Geometric Floquet Theory

In this problem set, our goal is to drive an alternative, geometric decomposition of the unitary evolution operator for periodically-driven systems described by the Hamiltonian $H(t+T)=H(t)$, and relate it to Floquet's theorem:

$$U(t,0)=\mathcal{T}\exp\left(-i\int_0^t H(s)\mathrm{d}s\right)\overset{\text{Floquet thm}}{=}P(t)\exp\left(-itH_F[0]\right), \tag{1}$$

where $P(t+T)=P(t)$ is the micromotion operator, and $H_F[0]$ is the Floquet Hamiltonian at Floquet gauge $t_0=0$ (see notes from class). We denote by $T=2\pi/\omega$ the period (frequency) of the drive.

1. Show that the Floquet Hamiltonian at Floquet gauge t, $H_F[t]$, satisfies the relation

$$H_F[t]=H(t)-(i\partial_t P(t))P^\dagger(t). \tag{2}$$

Argue that $H_F[t]=H_F[t+T]$ is periodic in the Floquet gauge.

2. Define $\mathcal{A}_F(t)=(i\partial_t P(t))P^\dagger(t)$ and show that $\mathcal{A}_F(t)$ is a proper gauge potential for the Floquet eigenstates $H_F[t]|n_F[t]\rangle=\varepsilon_n^F|n_F[t]\rangle$. What is the physical manifestation of the control parameter λ in this case?

3. Show that the lab-frame Hamiltonian $H(t)$,

$$H(t)=H_F[t]+\mathcal{A}_F(t)=H_{\mathrm{CD}}(t), \tag{3}$$

generates counterdiabatic driving for the Floquet states.

4. Recall the general $U(1)$ re-phasing gauge transformation for the adiabatic gauge potential,

$$|n[\lambda]\rangle\mapsto|n'[\lambda]\rangle=e^{i\chi_n(\lambda)}|n[\lambda]\rangle.$$

Now use the periodicity of the Floquet states $|n[t+T]\rangle=|n[t]\rangle$ and the properties of the quasienergies to show that the most general form of the gauge transformation which preserves the Floquet structure reads as

$$\chi_n(t)=m_n\omega t,\qquad m_n\in\mathbb{Z}. \tag{4}$$

What is the gauge group for $\mathcal{A}_F(t)$? What is the manifestation/meaning of the gauge transformation for $H_F[t]$?

5. Consider the gauge-invariant Kato potential, $\mathcal{A}_K(t)$, in the Floquet case. Using it, show the alternative "Kato" decomposition

$$H(t)=H_K(t)+\mathcal{A}_K(t), \tag{5}$$

and define the Kato Hamiltonian $H_K(t)$ and the Kato AGP $\mathcal{A}_K(t)$ in terms of the Floquet Hamiltonian $H_F[t]$ and the Floquet AGP $\mathcal{A}_F(t)$.

6a. Determine the eigenstates of the Kato Hamiltonian $H_K(t)$ and compare them to those of $H_F[t]$. Show that the Kato energies are given by $\varepsilon_n^K(t)=\langle n_F[t]|H(t)|n_F[t]\rangle$.

6b. Compare the Kato and Floquet AGPs: which phases do the Floquet states accumulate under evolution over one drive cycle, for each of the two AGPs?

7a. Like in CD driving, consider the change-of-frame transformation infinitesimally generated by $\mathcal{A}_K(t)$, and compute the co-moving frame Hamiltonian $\tilde{H}(t)$. Find an expression for the evolution operator in the co-moving frame.

7b. Apply the general expression derived in class for the unitary operator in the lab frame in terms of the unitary operator in the co-moving frame and the frame transformation itself, to show that the lab-frame evolution operator can be decomposed as

$$U(t,0) = \mathcal{W}(t,0)\exp\left(-it\text{Æ}(t,0)\right), \tag{6}$$

where the Average Energy operator is given by

$$\text{Æ}(t,0) = \sum_n \text{æ}_n(t)\,|n_F[0]\rangle\rangle\langle\langle n_F[0]|\,, \qquad \text{æ}_n(t) = \frac{1}{t}\int_0^t \varepsilon_n^K(s)\mathrm{d}s,$$

and the Wilson line operator is

$$\mathcal{W}(t,0) = \mathcal{T}\exp\left(-i\int_0^t \mathcal{A}_K(s)\mathrm{d}s\right)$$

Is the Wilsone line operator $\mathcal{W}(t,0)$ a valid micromotion operator?

8. Now consider stroboscopic times $t = \ell T$, $\ell \in \mathbb{N}$, where

$$U(\ell T,0) = W(\ell T)\exp\left(-i\ell T\text{Æ}(T,0)\right).$$

Argue that the Wilson loop operator $\mathcal{W}(T) = \mathcal{W}(T,0)$ is a unitary operator that does not depend on the Floquet gauge/initial time; show that its eigenvalues are given by the geometric phases $\gamma_n(T)$ of the Floquet states. Use this to argue that the average energy operator over one period generates the dynamical phases $\phi_n(t)$ and relate those to its eigenvalues $\text{æ}_n(t)$.

9a. Show the quasienergy decomposition

$$T\varepsilon_n^F = T\text{æ}_n(T) + \gamma_n(T). \tag{7}$$

Argue that this decomposition holds at arbitrary times t.

9b. Why is defining a Floquet ground state using the quasienergies is inappropriate? Argue that the eigenvalues $\text{æ}_n(T)$ can be used to unambiguously sort the Floquet states and define a Floquet ground state.

Equation (6) is called the geometric Floquet decomposition. It appears that inherently nonequilibrium effects, such as Floquet anomalous topological insulators and Floquet time crystals [later in the course], are geometric in their origin as their physics is determined by the geometric phases.