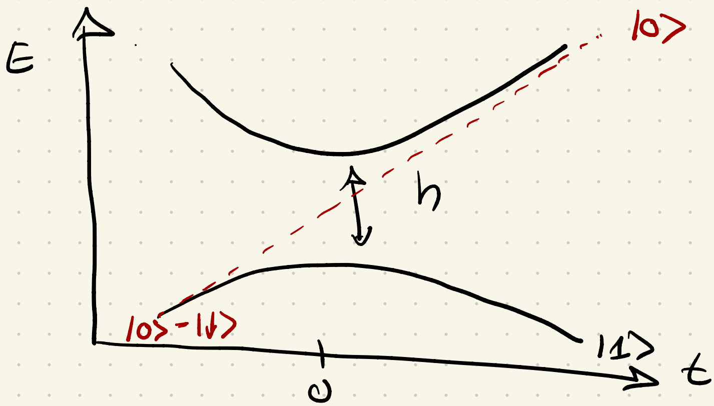
$$H_{\mathrm{LZ}}(t) = \frac{vt}{2}\sigma^z + \frac{h}{2}\sigma^x = \begin{pmatrix} \frac{vt}{2} & \frac{h}{2} \\ \frac{h}{2} & -\frac{vt}{2} \end{pmatrix},$$

$\hookrightarrow t \to \pm\infty$



1.1

$$i \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle$$

$$|\psi(t)\rangle = c_0(t) |0\rangle + c_1(t) |1\rangle$$

$$\text{I} \quad \dot{c}_1(t) = -i \frac{vt}{2} c_1 - i \frac{h}{2} c_0 \qquad / \partial_t$$

$$\text{II} \quad \dot{c}_0(t) = -i \frac{h}{2} c_1 + i \frac{vt}{2} c_0$$

$$\ddot{c}_1 = -i \frac{v}{2} c_1 - \frac{ivt}{2} \dot{c}_1 - i \frac{h}{2} \dot{c}_0$$

$$\text{I}: \; -i \frac{vt}{2} c_1 - i \frac{h}{2} \cancel{c_0} \qquad \text{II}: \; -i \frac{h}{2} c_1 + i \frac{vt}{2} \cancel{c_0}$$

$$\ddot{c}_1 + \left(\frac{iv}{2} + \frac{h^2}{4} + \frac{v^2 t^2}{4} \right) c_1 = 0 \qquad ✓$$

1.2) Consider $t \to \pm\infty$

- $H(t) = \frac{vt}{2}\sigma^z + \underline{\frac{h}{2}\sigma^x} \xrightarrow{t\to\pm\infty} \frac{vt}{2}\sigma^z$

so $t \to \pm\infty$

$$c_1 \approx |c_1|\, e^{i\varphi(t)}$$

- Eq (1): $\quad \varphi \in \mathbb{R}$

$$\left[-i\ddot{\varphi} - \dot{\varphi}^2 + \frac{iv}{2} + \frac{h^2}{4} + \left(\frac{vt}{2}\right)^2\right] c_1 = 0 \qquad / \ \mathrm{Im},\ \mathrm{Re}$$

$$\Rightarrow \quad \dot{\varphi} = \pm \frac{1}{2}\sqrt{h^2 + (vt)^2}$$

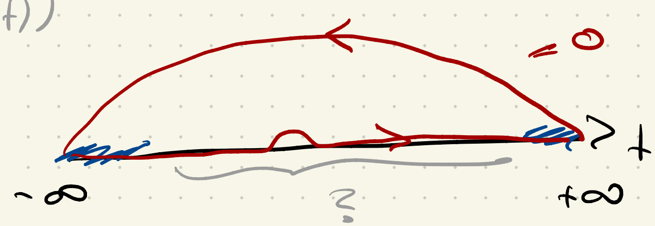
$$\ddot{\varphi} = +\frac{v}{2}$$

$$\Rightarrow \quad \dot{\varphi} = \pm \frac{v|t|}{2}\sqrt{1 + (h/vt)^2}$$

$$= \frac{vt}{2}\left(1 + \frac{1}{2}\left(\frac{h}{vt}\right)^2 + \dots\right)$$

$$= \frac{vt}{2} + \frac{h^2}{2vt} + \dots$$

1.3) $\displaystyle \int_{-\infty}^{\infty} dt \, \underbrace{\frac{\dot{c}_1(t)}{c_1(t)}}_{\frac{d}{dt}\log(c_1(t))} = \log \frac{c_1(+\infty)}{c_1(-\infty)}$



- $$\frac{\dot{c}_1(t)}{c_1(t)} \overset{|t|\to\infty}{\sim} -i\dot{\varphi}\Big|_{t\to\pm\infty} \approx -i\left(\frac{vt}{2} + \frac{1}{4}\frac{h^2}{vt}\right)$$

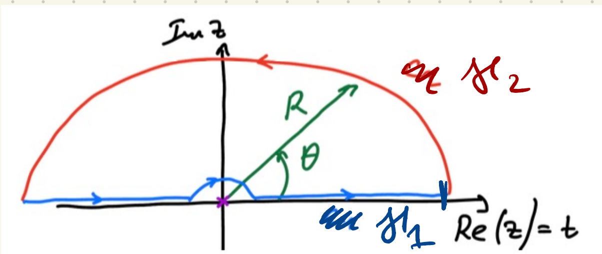
$c_1 \sim |c_1| e^{i\varphi}$

$|\psi(-\infty)\rangle = |1\rangle$

$c_1(-\infty) = 1$

$$P_{LZ} = |\langle 1 | \psi(+\infty)\rangle|^2$$
$$= |c_1(+\infty)|^2 = \left|\frac{c_1(+\infty)}{c_1(-\infty)}\right|^2 = e^{\log(\dots)}$$

1.4)



$$0 = \int_{\gamma_1} \frac{c_1'(z)}{c_1(z)}\, dz + \int_{\gamma_2} \frac{c_2'(z)}{c_2(z)}\, dz$$

$\partial_z c_2 = \frac{\partial z}{\partial \theta} \partial_\theta c_2$

$\log \frac{c_1(+\infty)}{c_1(-\infty)}$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\dot{c}_1}{c_1}\, dt = -\int_{\gamma_2} dz\, \frac{c_2'(z)}{c_2(z)}$$

$z = Re^{i\theta}$

$$= \lim_{R\to\infty} \int_0^{\pi} d\theta\, \frac{\partial z}{\partial \theta} \frac{\partial_\theta c_2}{c_2}$$

$$= +i \lim_{R\to\infty} \int_0^{\pi} d\theta\, i Re^{i\theta} \left(\frac{1}{2} v Re^{i\theta} + \frac{1}{4}\frac{h^2}{vRe^{i\theta}}\right)$$

$$= - \lim_{R \to \infty} \int_0^{\pi} d\theta \left(\underbrace{\frac{1}{2} v R^2 e^{i2\theta}}_{0} + \frac{1}{4} \frac{h^2}{v} \right)$$

$$= - \pi \frac{h^2}{4v}$$

$$\Longrightarrow \quad P_{LZ} = \left| \frac{c_1(\infty)}{c_1(-\infty)} \right|^2 = e^{-2\pi \frac{h^2}{4v}}$$

$$= e^{-2\pi \frac{(h/2)^2}{v}}$$

cpl^2

velocity

$\theta = \vartheta$

$\phi = \varphi$

$\vec{V} = \underline{V}$

2)

2.1) $H_{ctrl}[\theta, \phi] = E\,\hat{n}(\theta, \phi) \cdot \underline{\sigma}$

$$\hat{n} = \begin{pmatrix} \cos\phi \ \cos\theta \\ \sin\phi \ \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\hat{n}\underline{\sigma} = \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}$$

$Tr(\hat{n}\underline{\sigma}) = 0 \qquad \det(\hat{n}\underline{\sigma}) = -1$

$\lambda_{1/2} = \pm 1$

$\Rightarrow \underline{\lambda = +1}$:

$$\begin{pmatrix} \cos\theta - 1 & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta - 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \qquad (*)$$

<u>ST</u>:

$\cos\theta - 1 = -2\sin\frac{\theta}{2}\sin\frac{\theta}{2} \qquad 1 + \cos\theta = 2\cos^2\frac{\theta}{2}$

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$$2\begin{pmatrix} -\sin^2\frac{\theta}{2} & e^{i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & -\cos^2\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \qquad (*)$$

$$|\psi_+[\theta,\phi]\rangle = \cos\frac{\theta}{2}\,|\uparrow\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|\downarrow\rangle$$

by orthogonality

$$|\psi_-[\theta,\phi]\rangle = e^{-i\phi}\sin\frac{\theta}{2}\,|\uparrow\rangle - \cos\frac{\theta}{2}\,|\downarrow\rangle$$

– Compute U: $\quad U^\dagger H U = \mathrm{diag}(+E, -E)$

$$U = (|\psi_+\rangle, |\psi_-\rangle)$$

$$= \begin{pmatrix} \cos\theta/2 & e^{-i\phi}\sin\theta/2 \\ e^{i\phi}\sin\theta/2 & -\cos\theta/2 \end{pmatrix} \equiv \hat{v}\sigma$$

$$v = \begin{pmatrix} \sin\theta/2 & \cos\phi \\ \sin\theta/2 & \sin\phi \\ & \cos\theta/2 \end{pmatrix} \neq \hat{n}$$

2.2 $\quad (\theta(t), \varphi(t))$

$$\underset{U(t)}{\leadsto} \quad \tilde{H}_{ctrl}(t) = U^\dagger(t)\, H_{ctrl}(t)\, U(t) + i U^\dagger(t)\, \partial_t U(t)$$

$$= \mathrm{diag}(+E, -E) + \dot{\theta}\,\tilde{\mathcal{A}}_\theta + \dot{\varphi}\,\tilde{\mathcal{A}}_\varphi$$

$$\tilde{A}_\lambda = U^\dagger i\partial_\lambda U \xrightarrow[U[\;]U^\dagger]{} \mathcal{A}_\lambda = (i\partial_\lambda U)U^\dagger$$

2.3

$$\tilde{A}_\theta = u^\dagger i\partial_\theta u \qquad \tfrac{1}{2} e_\theta$$

$$= (\underline{v}\,\underline{\sigma})\, i(\partial_\theta \underline{v} \cdot \underline{\sigma})$$

e_r $\quad$ $e_r \times e_\theta \sim e_\phi$

$$= i(i\underline{v} \times \partial_\theta \underline{v}) \cdot \underline{\sigma}$$

$$= -\frac{1}{2} \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} \cdot \underline{\sigma}$$

$$(\underline{a}\,\underline{\sigma}) \cdot (\underline{b}\,\underline{\sigma}) = \underline{a}\,\underline{b} + i(\underline{a} \times \underline{b}) \cdot \underline{\sigma}$$

$$\tilde{A}_\phi = i(i\underline{v} \times \partial_\phi \underline{v})\,\sigma$$

$$= -\sin\frac{\theta}{2} \begin{pmatrix} -\cos\frac{\theta}{2} & \cos\phi \\ -\cos\frac{\theta}{2} & \sin\phi \\ & \sin\theta/2 \end{pmatrix} \underline{\sigma}$$

$$A_\theta = (i\partial_\theta u)\, u^\dagger = i(i\,\partial_\theta \underline{v} \times \underline{v})\,\underline{\sigma} = -\tilde{A}_\theta$$

$$A_\phi = -\tilde{A}_\phi$$

$$A_\phi\, |\psi_+[\theta,\phi]\rangle = \sin\frac{\theta}{2} \begin{pmatrix} \sin\frac{\theta}{2} & -e^{-i\phi}\cos\frac{\theta}{2} \\ -e^{i\phi}\cos\frac{\theta}{2} & \sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

$$= \sin\frac{\theta}{2} \begin{pmatrix} 0 \\ -e^{i\phi}\cos^2\frac{\theta}{2} - e^{i\phi}\sin^2\theta/2 \end{pmatrix}$$

$$= \sin\theta/2 \begin{pmatrix} 0 \\ -e^{i\phi} \end{pmatrix}$$

$$i\partial_\phi |\psi_+[\theta,\phi]\rangle = \begin{pmatrix} 0 \\ -e^{i\phi}\sin\theta/2 \end{pmatrix} = \mathcal{A}_\phi |\psi_+[\dots]\rangle$$

and similar for $|\psi_-[\theta,\phi]\rangle$

2-4) Kato AGP

$$\mathcal{A}_{K,\lambda} = \frac{1}{2}\sum_n [P_n, i\partial_\lambda P_n] \qquad P_n = |\psi_n\rangle\langle\psi_n|$$

- $P_\pm[\theta,\phi] = \frac{1}{2}(\mathbb{1} \pm n[\theta,\phi]\cdot\underline{\sigma})$

since,

- $[P_\pm, H_{ctrl}] = 0$ by definition

$\Rightarrow P_\pm = a_\pm \mathbb{1} + b_\pm n\underline{\sigma}$

- prop. of $P_\pm$

> $\mathrm{Tr}(P_\pm) = 1 = 2a_\pm \Rightarrow a_\pm = 1/2$

> $\pm E = \langle H_{ctrl}\rangle_{\psi_\pm} = \mathrm{Tr}(P_\pm H_{ctrl}) = 2b_\pm E$

$\Leftrightarrow b_\pm = \pm 1/2$ ✓

• $\mathcal{A}_\phi$:

$$[\mathbb{1}, \cdot] = 0$$

$$[P_\pm, i\partial_\phi P_\pm] = \frac{i}{4}[\mathbb{1} \pm \hat{n}\sigma, \pm\partial_\phi \hat{n}\cdot\sigma]$$

$[\sigma^\alpha, \sigma^\beta] = i\varepsilon^{\alpha\beta\gamma}\sigma^\gamma \cdot 2$

$$\overset{\curvearrowright}{=} \frac{i}{2}(i\hat{n} \times \partial_\phi \hat{n})\underline{\sigma}$$

$$= \frac{\sin\Theta}{2}\begin{pmatrix} -\cos\Theta \cos\phi \\ -\cos\Theta \sin\phi \\ \sin\Theta \end{pmatrix}\cdot\underline{\sigma}$$

$$\Rightarrow \mathcal{A}_{K,\phi} = \frac{1}{2}\left([P_+, i\partial_\phi P_+] + [P_-, i\partial_\phi P_-]\right)$$

$$= \frac{\sin\Theta}{2}\begin{pmatrix} -\cos\Theta \cos\phi \\ -\cos\Theta \sin\phi \\ \sin\Theta \end{pmatrix}\cdot\underline{\sigma}$$

$$\langle\psi_n|\mathcal{A}_{K,\lambda}|\psi_n\rangle = 0 \quad \forall n \quad \overset{2LS}{\Longleftrightarrow} \quad \mathrm{Tr}(\mathcal{A}_{K,\lambda} H) = 0$$

$$\mathcal{A}_{K,\lambda} \perp H$$

• For $\mathcal{A}_\Theta = \frac{1}{2}\begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}\cdot\underline{\sigma} \perp H = E\begin{pmatrix} \cos\phi \sin\Theta \\ \sin\phi \sin\Theta \\ \cos\Theta \end{pmatrix}\cdot\underline{\sigma}$

$$\Rightarrow \mathcal{A}_\Theta = \mathcal{A}_{K,\Theta}.$$

2.5) CD:

$$H_{CD} = H_{ctrl} + \dot{\phi}\, \mathcal{A}_{\phi} + \dot{\theta}\, \mathcal{A}_{\theta}$$

$$H_{K,CD} = H_{ctrl} + \dot{\phi}\, \mathcal{A}_{K,\phi} + \dot{\theta}\, \mathcal{A}_{K,\theta}$$

2.6)

$$|\psi_\pm[\theta,\varphi]\rangle \longmapsto e^{i\chi_\pm[\theta,\varphi]}\, |\psi_\pm[\theta,\varphi]\rangle$$

still valid eigenbasis/ eigenstate

$$U \longmapsto U' = \left(e^{i\chi_+} |\psi_+\rangle ,\ e^{i\chi_-} |\psi_-\rangle \right)$$

$$\mathcal{A}_\lambda \longmapsto \mathcal{A}'_\lambda = \left(i\partial_\lambda U'\right) U'^\dagger$$

$$= i\partial_\lambda \left(e^{i\chi_+} |\psi_+\rangle ,\ e^{i\chi_-} |\psi_-\rangle \right) \begin{pmatrix} \langle\psi_+| e^{-i\chi_+} \\ \langle\psi_-| e^{-i\chi_-} \end{pmatrix}$$

$$= -(\partial_\lambda \chi_+) |\psi_+\rangle\langle\psi_+| - (\partial_\lambda \chi_-) |\psi_-\rangle\langle\psi_-| \qquad (1)$$

$$\left. + |i\partial_\lambda \psi_+\rangle\langle\psi_+| + |i\partial_\lambda \psi_-\rangle\langle\psi_-| \right\} = \mathcal{A}_\lambda \qquad (2)$$

$$= \mathcal{A}_\lambda - \sum_{n=\pm} (\partial_\lambda \chi_n)\, |\psi_n\rangle\langle\psi_n|$$

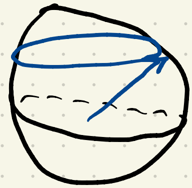
$$P_n = |\psi_n\rangle\langle\psi_n| \longmapsto \cancel{e^{i\chi_n}} |\psi_n\rangle\langle\psi_n| \cancel{e^{-i\chi_n}} = P_n$$

$$\mathcal{A}_{K,\lambda} \longmapsto \mathcal{A}_{K,\lambda} \qquad \checkmark$$

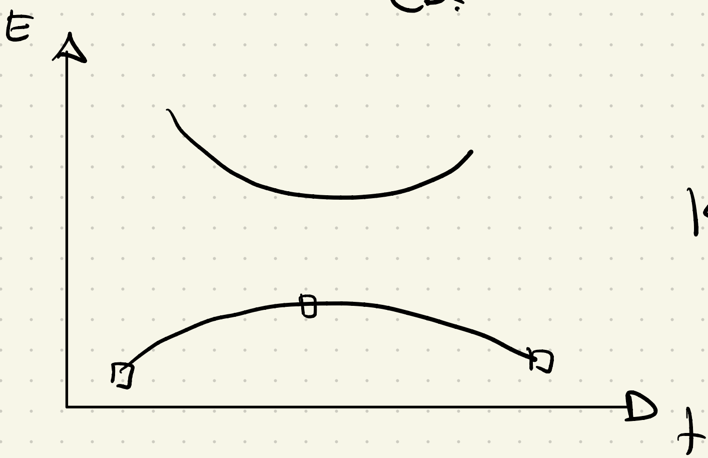
2.7)

$\Theta(t) = \Theta_* = \pi/3$

$\varphi(t) = \omega t = \varphi(t+T) \quad T = \frac{2\pi}{\omega}$



CD?



$H_{ctrl} + \dot{\lambda}\, A_\lambda$

$|\langle \psi(t) | \psi_0 [\lambda(t)] \rangle|^2 = 1$

$|\psi(t)\rangle = e^{-i\, \text{phase}(t)} |\psi_0 [\lambda(t)]\rangle$

- adiabatic limit:

$\text{phase}(t) = \gamma_n + \phi_n$

$$\gamma_n = -\int_{\lambda(0)}^{\lambda(t)} \langle \psi_\pm(\lambda) | i\partial_\lambda \psi_\pm(\lambda)\rangle \, d\lambda$$

here
$\lambda = \phi$

$$= -\int_0^{\omega t} \langle \psi_\pm(\phi) | \underline{i\partial_\phi \psi_\pm(\phi)} \rangle \, d\phi$$

$\boxed{2.3}$

$$= \boxed{\pm \sin^2\left(\frac{\Theta}{2}\right) \omega t = \gamma_\pm}$$

$$\phi_n = \int_0^t \underbrace{E_\pm(\lambda(s))}_{\omega = \pm E} ds$$

$$= \boxed{\pm E t = \phi_\pm}$$

- CD:

> dynamic gauge:

$$H_{CD} = H_{ctrl} + \dot\lambda i \underbrace{u^\dagger \partial_\lambda u}_{\mathcal{A}_\lambda} + \sum_n (\partial_\lambda \chi_n) P_n[\lambda]$$

go to rot. frame wrt u

$$\tilde H_{CD} = \tilde H_{ctrl} + \dot\lambda \tilde{\mathcal{A}}_\lambda$$

$$= \text{diag}(\dots) \xcancel{- \dot\lambda \tilde{\mathcal{A}}_\lambda + \dot\lambda \mathcal{A}_\lambda} \rightarrow 0$$

$$= \mathcal{D}(t)$$

$$\Leftrightarrow \quad |\tilde\psi(t)\rangle = e^{-i\int_0^t E_n(s) + \partial_\lambda \chi_n ds} |\tilde\psi_n\rangle$$

$$\Leftrightarrow \quad |\psi(t)\rangle = u|\tilde\psi(t)\rangle$$

$$= e^{-i\int_0^t E_n(s) ds} e^{i\chi_n} \underbrace{u|\tilde\psi_n\rangle}_{|\psi_n[\lambda(t)]\rangle}$$

$$= e^{-i\phi_n} \dots$$

phase$(t) = \phi_n$

- Kato: $\partial_\lambda \chi_n = \langle \psi_n | \partial_\lambda \psi_n \rangle \longrightarrow \int \; = \gamma_n(t)$