Inverse frequency expansions

- Floquet thm: $H(t+T) = H(t)$

$$\Rightarrow U_{lab}(t,0) = P(t)\, \underbrace{e^{-itH_F}}_{= U_{rot}(t,0)}\, \underbrace{P^\dagger(0)}_{= \mathbb{1}} \;,\quad b/c\;\; U(0,0) = \mathbb{1}$$

$\rightarrow$ looks like transf. law for evo. op's b/w lab & rot frames!

$\Rightarrow P(t)$: rot $\longrightarrow$ lab

$\Rightarrow$ Hamiltonian in rot frame :

$$H_{rot}(t) = P^\dagger(t) H_{lab}(t) P(t) - i P^\dagger(t)\partial_t P(t) \overset{\text{Floquet thm}}{=} H_F$$

time-indep

- Floquet's thm: statement about existence of a reference frame in which the dynamics is governed by the time indep. Hamiltonian H_F; moreover this reference frame is a rotating frame : $P(t+T) = P(t)$

note : thm doesn't tell us how to compute H_F (nor how to find that special rot. frame)!

- notation:

- effective / Floquet Hamiltonian : H_F (H_{eff})

$\rightarrow$ this is NOT the same as H_0, $H(t=0)$

- Floquet unitary: $U_F = U(T,0) = e^{-iTH_F}$

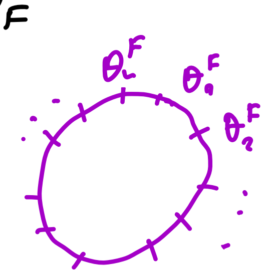
$$\Rightarrow U(\ell T, 0) = e^{-i\ell T H_F} = U_F^\ell$$

$\rightarrow$ evo. op. over one drive cycle / period T

$\rightarrow$ eigendecomposition: $U_F = \sum_n e^{i\theta_n^F} |n_F\rangle\langle n_F|$

Floquet phases

Floquet states



spectrum of U_F

$\{\theta_n^F\}$ lives on unit circle

$$\Rightarrow H_F = \sum_n \varepsilon_n^F |n_F\rangle\langle n_F| \;,\qquad T\varepsilon_n^F = \theta_n^F$$

quasi-energies

q'energies defined mod ω : $e^{iT(\varepsilon_n^F + l\omega)} = e^{iT\varepsilon_n^F}$ $\forall l \in \mathbb{Z}$

$\Rightarrow$ Floquet Hamiltonian H_F is <u>not</u> unique (depends on the choice for branch cut of log)

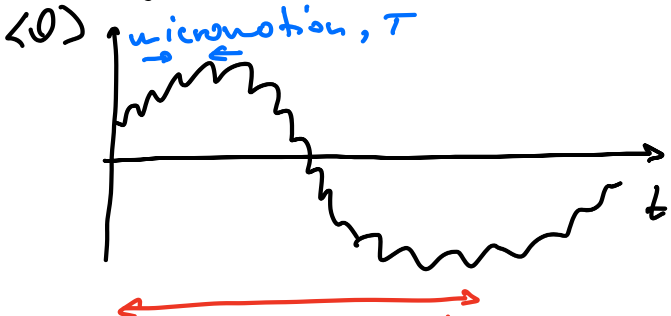
$\rightarrow$ analogy to quasi-momentum (Bloch's thm. for $V(\vec{x}) = V(\vec{x}+\vec{a})$) in systems w/ spatial periodicity

- U_F is unique

– <u>kick operator</u> : $K(t)$: $\underbrace{P(t)}_{\text{micromotion operator}} = e^{-iK(t)}$ ← kick operator

$K(t+T) = K(t)$

- at high drive freq. ($\omega \gg \omega_0$), $K(t)$ governs micromotion

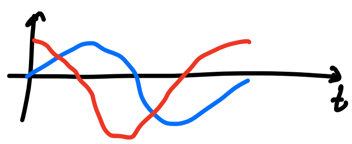


long time scale, governed by H_F

micromotion: dynamics within drive cycle

<u>note</u>: $K(t)$ is instantaneous micromotion generator, i.e.
$$K(t) \neq (i\partial_t P(t))P^\dagger(t)$$

– <u>Floquet "gauge"</u> : dependence on the initial time t_0 (or phase of drive)



$$u(t,t_0) = \underbrace{e^{-iK(t,t_0)}}_{\Rightarrow P(t,t_0)} e^{-i(t-t_0)H_F[t_0]}$$

↳ Floquet gauge

$\rightarrow$ $H_F[t_0]$ matrix depends on t_0

<u>but</u> ε_n^F are <u>indep.</u> of t_0 ($\rightarrow$ hence t_0 is a gauge)

$\Rightarrow$ Floquet states also depend on t_0 : $|u_F[t_0]\rangle$

$$H_F[t_0]\,|u_F[t_0]\rangle = \varepsilon_n^F\,|u_F[t_0]\rangle$$

(HW): can show: $P(t, t_0)|u_F[t_0]\rangle = |u_F[t]\rangle$
→ micromotion op. moves b/w inst. Floquet states

note: Floquet "gauge" is physical; it is set by initial time/condition

Q: How do we compute H_F & K in practice?
→ inverse-frequency expansions: $\omega \gg \omega_0$, $\omega \to \infty$ (or $T \to 0$)

1) simple case: step drives (similar for kicked drives)

$$U_F = e^{-i\frac{T}{2}H_1} e^{-i\frac{T}{2}H_0} \overset{!}{=} e^{-iTH_{FM}}$$

$$\Rightarrow H_{FM} = \frac{i}{T}\log\left(e^{-i\frac{T}{2}H_1} e^{-i\frac{T}{2}H_0}\right)$$

$$= \frac{i}{T}\Big(-i\frac{T}{2}(H_0+H_1) + \left(\frac{-iT}{2}\right)^2 \frac{1}{2}[H_0, H_1]$$
$$+ \left(\frac{-iT}{2}\right)^3 \frac{1}{12}\left([H_0,[H_0,H_1]] + 0 \leftrightarrow 1\right)$$
$$+ \dots\Big)$$

$$= \frac{1}{2}(H_0 + H_1) = \mathcal{O}(T^0) := H_{FM}^{(0)}$$
$$- \frac{T}{8} i [H_0, H_1] =: H_{FM}^{(1)} = \mathcal{O}(T^1)$$
$$- \frac{T^2}{96}\left([H_0,[H_0,H_1]] + [H_1,[H_1,H_0]]\right) := H_{FM}^{(2)} = \mathcal{O}(T^2)$$
$$+ \dots$$

$$H_{FM} = \sum_{n=0}^{\infty} H_{FM}^{(n)} \quad ; \quad H_{FM}^{(n)} \propto \frac{1}{\omega^n} \propto T^n$$

Baker-Campbell-Hausdorff (BCH) formula

2) generic time-periodic dependence
<u>Floquet-Magnus expansion</u>
(FM)

$$\text{LSH} = \mathcal{T} e^{-i\int_0^T dt\, H(t)} = U_F \overset{\text{Floquet}}{=} e^{-iT H_{FM}} = \text{RHS}$$

assume: $H_{FM} = \sum_{n=0}^{\infty} H_{FM}^{(n)} \propto \frac{1}{\omega^n}$ & $\omega \to \infty$

notation: $H(t_j) = H_j$

$$\text{LHS} = \mathbb{1} - i \underbrace{\int_0^T dt_1\, H_1}_{\sim T} - \underbrace{\int_0^T dt_1 \int_0^{t_1} dt_2\, H_1 H_2}_{\sim T^2}$$

$$\text{RHS} = \mathbb{1} - iT\left(H_{FM}^{(0)} + H_{FM}^{(1)} + H_{FM}^{(2)} + \dots \right) + \frac{(iT)^2}{2}\left(H_{FM}^{(0)} + H_{FM}^{(1)} + \dots \right)\left(H_{FM}^{(0)} + H_{FM}^{(1)} + \dots \right) + \dots$$

$$H_{FM}^{(0)} = \frac{1}{T}\int_0^T dt\, H(t)$$ time-averaged Hamiltonian

$$-\int_0^T dt_1 \int_0^{t_1} dt_2\, H_1 H_2 \overset{!}{=} -iT H_{FM}^{(1)} - \frac{T^2}{2}\left(H_{FM}^{(0)}\right)^2$$

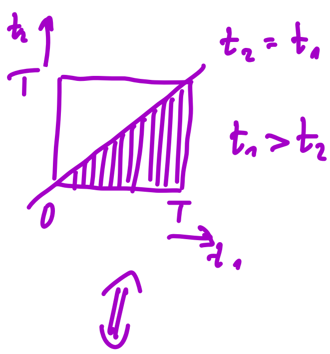
$$\Rightarrow H_{FM}^{(1)} = \frac{i}{T}\left\{ \frac{\cancel{T^2}}{2}\left(\frac{1}{\cancel{T}}\int_0^T dt\, H(t)\right)^2 - \int_0^T dt_1 \int_0^{t_1} dt_2\, H_1 H_2 \right\}$$

$$= \frac{i}{T}\left\{ \frac{1}{2}\int_0^T dt_1 \int_0^T dt_2\, H_1 H_2 - \int_0^T dt_1 \int_0^{t_1} dt_2\, H_1 H_2 \right\}$$

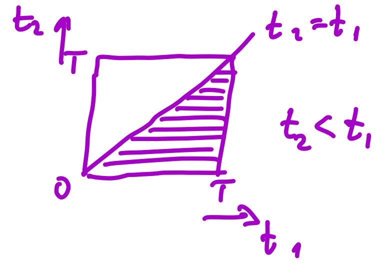
$$= \frac{i}{T}\left\{ \frac{1}{2}\int_0^T dt_1 \left(\int_0^{t_1} dt_2 + \int_{t_1}^T dt_2 \right) H_1 H_2 - 1 \int_0^T dt_1 \int_0^{t_1} dt_2\, H_1 H_2 \right\}$$

$$= -\frac{i}{2T}\left\{ \int_0^T dt_1 \int_0^{t_1} dt_2\, H_1 H_2 - \underbrace{\int_0^T dt_1 \int_{t_1}^T dt_2\, H_1 H_2} \right\}$$

$$\overset{1 \leftrightarrow 2}{=} \int_0^T dt_2 \int_{t_2}^T dt_1\, H_2 H_1$$



$$= \int_0^T dt_1 \int_0^{t_1} dt_2 \; H_2 H_1$$



$$\Rightarrow \quad H_{FM}^{(1)} = -\frac{i}{2T} \int_0^T dt_1 \int_0^{t_1} dt_2 \, [H(t_1), H(t_2)]$$

similarly:

$$H_{FM}^{(2)} = \frac{1}{3!\, T\, i^2} \int_0^T dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \, [H_1, [H_2, H_3]] + [H_3, [H_2, H_1]]$$

FM expansion generalizes BCH formula to arbitrary periodic drives

– determined H_{FM}, next comes $K_{FM}(t)$

$$\mathcal{T} e^{-i \int_0^t ds\, H(s)} = e^{-i K_{FM}(t)} e^{-it \underbrace{H_{FM}}_{\text{known expansion}}}$$

ansatz: $K_{FM}(t) = \sum_{n=0}^{\infty} K_{FM}^{(n)}(t) \propto \frac{1}{\omega^n}$

plugging in the expansion for $H_{FM}^{(n)}$:

$$\Rightarrow \quad K_{FM}^{(0)}(t) \equiv 0$$

$$K_{FM}^{(1)}(t) = \int_0^t ds \left(H(s) - H_{FM}^{(0)}\right) \propto \mathcal{O}\left(\frac{1}{\omega}\right)$$

→ Floquet-Magnus boundary condition for kick op:

$$K_{FM}(\ell T) = 0 \; , \quad \ell \in \mathbb{Z}$$

3) van Vleck expansion:

$$U(t, t_0) = e^{-i K_{vV}(t; t_0)} e^{-it H_{vV}} e^{+i K_{vV}(t; t_0)}$$

indep. of Floquet gauge

– defining properties:

(i) H_{vV} indep. of t_0 (all dependence is in kick op. $K_{vV}(t; t_0)$)

(ii) $\int_0^T K_{vV}(t; t_0)\, dt = 0$ (boundary condition) want to expand: $H_{\nu V} = \sum_n H_{\nu V}^{(n)} \propto \frac{1}{\omega^n}$

$$K_{\nu V}(t) = \sum_n K_{\nu V}^{(n)}(t) \propto \frac{1}{\omega^n}$$

<u>recall</u>: Floquet's thm defines a rot. ref. frame where dynamics is generated by static $H_{\nu V}$

<u>now</u>: construct this frame transformation order by order in $1/\omega$ $\hookrightarrow$ vV expansion

<u>identities</u>:

$$e^{iK} H e^{-iK} = H + i[K,H] - \frac{1}{2}[K,[K,H]] + \dots$$

$$-i e^{iK} \partial_t e^{-iK} = -\partial_t K - \frac{i}{2}[K, \partial_t K] + \frac{1}{6}[K,[K,\partial_t K]] + \dots$$

<u>starting point</u>:

$$H_{\nu V} = e^{iK_{\nu V}(t)} H(t) e^{-iK_{\nu V}(t)} - i e^{iK_{\nu V}(t)} \partial_t e^{-iK_{\nu V}(t)}$$

$$= H(t) + i[K^{(1)}(t), H(t)] + \mathcal{O}(\omega^{-2})$$

$$- \frac{i}{2}[K^{(1)}(t), \partial_t K^{(1)}(t)] - \partial_t \left(K^{(1)}(t) + K^{(2)}(t)\right) + \mathcal{O}(\omega^{-2})$$

time-dep on RHS has to vanish order by order in $1/\omega$ since LHS is time-indep.

expand: $H(t) = \sum_{l=-\infty}^{\infty} H_l e^{+il\omega t}$ ← operator valued coefficients

$$H_{\nu V} = H_0 + \underbrace{\sum_{l\neq 0} H_l e^{il\omega t} - \partial_t K^{(1)}(t)}_{\stackrel{!}{=}0}$$

$$- \partial_t K^{(2)} + i[K^{(1)}(t), H(t)] - \frac{i}{2}[K^{(1)}(t), \partial_t K^{(1)}(t)]$$

$$\Rightarrow \partial_t K^{(1)} \stackrel{!}{=} \sum_{l\neq 0} H_l e^{il\omega t}$$

boundary cond. $\int_0^T K(t)\,dt = 0$

$$\Rightarrow K^{(1)}(t) = \sum_{l\neq 0} \frac{1}{il\omega} H_l e^{il\omega t} + 0$$

$$\stackrel{!}{=} H_0 + \cancel{i}\Big[\overbrace{\sum_{l\neq 0}\frac{1}{\cancel{i}l\omega} H_l e^{il\omega t}}^{=K^{(1)}(t)}, \overbrace{\underline{\sum_m H_m e^{im\omega t}}}^{=H(t)}\Big]$$

$$-\frac{\cancel{i}}{2}\Big[\underbrace{\sum_{l\neq 0}\frac{1}{\cancel{i}l\omega} H_l e^{il\omega t}}_{=K^{(1)}(t)}, \underbrace{\sum_{m\neq 0} H_m e^{im\omega t}}_{=\partial_t K^{(1)}(t)}\Big]$$

$$-\partial_t K^{(2)}(t) + \mathcal{O}(1/\omega^2)$$

split $\sum_m = (m=0) + \sum_{m\neq 0}$

combine w/ 2^{nd} line: $\frac{1}{\omega} - \frac{1}{2\omega} = +\frac{1}{2\omega}$

$$= H_0 + \frac{1}{\omega}\sum_{l\neq 0}\frac{1}{l} e^{il\omega t}[H_l, H_0] + \frac{1}{2\omega}\sum_{\substack{l\neq 0\\ m\neq 0}}\frac{1}{l} e^{i(l+m)\omega t}[H_l, H_m] - \partial_t K^{(2)}(t) + \mathcal{O}(1/\omega^2)$$

separate out $m=-l$ term, since its t-indep.

$$= H_0 + \frac{1}{2\omega}\sum_{l\neq 0}\frac{1}{l}[H_l, H_{-l}] + \Big[\frac{1}{2\omega}\sum_{l\neq 0}\sum_{m\neq -l,0}\frac{1}{l} e^{-i(l+m)\omega t}[H_l, H_m] + \frac{1}{\omega}\sum_{l\neq 0}\frac{1}{l} e^{il\omega t}[H_l, H_0] - \partial_t K^{(2)}(t)\Big] \stackrel{!}{=} 0 \Rightarrow K^{(2)}(t)$$

$$+ \mathcal{O}(\omega^{-2})$$

$\Rightarrow$

$$K^{(1)}_{vv}(t) = \frac{1}{\omega}\sum_{l\neq 0}\frac{1}{il} H_l e^{il\omega t} = \dots = -\frac{1}{2}\int_t^{T+t} dt' \Big(1 + 2\frac{t-t'}{T}\Big) \bmod 2 \; H(t)$$

$$K^{(2)}_{vv}(t) = \frac{1}{\omega^2}\sum_{l\neq 0}\frac{1}{il^2}[H_l, H_0] e^{il\omega t} + \frac{1}{2\omega^2}\sum_{\substack{l\neq 0\\ m\neq -l,0}}\frac{1}{l(l+m)}[H_l, H_m] e^{i(l+m)\omega t}$$

$$H_{vV}^{(0)} = H_0 = \frac{1}{T}\int_0^T dt\ H(t)$$

$$H_{vV}^{(1)} = \frac{1}{2\omega}\sum_{\ell\neq 0}\frac{1}{\ell}[H_\ell, H_{-\ell}] = \ldots = -\frac{1}{2}\int_0^T dt_1 \int_0^{t_1} dt_2 \left(1 - 2\,\frac{t_1 - t_2}{T}\right) \bmod 2\ [H(t_1), H(t_2)]$$

4) Brillouin - Wigner expansion
↳ PRB 93, 144307 (2016)

<u>Remarks</u>:

1) all expansions produce the same Floquet-gauge invariants (q'energies) up to the order of approx., but may disagree beyond

2) FM: $H_{FM}[t_0]$ depends on Floquet gauge
vV: H_{vV} is <u>indep.</u> of t_0
(entire gauge dependence is in $K_{vV}(t, t_0)$)

3) <u>vV</u> is preferred when interested in properties of the q'energy spectrum (→ spectral engineering, e.g. topo insulators)

4) <u>FM</u> is preferred when discussing strobo. time evo, since it does not contain an initial kick:

$$U(t,0) = e^{-iK_{FM}(t)}\, e^{-it H_{FM}[0]}\, \underbrace{e^{+iK_{FM}(0)}}_{=\mathbb{1}\ \to \text{ no kick}}$$

$$U(t,0) = e^{-iK_{vV}(t)}\, e^{-it H_{vV}}\, \underbrace{e^{+iK_{vV}(0)}}_{\neq \mathbb{1} \text{ in general}}$$

→ initial kick

5) <u>note</u>: time-ordered integrals in the def. of exp. coefficients <u>decouple</u> from commutator structure
→ different types of drives (e.g. harmonic vs. square) have similar series w/ different coefficients (time-ordered integrals)

6) FM: if $H(t) = H(T-t)$, e.g. cos-like phase

$\Rightarrow$ all odd-order terms vanish: $H_{FM}^{(2n+1)} \equiv 0 \quad \forall n \in \mathbb{N}$

(caveat: this not true for vV expansion)

7) expansions have different names across different fields:

$\rightarrow$ inverse / high freq. exp. (non-equil. dynamics)

$\rightarrow$ Schrieffer-Wolff transformation (condensed matter)

$\rightarrow$ adiabatic elimination (rotating wave approx., etc.)

$\hookrightarrow$ quantum optics