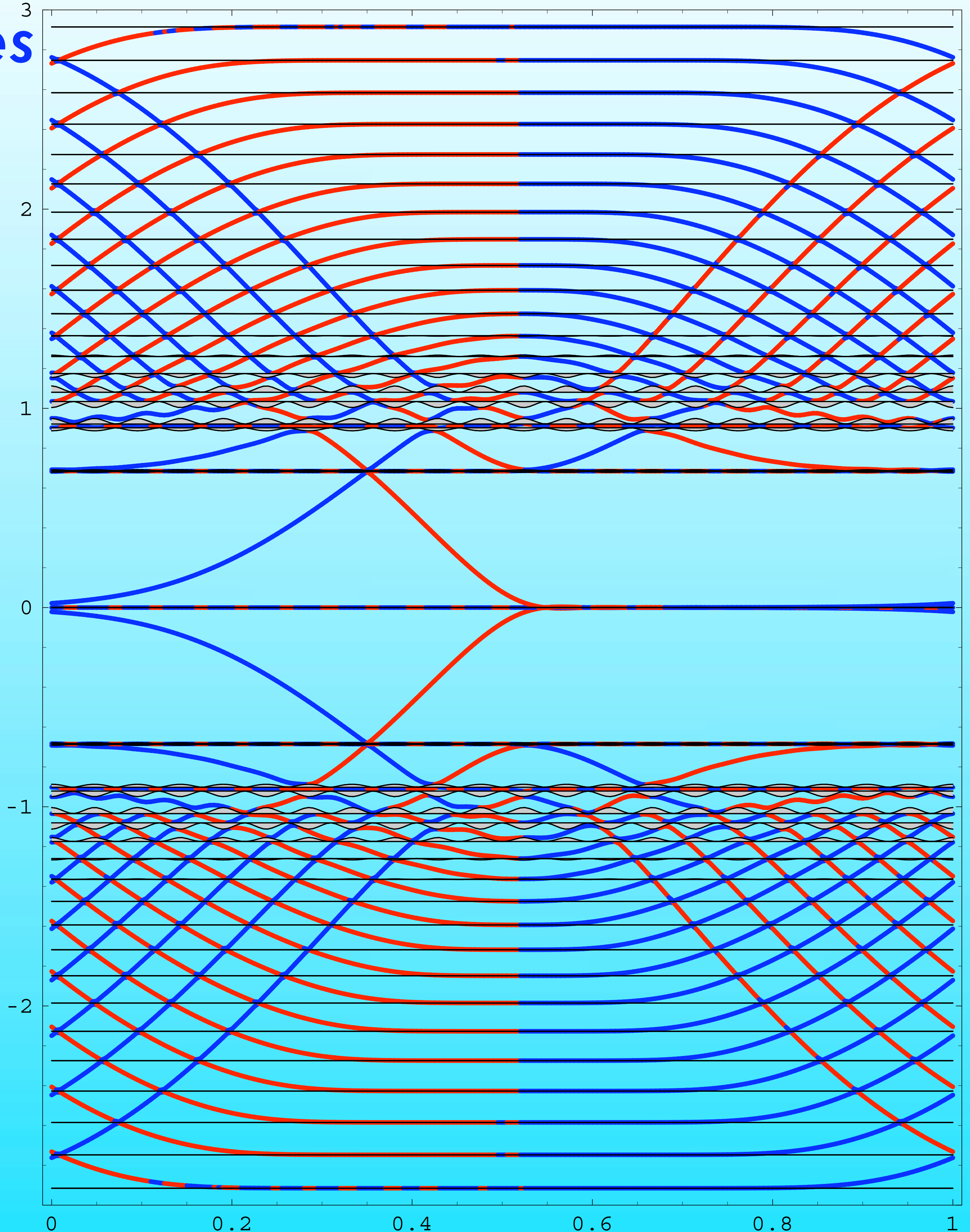
Edge States of Graphene

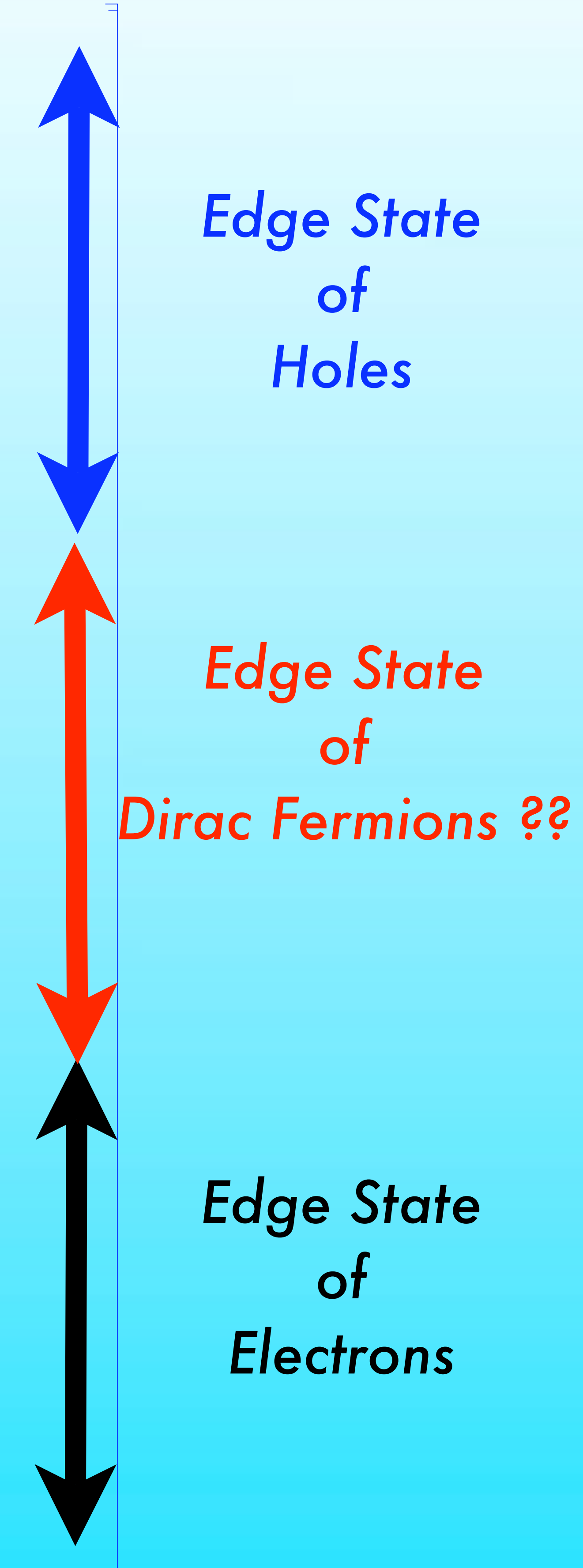
☆ **Zigzag Edges**

Full Spectrum

$\phi = 1/21$

Weak Field





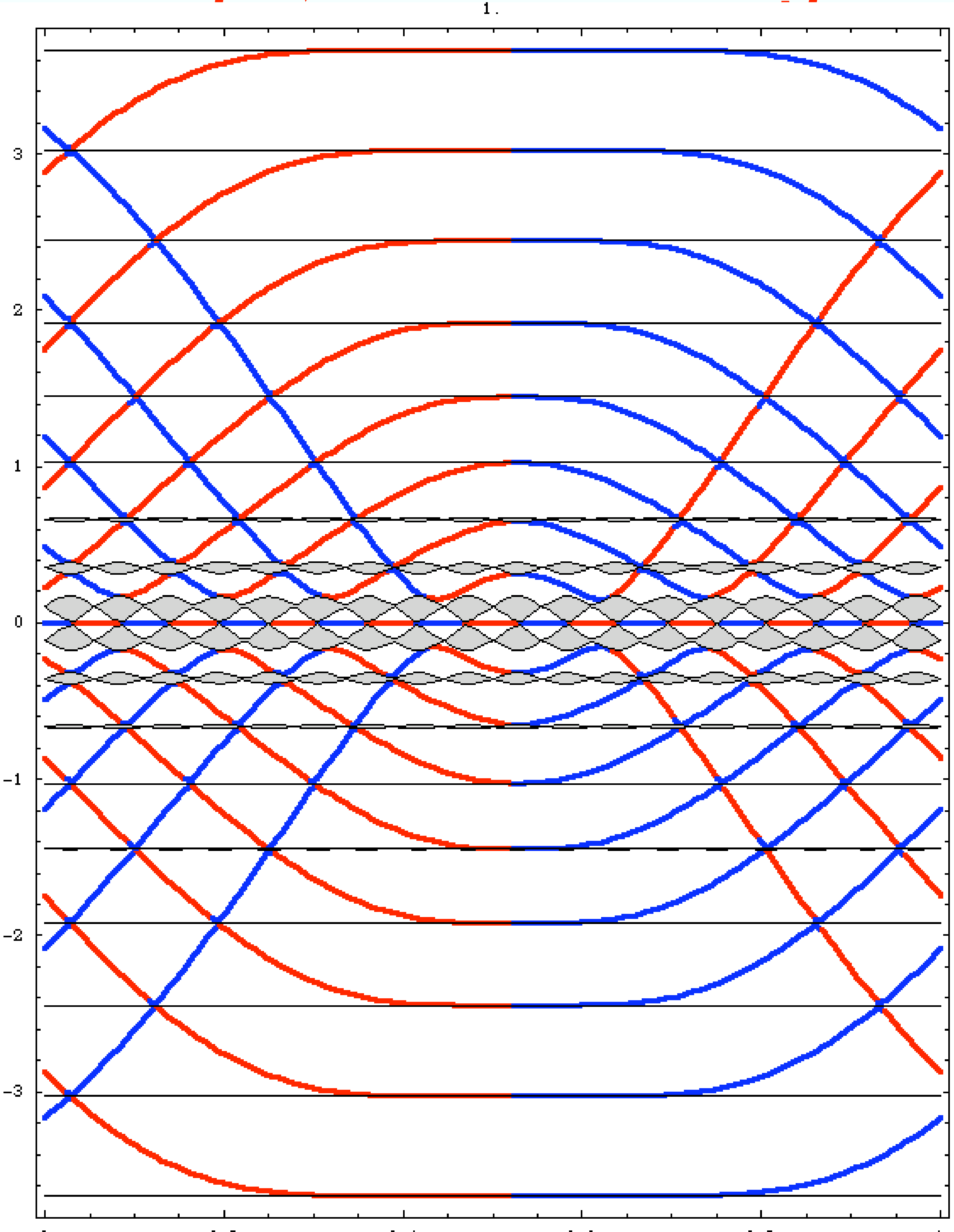
Adiabatic Equivalences of Edge States

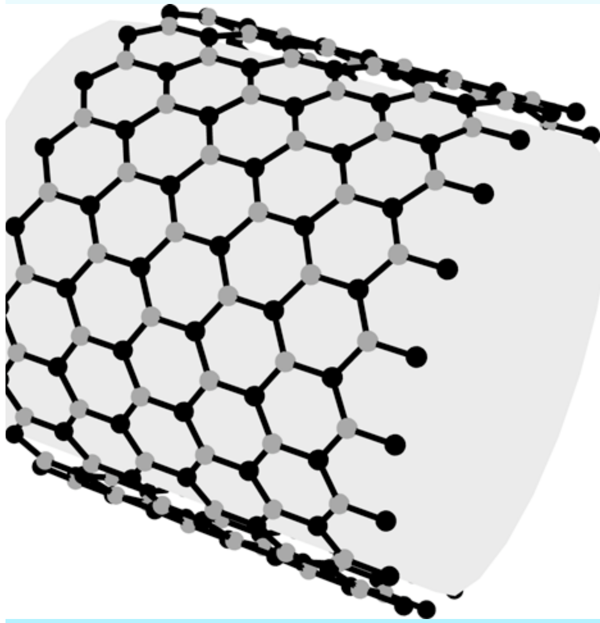
☆ Zigzag Edge

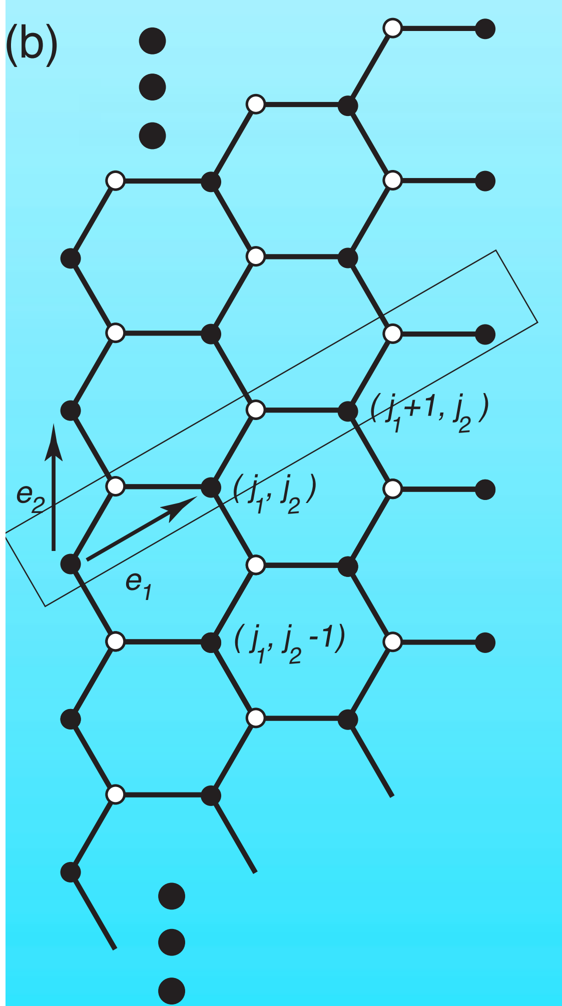
$\phi = 1/9$

Near Zero

Two Topological Equivalences Near the Zero and Near the BandEdges







$t'/t = 1$: Square Lattice

$t'/t = 0$:Honeycomb Lattice

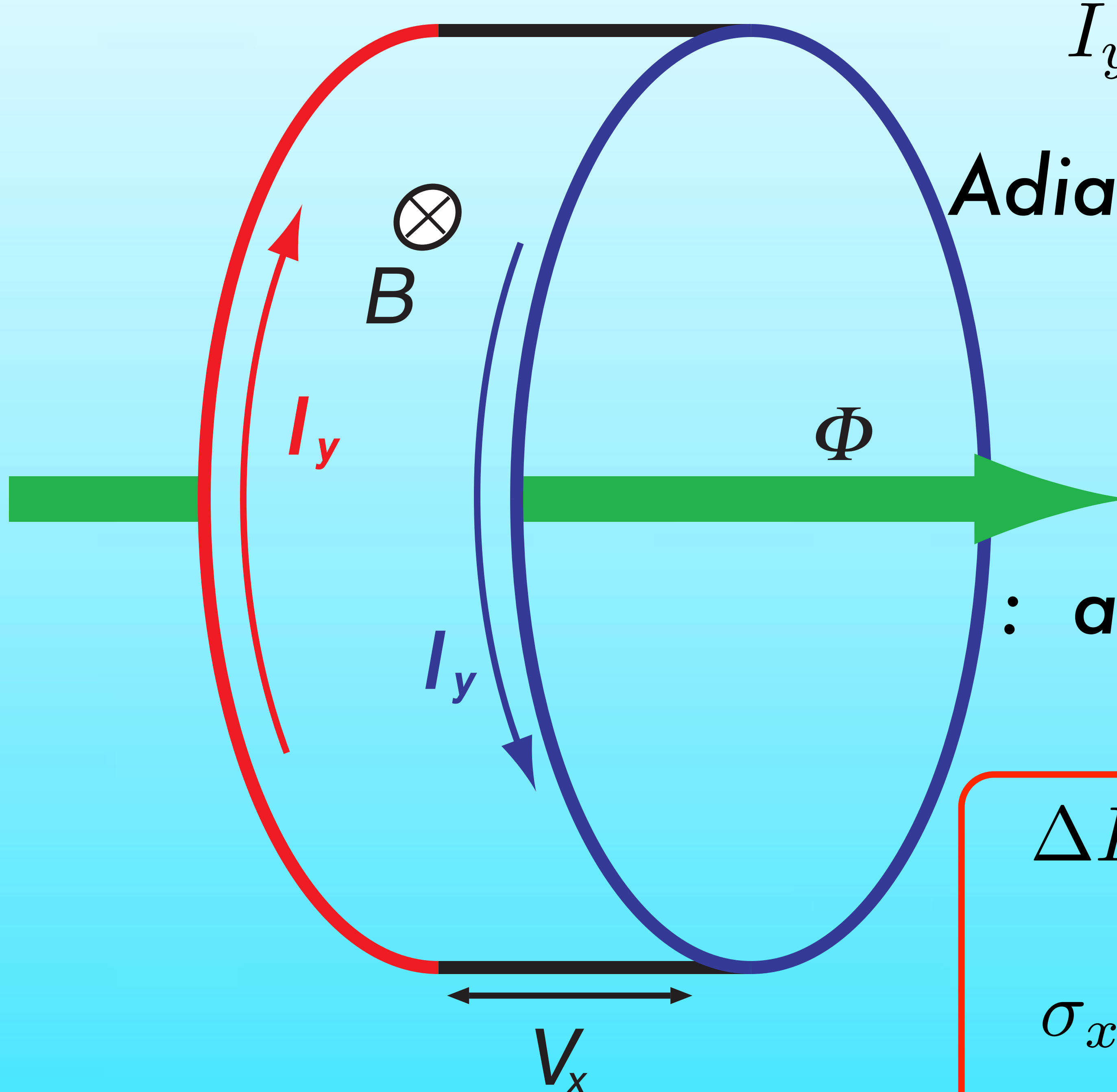
$t'/t = -1$: π Flux State

How the edge states determine ?

How to calculate σ_{xy} by the edge states?

Laughlin's Argument & Edge States

☆ Gauge Invariance & Byers-Yang' Formula



$$I_y = \frac{\Delta E}{\Delta \Phi} = \sigma_{xy} V_x$$ Byers-Yang

Adiabatic increase by $\Delta\Phi = \Phi_0 = \frac{h}{e}$

→ Insulating System goes back to the Original State

: assume n electrons are carried from the left to the right

$$\Delta E = neV_x$$

$$\sigma_{xy} = \frac{e^2}{h} n$$

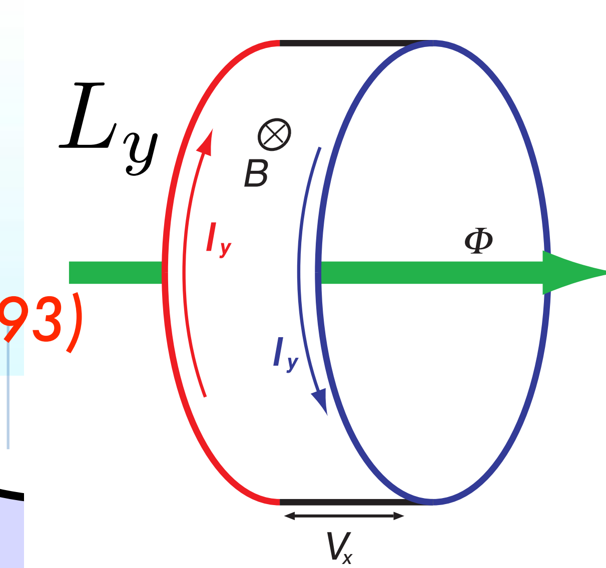
n is an integer but unknown

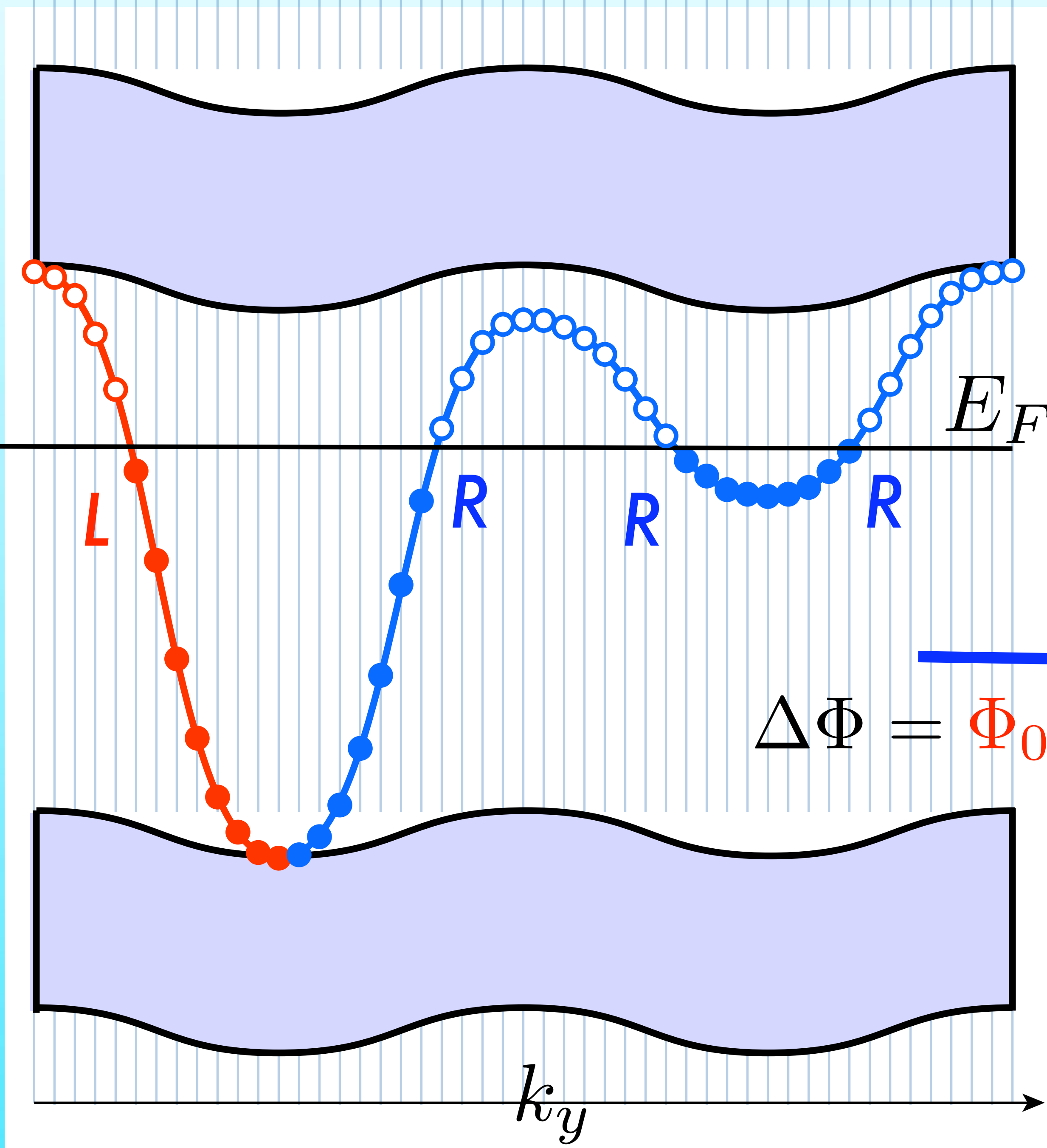
Quantization of σ_{xy} by Edge states

Edge States & Hall Conductance

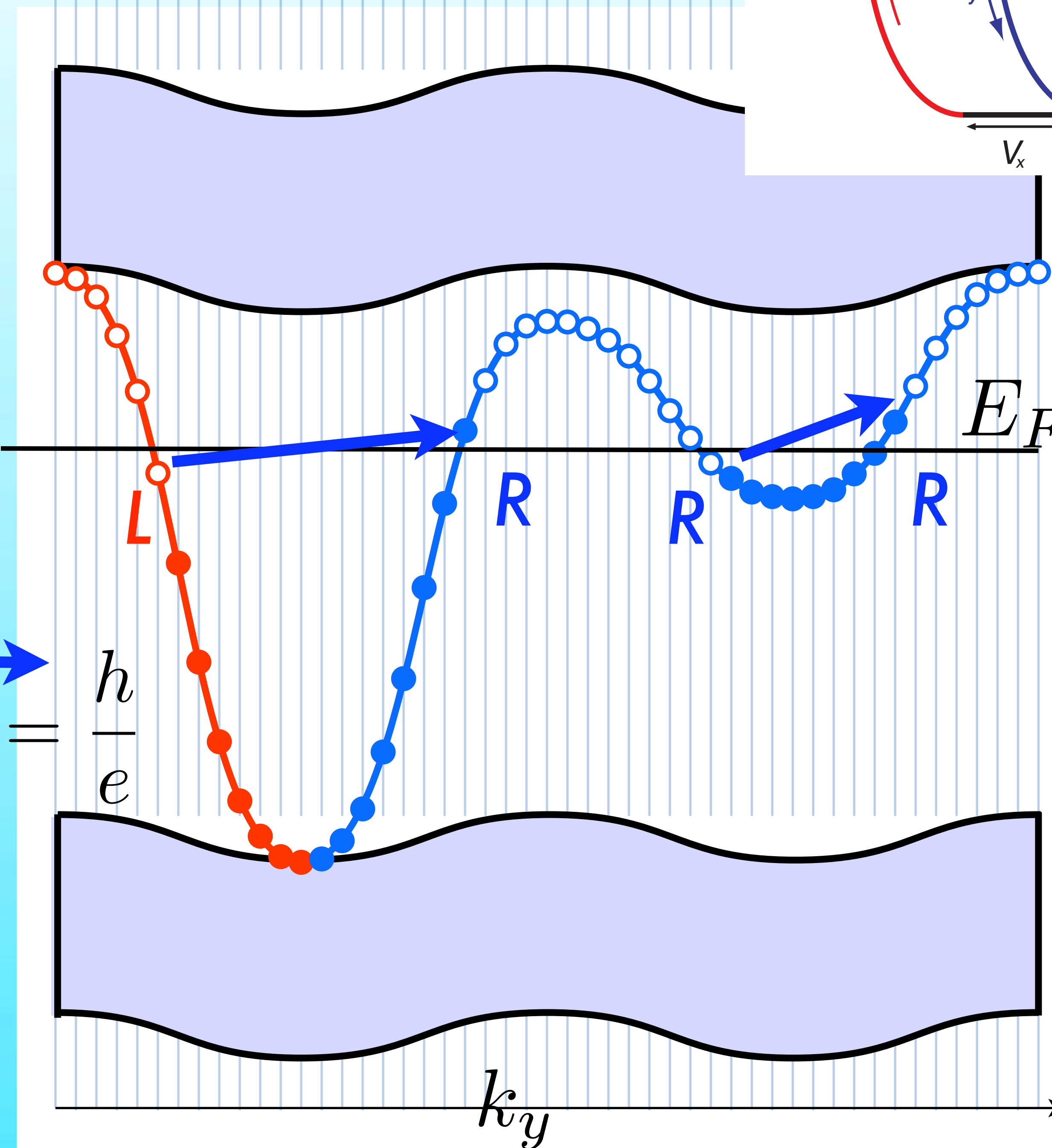
☆ Adiabatic Charge Transfer

Y.H., Phys. Rev. B 48, 11851 (1993)





$$\Delta\Phi = \Phi_0 = \frac{h}{e}$$



$$k_y = 2\pi \frac{n + \frac{\Phi}{\Phi_0}}{L_y}, \quad n: \text{ integers}$$

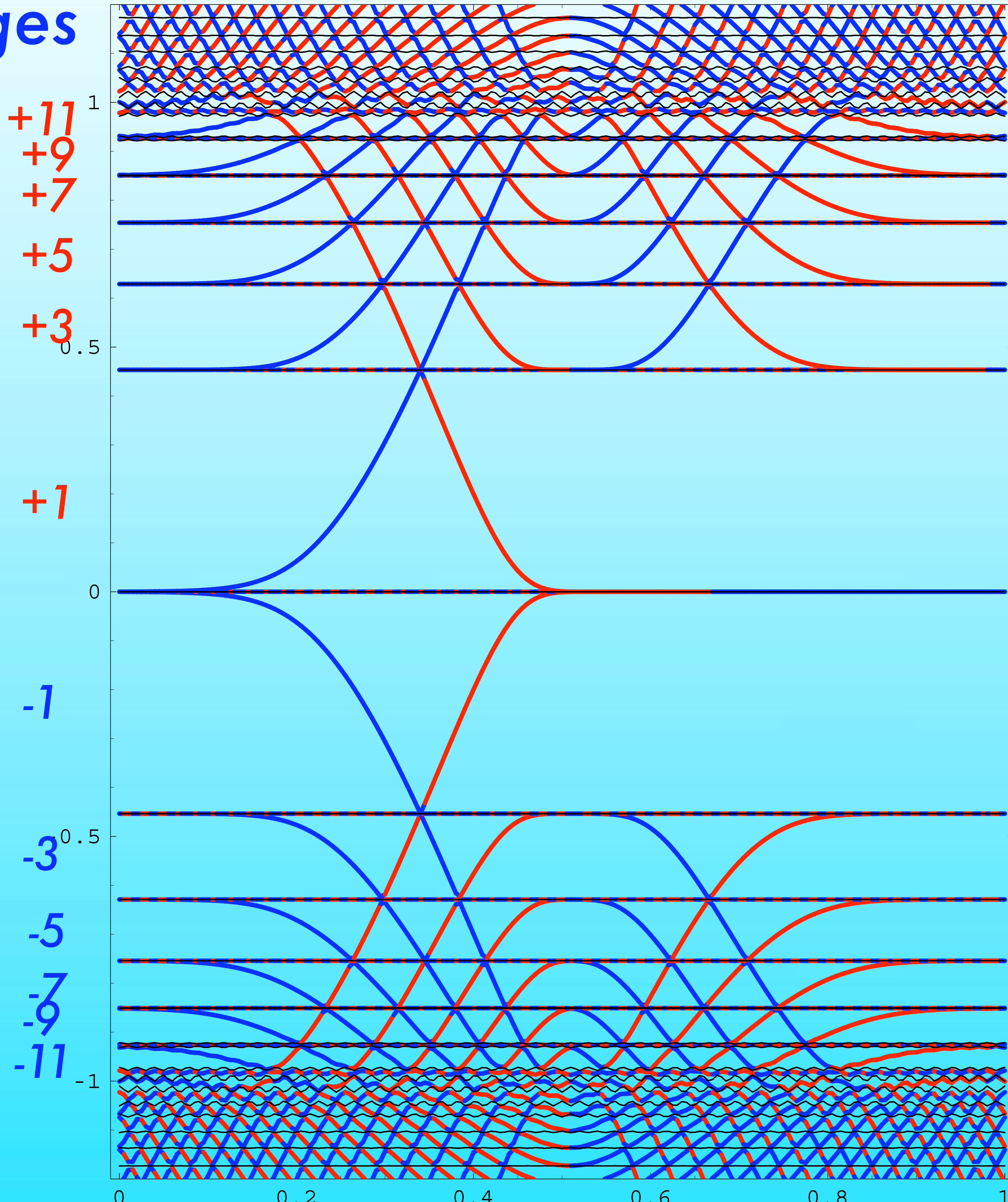
1 Electron is carried from the Left to the right in this case

$$\sigma_{xy} = \frac{e^2}{h} \cdot 1$$

Edge States of Graphene

☆ Zigzag Edges

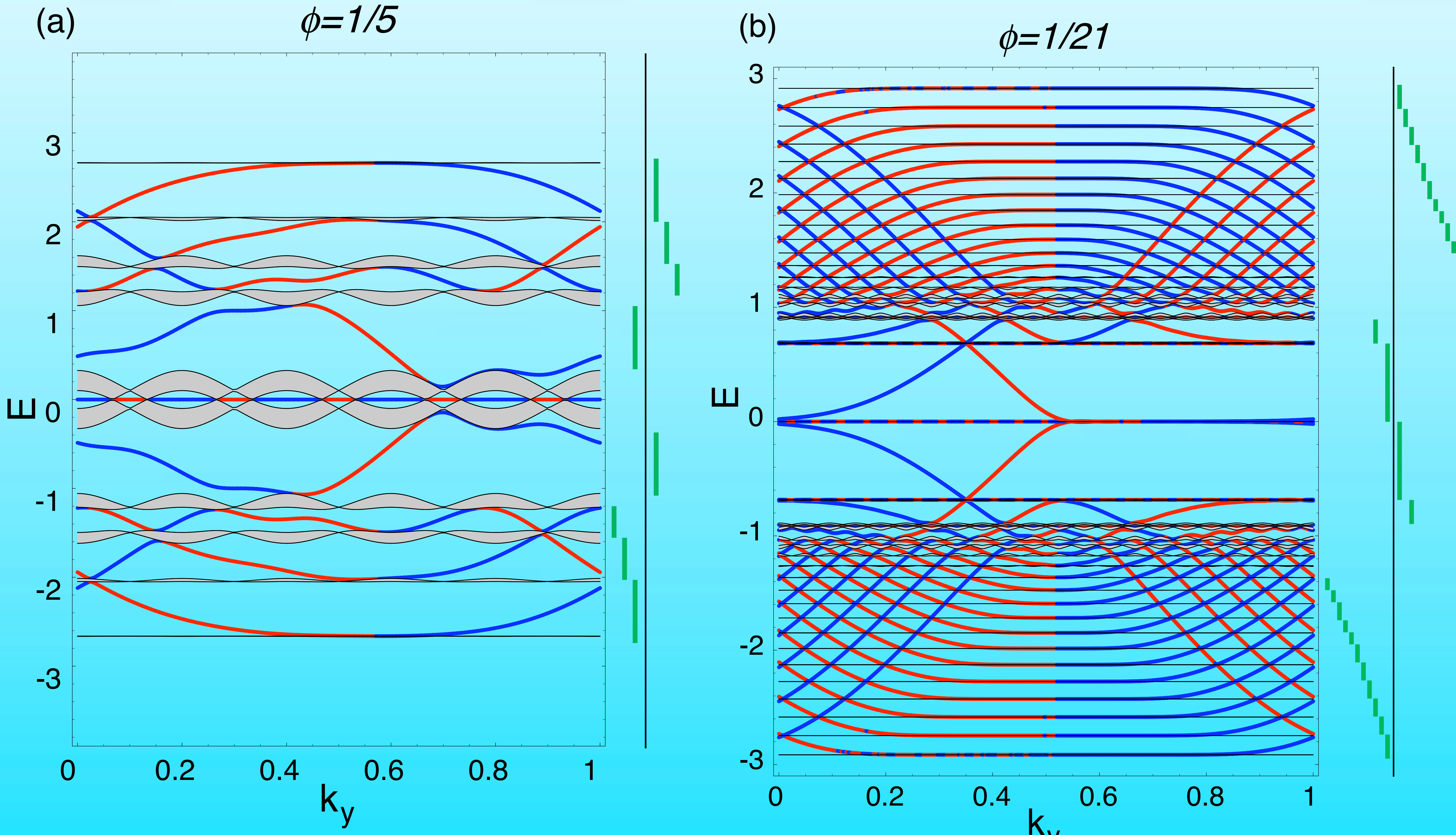
$\phi = 1/51$



Near Zero Energy

Edge States being consistent with Dirac Type Quantization appear

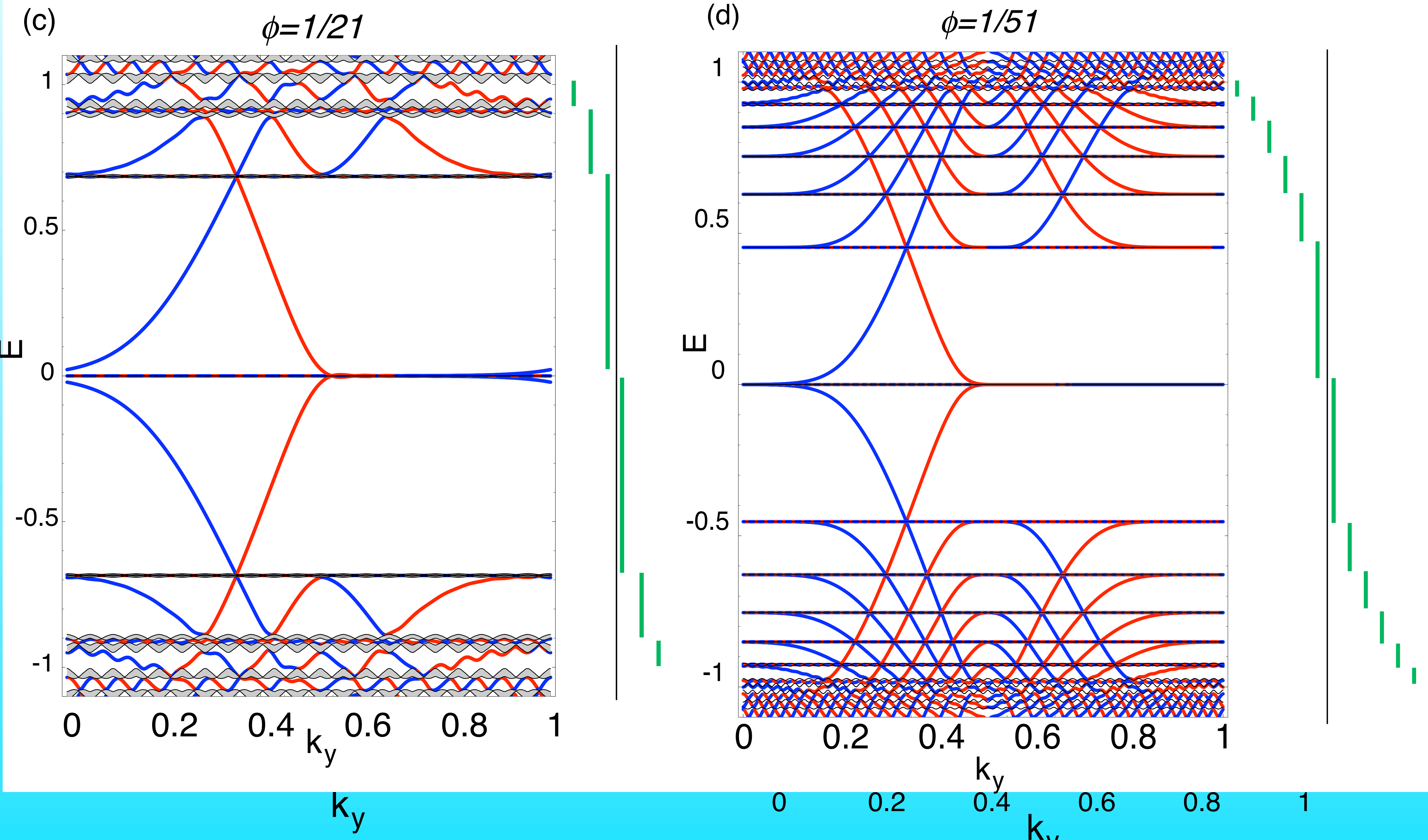
Bulk – Edge Correspondence ?



Bulk – Edge Correspondence ?

☆ ***Numerically*** $\sigma_{xy}^{\text{bulk}} = \sigma_{xy}^{\text{edge}}$

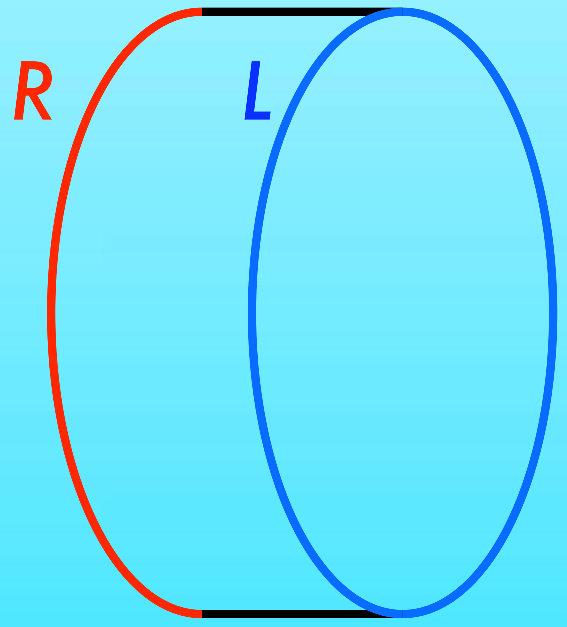
Near Zero

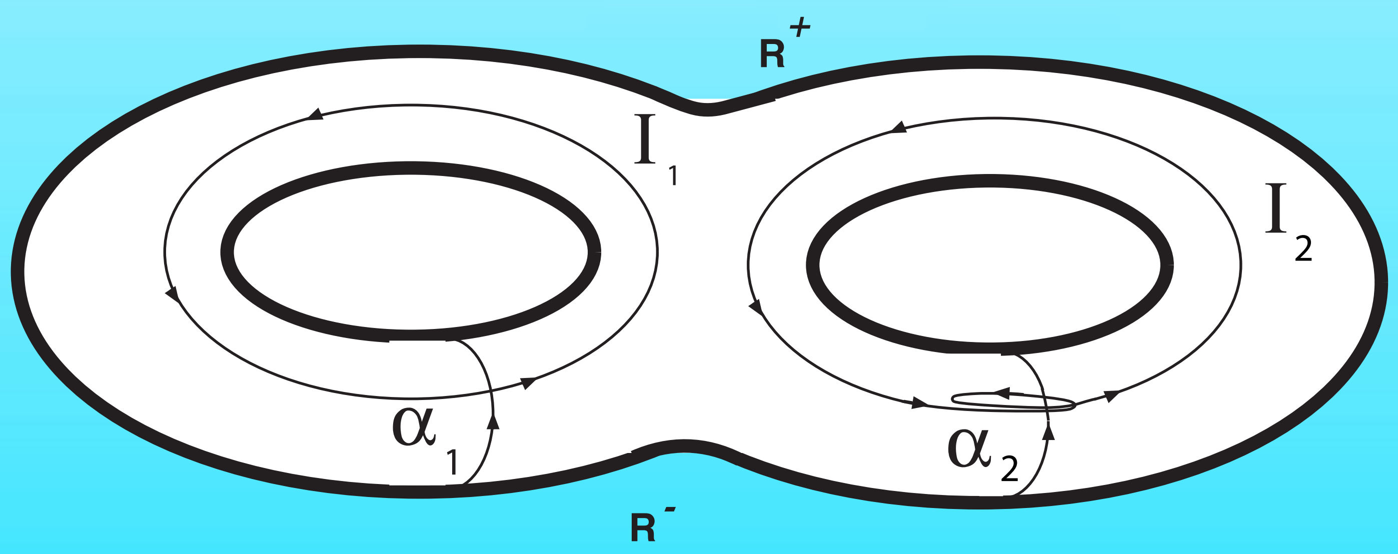


Analytical Consideration of edge states in Graphene

☆ *Followed by the discussion on a square lattice*

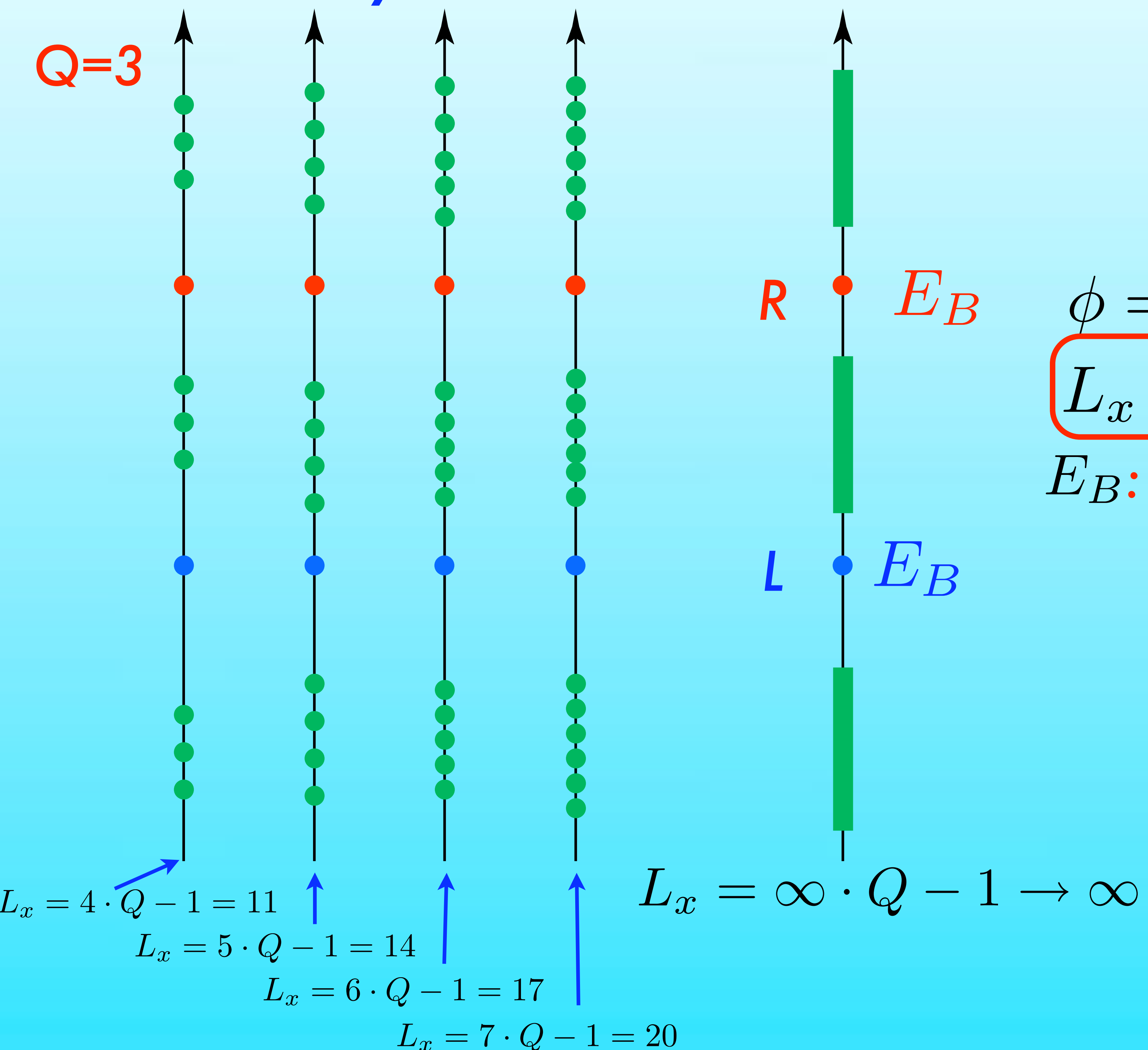
Y.H., Phys. Rev. B 48, 11851 (1993)
Phys. Rev. Lett. 71, 3697 (1993)

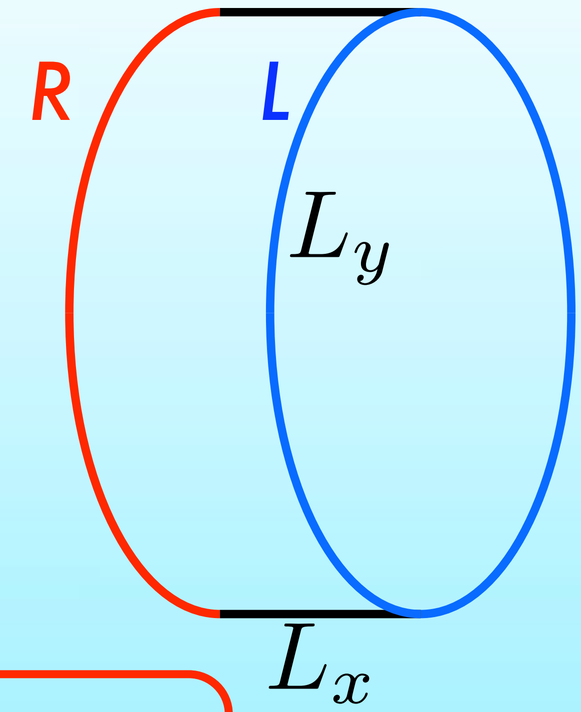




Width (L_x) dependence of the spectrum

☆ 1D system with Boundaries of length L_x





$$\phi = P/Q$$

$$L_x \equiv -1, \quad \mod Q$$

E_B: independent of L_x

Q=2q: graphene

Infinite size limit
with Boundaries

Edge State and Bloch State

☆ *reduced 1D system and transfer matrix*

$$H = \sum_{k_y} H_{\rm 1D}(k_y)$$

Y.H., Phys. Rev. B 48, 11851 (1993)
Phys. Rev. Lett. 71, 3697 (1993)

$$|E,k_y\rangle = \sum_{j_x}\left[\psi_\bullet(E,j_x,k_y)c_\bullet^\dagger(j_x,k_y)|0\rangle + \psi_\circ(E,j_x,k_y)c_\circ^\dagger(j_x,k_y)|0\rangle\right],$$

$$H_{\rm 1D}(k_y)|z,k_y\rangle = z|z,k_y\rangle,\ \ z=E$$

$$M_{\circ\bullet}(j_x) = \begin{pmatrix} \frac{E}{t^*_{\circ\bullet}(j_x)} & -\frac{t_{\bullet\circ}(j_x-1)}{t^*_{\circ\bullet}(j_x)} \\ 1 & 0 \end{pmatrix}$$

$$M_{\bullet\circ}(j_x) = \begin{pmatrix} \frac{E}{t^*_{\bullet\circ}(j_x)} & -\frac{t_{\circ\bullet}(j_x)}{t^*_{\bullet\circ}(j_x)} \\ 1 & 0 \end{pmatrix}$$

Transfer matrix $\boldsymbol{\psi}(j_x+1) = M_{\rm t}(j_x)\boldsymbol{\psi}(j_x)$

$$\boldsymbol{\psi}(j_x) = \begin{pmatrix} \psi_\bullet(j_x) \\ \psi_\circ(j_x-1) \end{pmatrix} \qquad M_{\rm t}(j_x) = M_{\bullet\circ}(j_x)M_{\circ\bullet}(j_x)$$

$$t_{\circ\bullet}(j_x,k_y) = t\left(1+e^{ik_y - i2\pi\phi j_x}\right)$$
$$t_{\bullet\circ}(j_x,k_y) = t\left[1+(t'/t)e^{ik_y - i2\pi\phi(j_x+1/2)}\right]$$

How these two are related ??

Bloch State

$$\boldsymbol{\psi}_B(q) = M\boldsymbol{\psi}_B(0) = \rho\boldsymbol{\psi}_B(0)$$
$$|\rho| = 1$$

Edge State

$$\boldsymbol{\psi}_E(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\quad \boldsymbol{\psi}_E(q) = \begin{pmatrix} * \\ 0 \end{pmatrix}$$

$$M = M_{\rm t}(q-1)M_{\rm t}(q-2)\cdots M_{\rm t}(0)$$

Analytic Continuation of the Bloch State

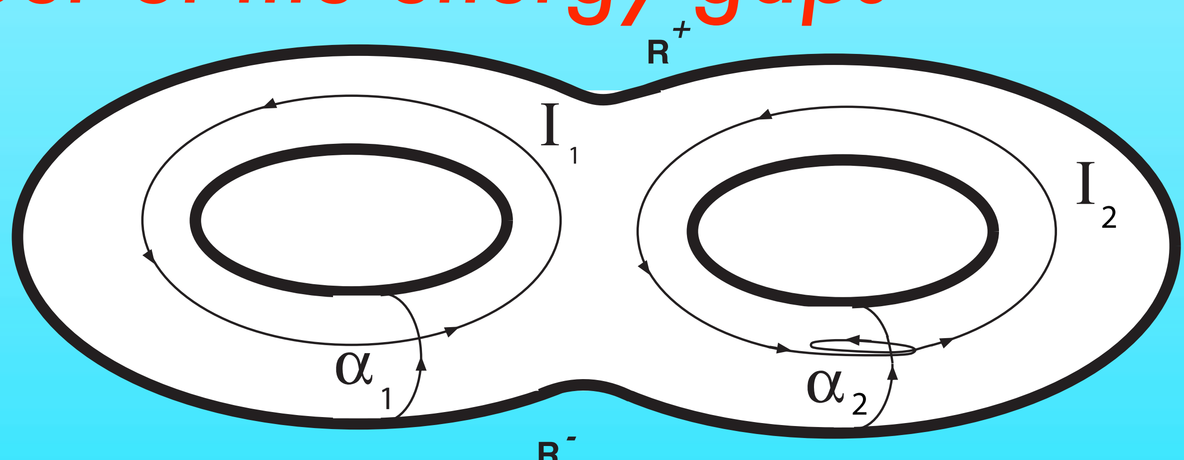
- The Edge State is obtained from the Bloch State by Analytical continuation

Y.H., Phys. Rev. B 48, 11851 (1993)
Phys. Rev. Lett. 71, 3697 (1993)

  - Energy of the Bloch state ψ_B is in the band
  - Energy of the edge state ψ_E is in the gap
- Complex energy surface is required

ψ_B & ψ_E :Unified on Complex Energy surface

  - Energy bands : branch cuts, 2 Riemann sheets required
  - Q branch cuts
  - genus (number of holes) g=Q-1 Riemann surface
  - g : number of the energy gaps



$$\phi = P/Q$$

$$\sqrt{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_{2Q-1})(z-\lambda_{2Q})}$$

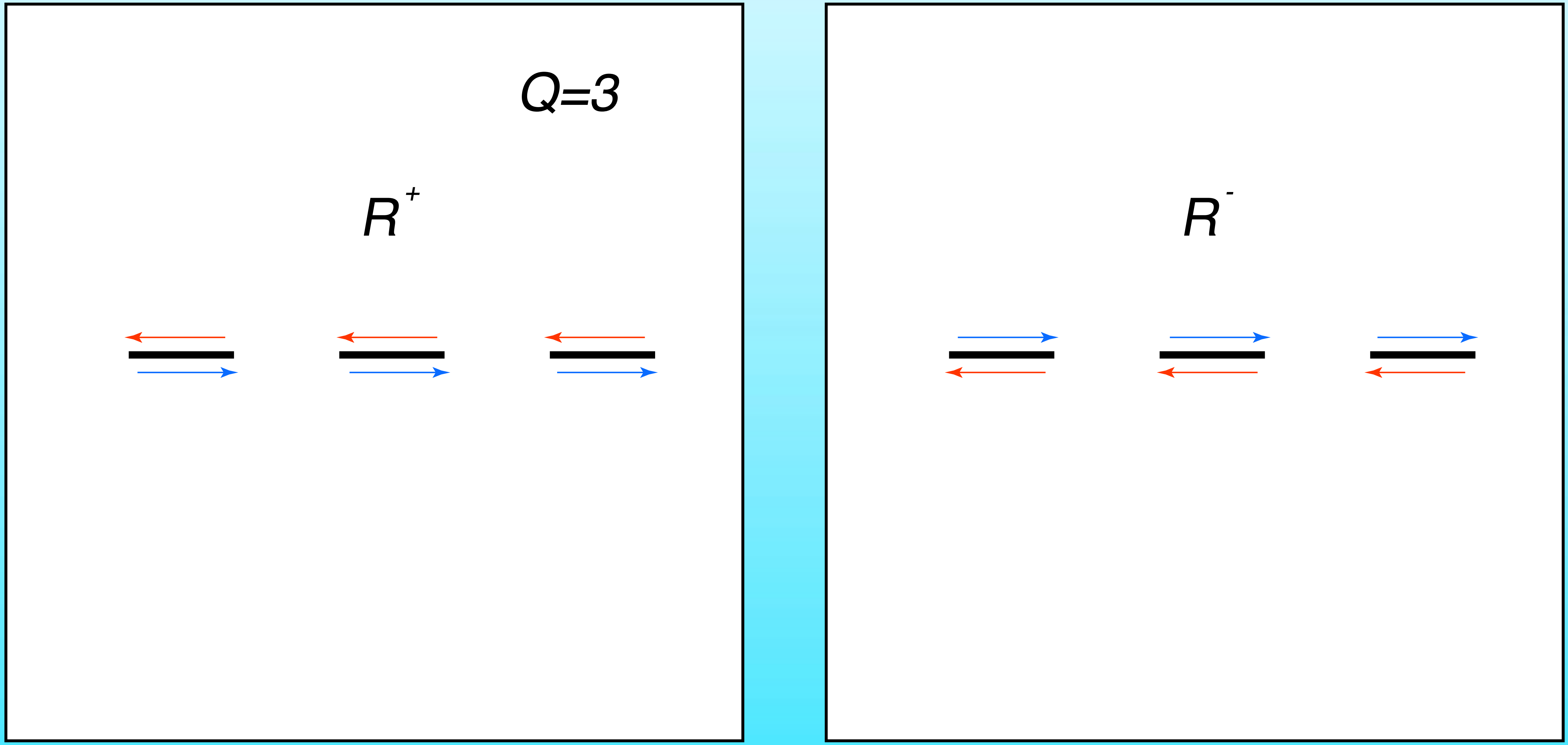
Construction of the Riemann surface

$\Phi = P/Q, \quad Q = 3$

$$\sqrt{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_{2Q-1})(z-\lambda_{2Q})}$$

- Glue 2 complex planes with Q branch cuts

Q=3 energy bands: Q=3 branch cuts



g=Q-1 holes

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- ☆ Glue 2 complex planes with Q branch cuts

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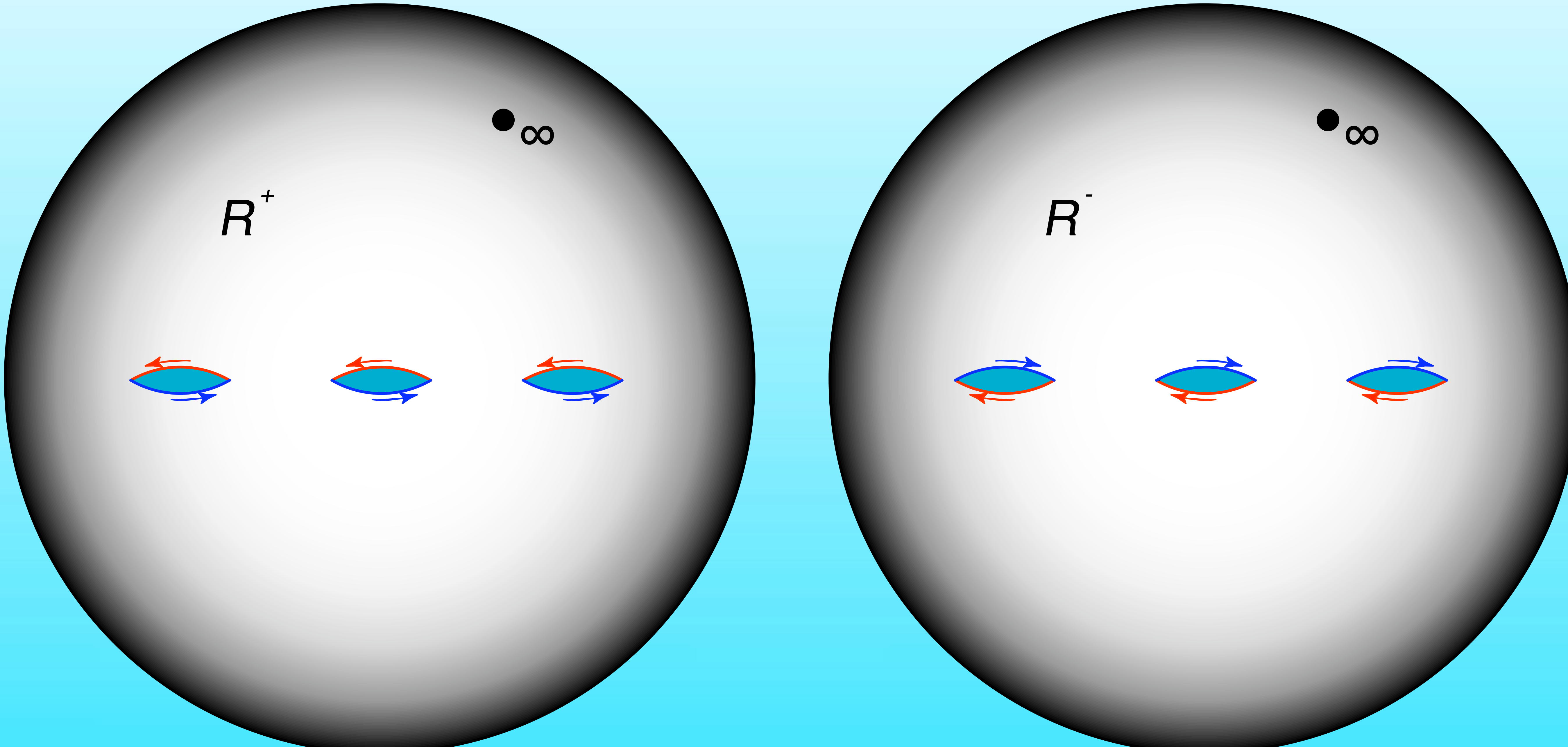

g=Q-1 holes

Construction of the Riemann surface

$\Phi = P/Q, \quad Q = 3$ $\quad \sqrt{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_{2Q-1})(z-\lambda_{2Q})}$

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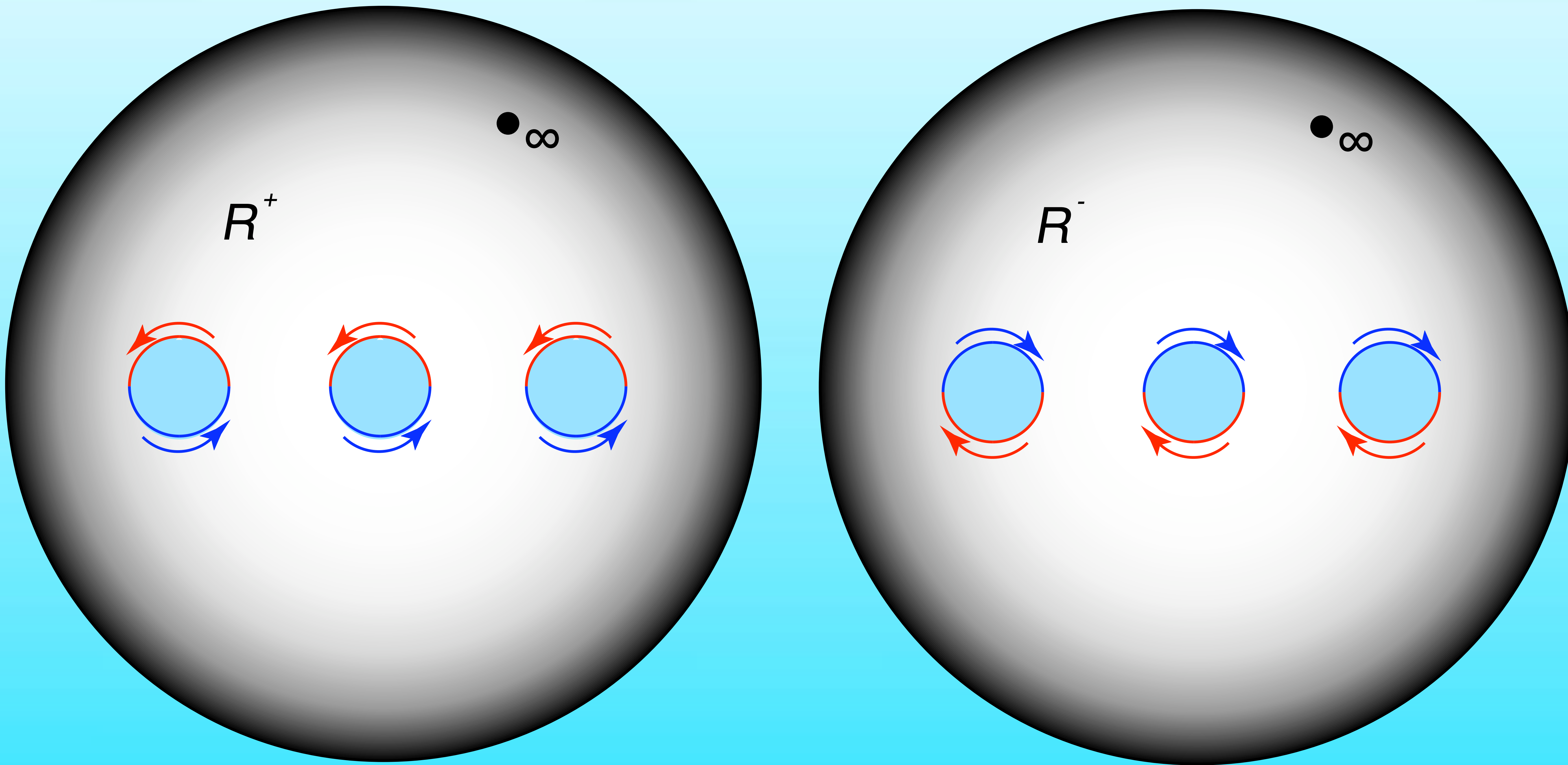

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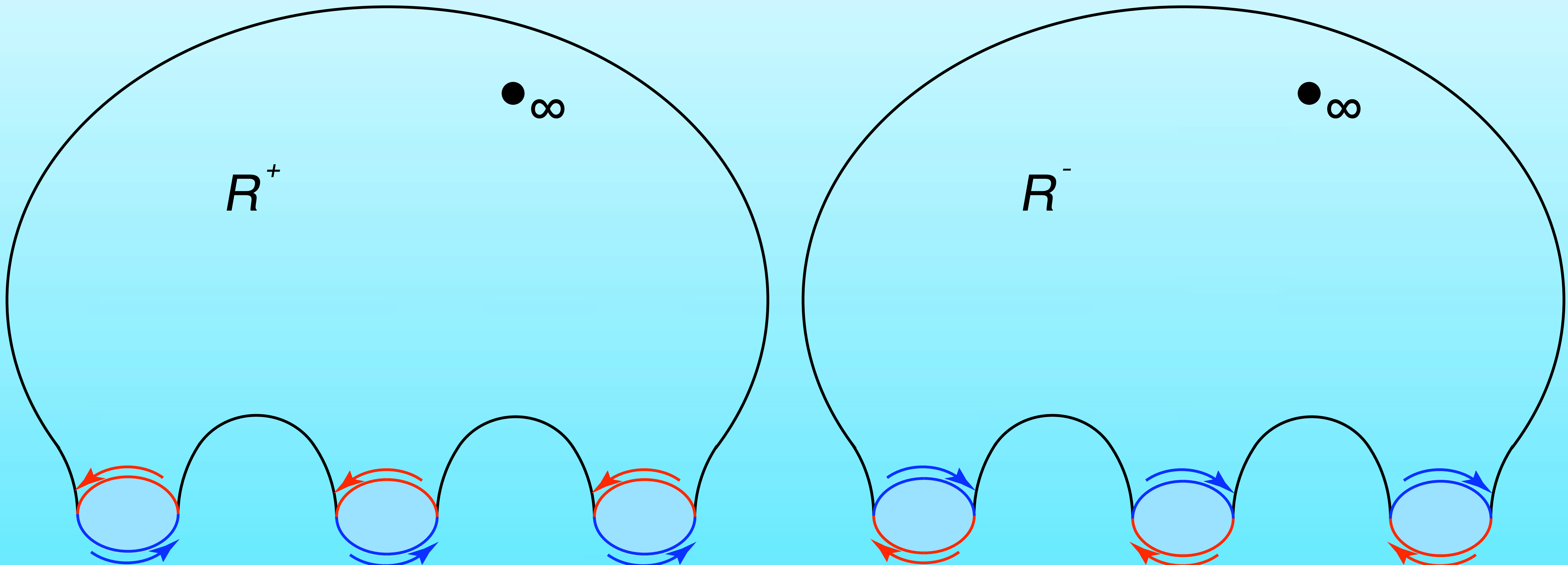

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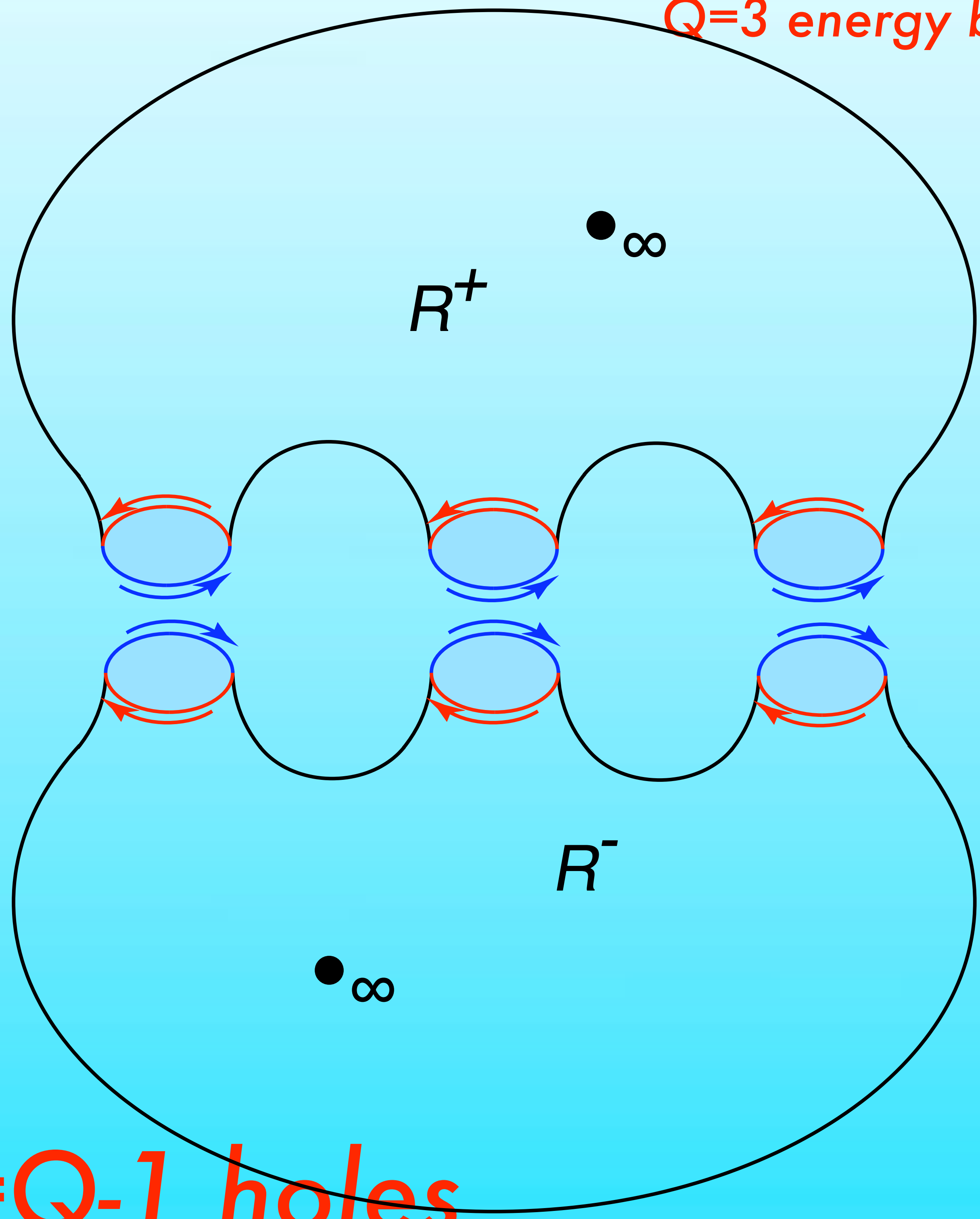

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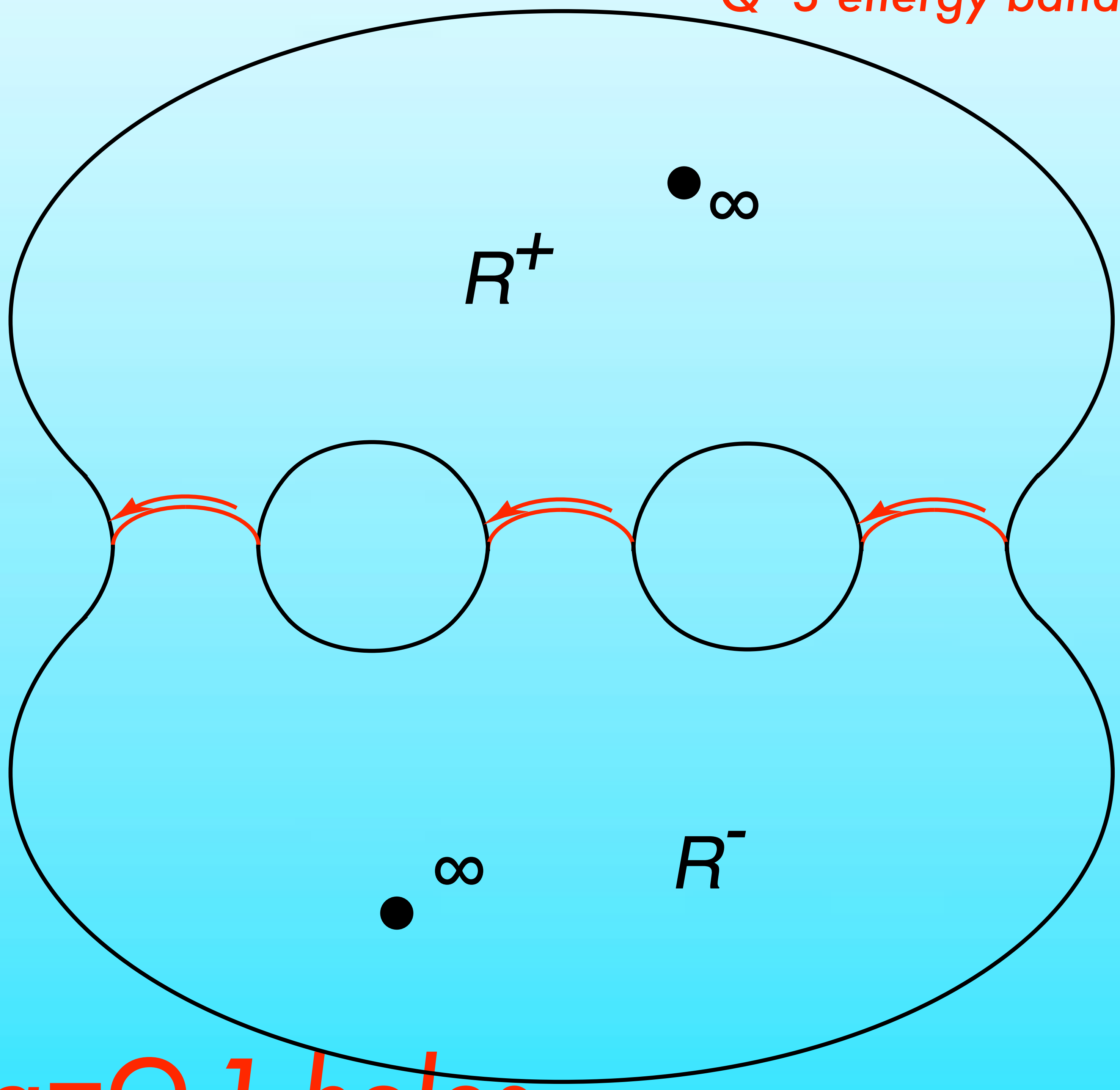

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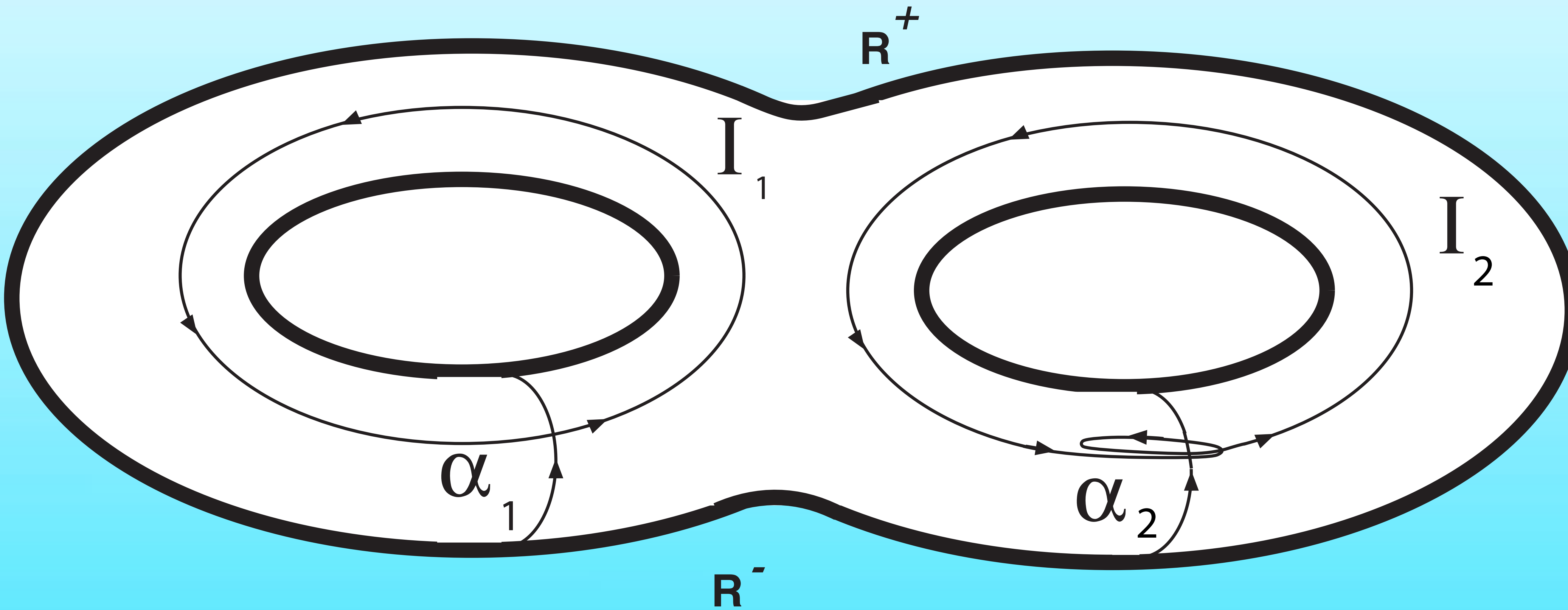
Q=3 energy bands: Q=3 branch cuts



g=Q-1 holes

Wave function & Riemann Surface

As for fixed ky of the 1D systems

- Zeros of the Bloch fn. defines the Edge State Energies

Energy bands	⟷	Branch cuts
Energy gaps	⟷	Holes

W. fn. is localized at		The zero of the Bloch fn. is on
the left edge	⟷	the upper Riemann Surface R^+
the right edge	⟷	the upper Riemann Surface R^-

- Changing $k_y \in [0, 2\pi]$, the zero in the j-th gap makes a closed loop

Wave function & Riemann Surface

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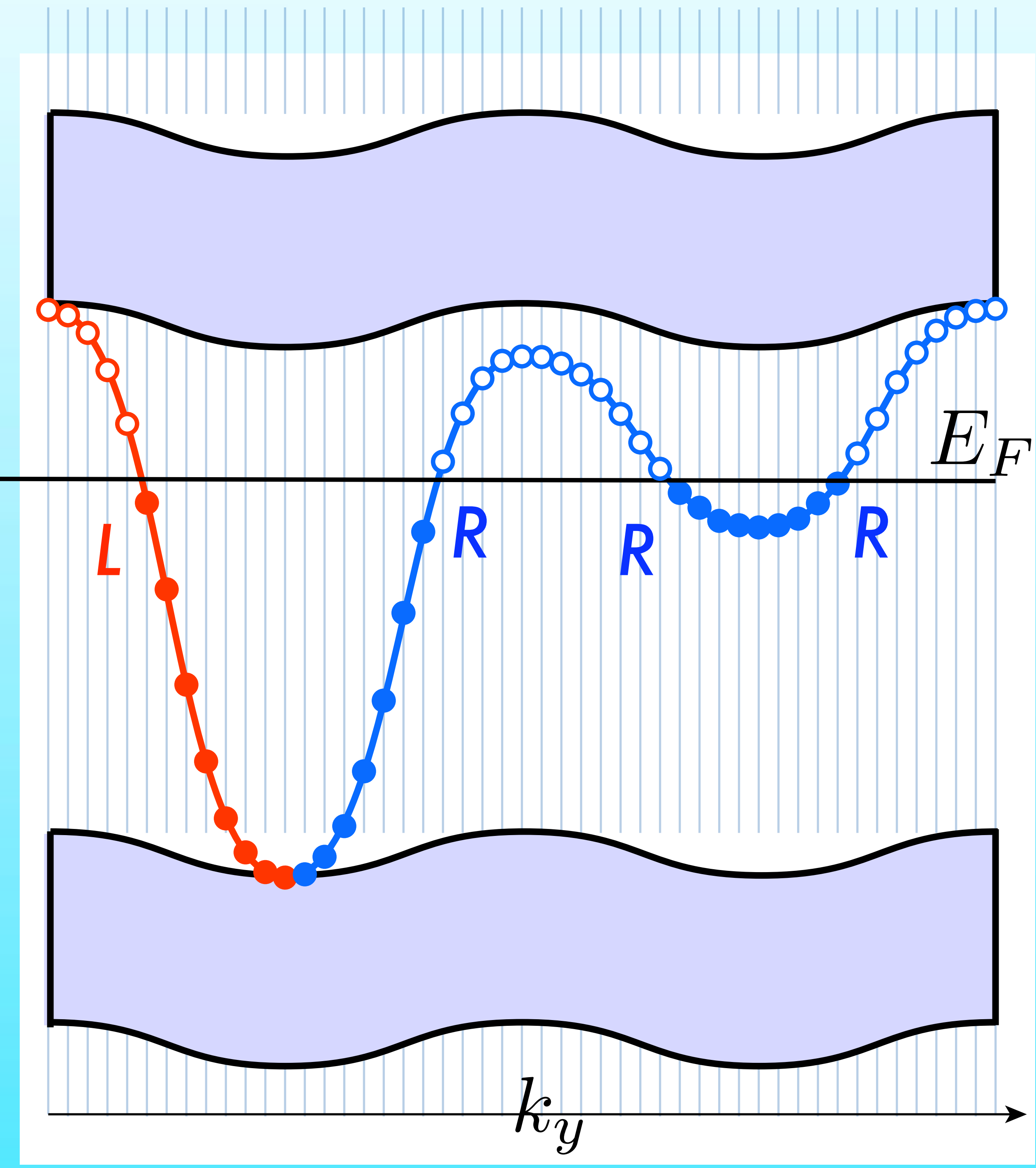
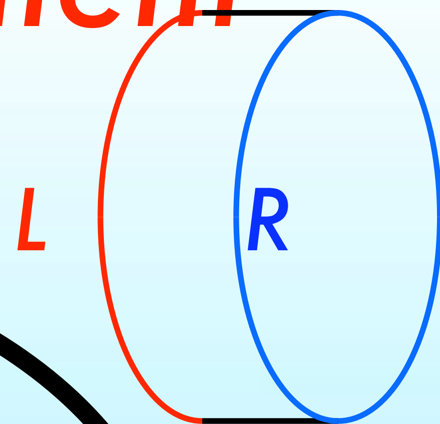
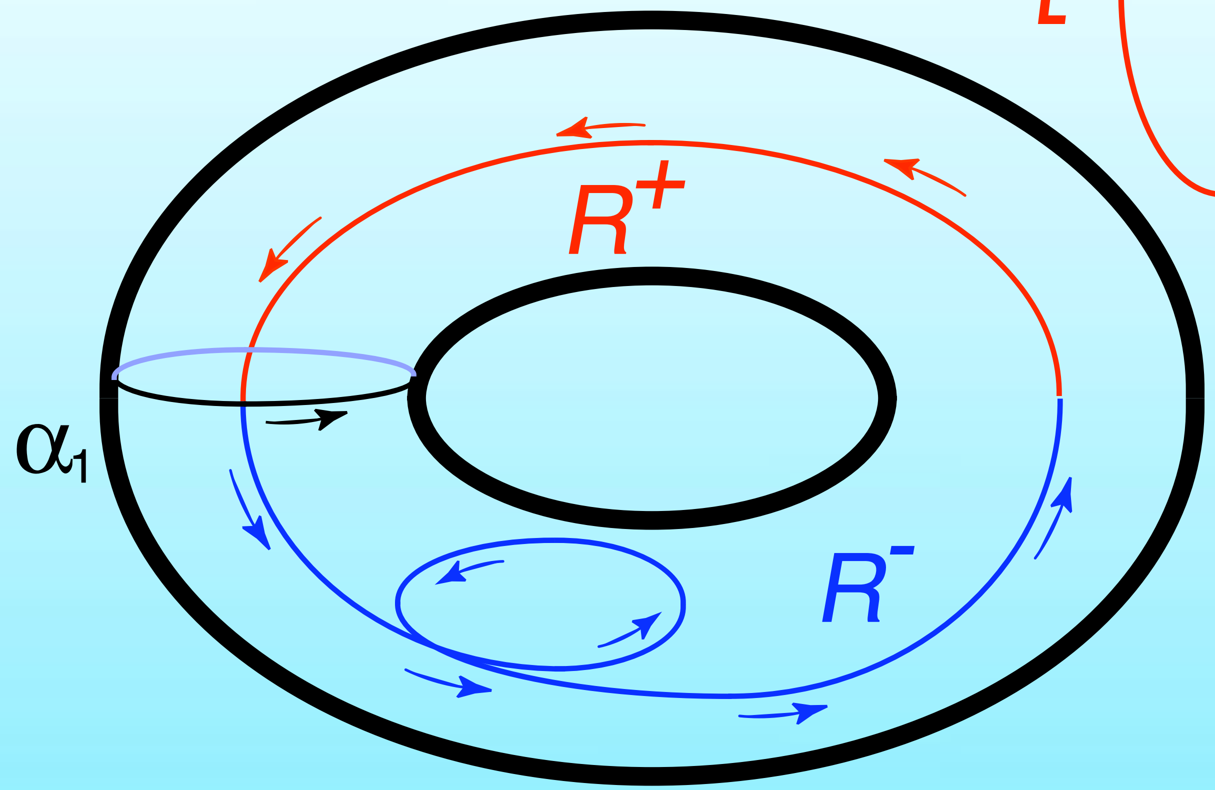
Riemann surface & Laughlin's Argument







$$I(\alpha_j, L^j_{\text{edge}}) = +1, \;\; j = 1$$

Winding number

or

Intersection number with canonical loop

$$\sigma_{xy}^{j,\text{Edge}} = \frac{e^2}{h} \cdot I(\alpha_j, C^j_{\text{edge}})$$

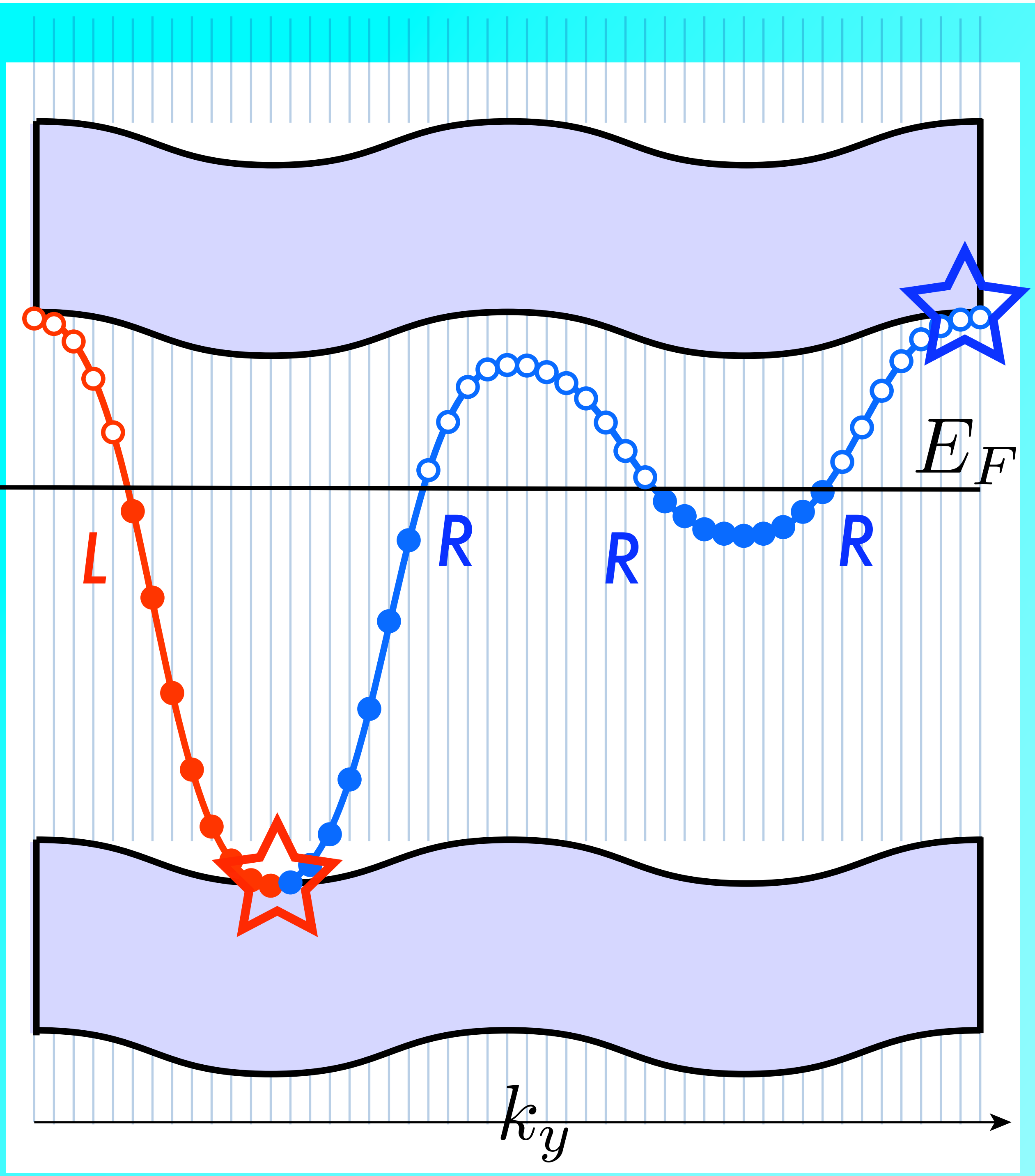
Y.H., Phys. Rev. B 48, 11851 (1993)

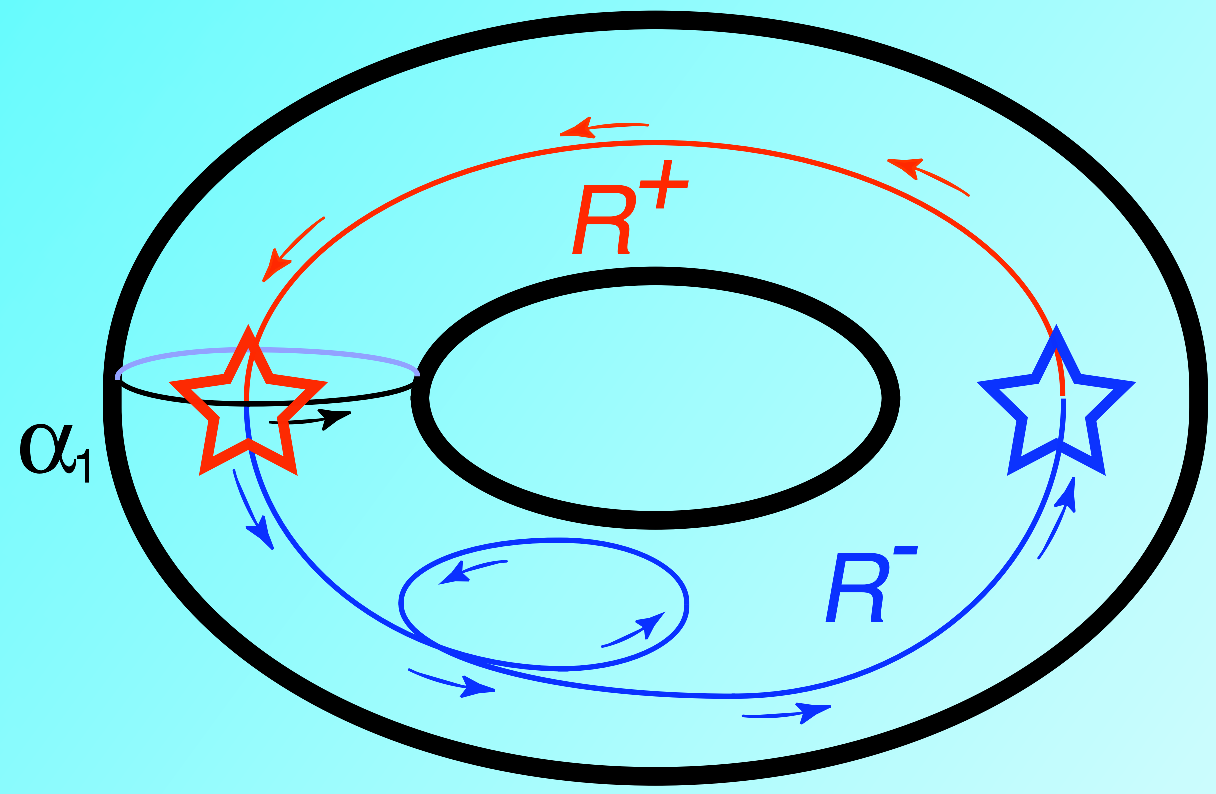
Bulk – Edge Correspondence

- *Hall Conductance of the Bulk States* $\sigma_{xy}{}^{\rm bulk}$
  - Chern Number, $C^j_{\rm FS}$
- *Hall Conductance of the Edge States* $\sigma_{xy}{}^{\rm edge}$
  - Intersection number, $I(\alpha_j, C^j_{\rm edge})$

Their relation:

Edge State make a vortex when it touches to the bands





Y.H., Phys. Rev. Lett. 71, 3697 (1993)

☆ ☆ The touching point makes a vortex in the eneryg band

Which contribute to the Chern number of the Bulk

$$C^j_{\mathrm{FS}} = I(\alpha_j, C^j_{\mathrm{edge}})$$

Bulk – Edge Correspondence

- Hall Conductance of the Bulk States $\sigma_{xy}{}^{\text{bulk}}$
  - Chern Number, C^j_{FS}
- Hall Conductance of the Edge States $\sigma_{xy}{}^{\text{edge}}$
  - Intersection number, $I(\alpha_j, C^j_{\text{edge}})$

Their relation:

Edge State make a vortex when it touches to the bands

As for topological quantities

$$C^j_{\text{FS}} = I(\alpha_j, C^j_{\text{edge}})$$

Its physical outcome

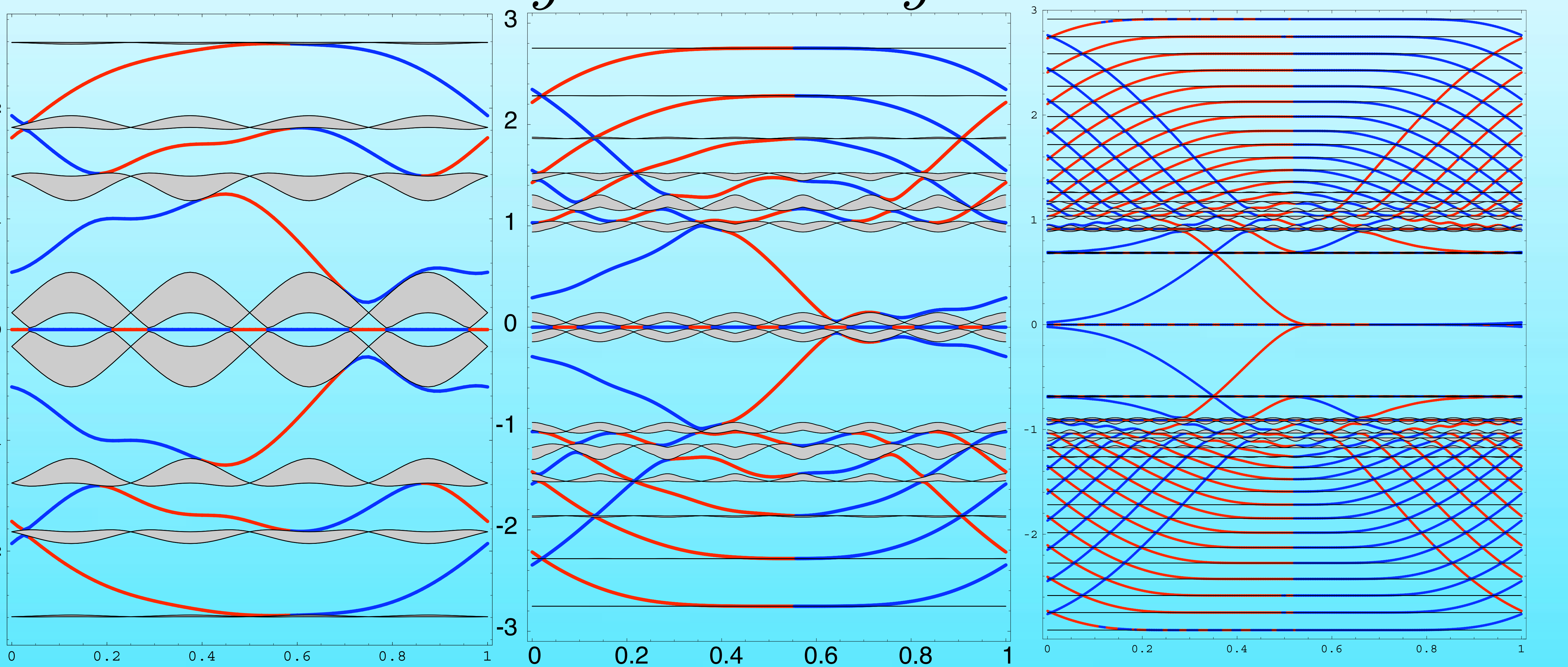
Y.H., Phys. Rev. Lett. 71, 3697 (1993)

$$\sigma_{xy}{}^{\text{bulk}} = \sigma_{xy}{}^{\text{edge}}$$

Justified in Graphene as well

Edge states & Intersection number of Edge State Loops

$$\sigma_{xy}^{\text{bulk}} = \sigma_{xy}^{\text{edge}}$$

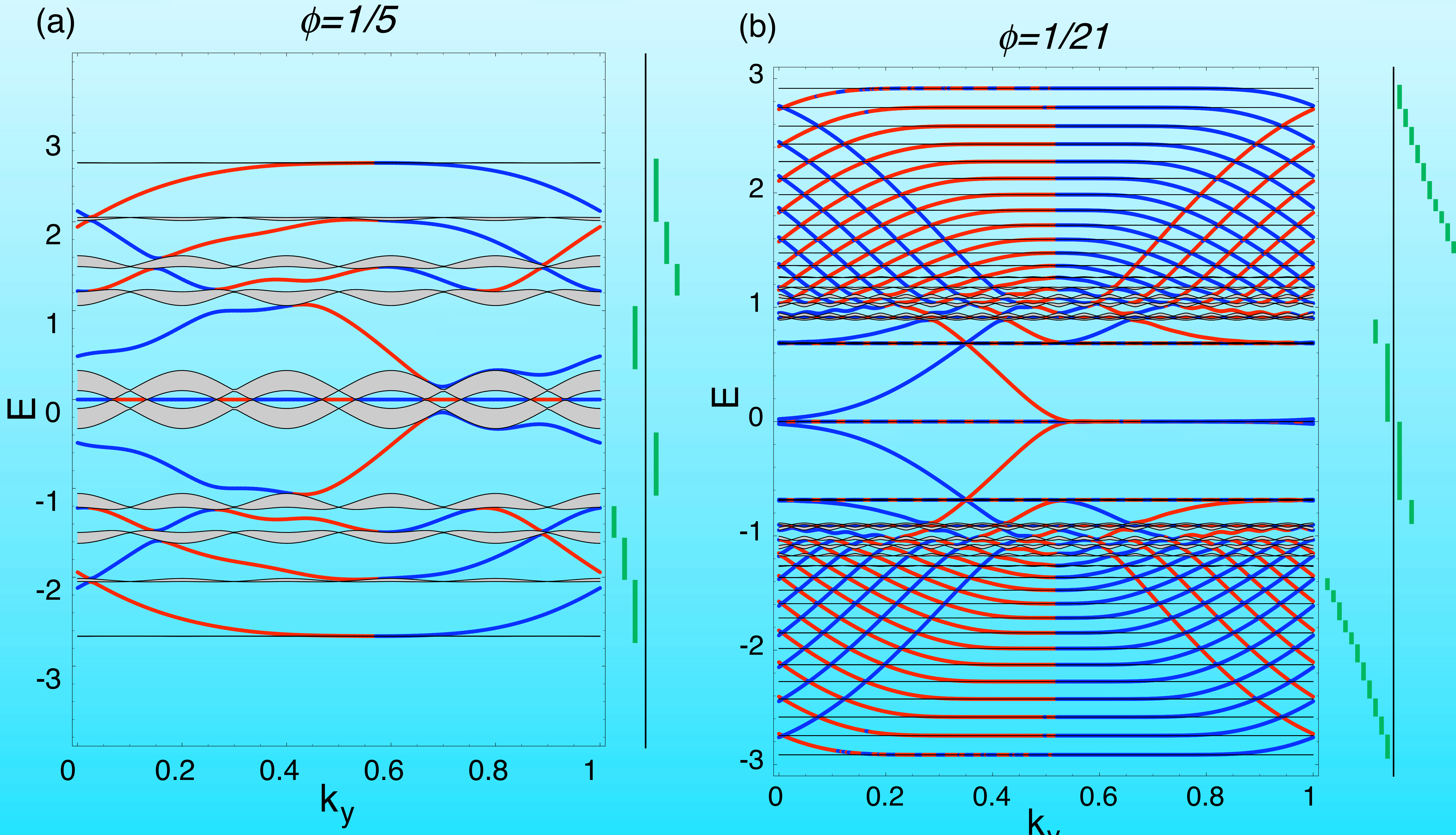


$\phi = 1/4$ $\phi = 1/7$ $\phi = 1/21$

Imagine loops on the Riemann surrface

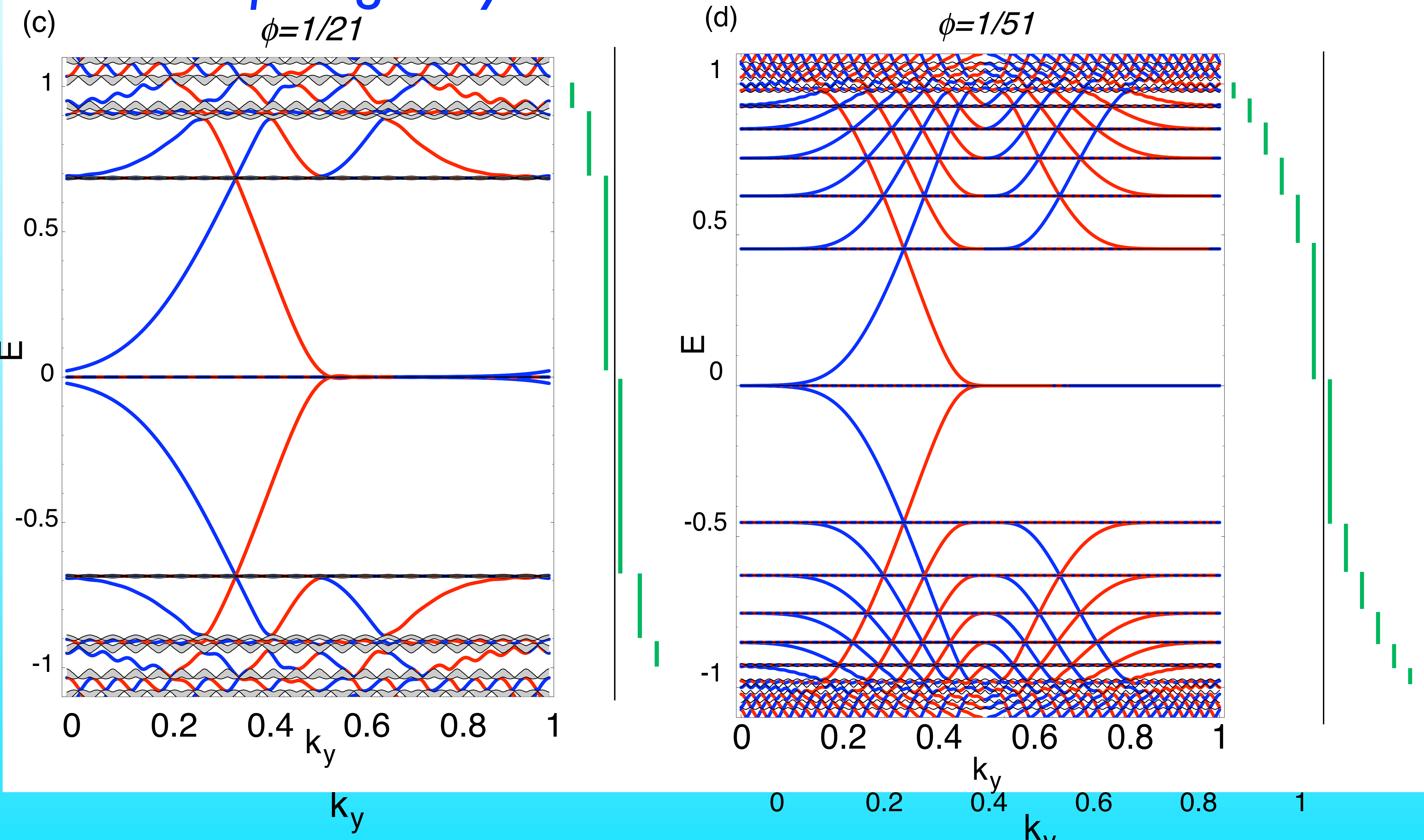
Bulk – Edge Correspondence

- ☆ Analytically $\sigma_{xy}{}^{\text{bulk}} = \sigma_{xy}{}^{\text{edge}}$
- ☆ Topologically



Bulk – Edge Correspondence

- ☆ *Analytically* $\sigma_{xy}{}^{\text{bulk}} = \sigma_{xy}{}^{\text{edge}}$
- ☆ *Topologically* **Near Zero**



Summary

- *Topological Aspects of Graphene (Bulk)*
  - *Topological Stability of the Dirac Fermions*
  - *Topological Stability of the Anomalous QHE*
    - *Adiabatic Principle and Topological Equivalence*
    - *Quantum phase Transition by chemical potential shift*
    - *Technical development for calculating Chern numbers (Lattice Gauge Theo*
- *Topological Aspects of Graphene (Edge)*
  - *Without Magnetic field (old work)*
    - *Topological Origin of Zero Modes*
  - *With Magnetic field*
    - *Edge States of Dirac Fermions*
- *Bulk – Edge Correspondence*
  - *Analytially and numerically*