Periodically Driven Systems

- models using time-dep. Hamiltonian:

$$H(t) = H_0 + f(t)\,V \,, \quad V = V^\dagger$$

$$H(t) = H_0 + g(t)\,V + g^*(t)\,V^\dagger$$

$$H(t) = H_0 + H_1(t)$$

- time evo: solve Schrödinger's eq.

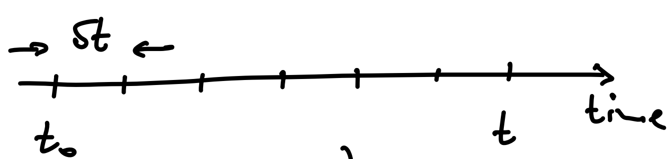
$$i\partial_t |\psi(t)\rangle = H(t)\,|\psi(t)\rangle \,; \quad [H(t_1), H(t_2)] \neq 0$$

$\hookrightarrow$ soln: time-ordered exponential

$$\delta t = \frac{t-t_0}{N}$$

$$|\psi(t)\rangle = U(t,t_0)\,|\psi(t_0)\rangle$$

$$U(t,t_0) = \mathcal{T}\exp\left(-i\int_{t_0}^{t} ds\; H(s)\right)$$



computational definition

$$= \lim_{N\to\infty} e^{-i\frac{t-t_0}{N} H\left(N\frac{t-t_0}{N}\right)}\, e^{-i\frac{t-t_0}{N} H\left((N-1)\frac{t-t_0}{N}\right)} \cdots e^{-i\frac{t-t_0}{N} H\left(\frac{t-t_0}{N}\right)}$$

$$= \lim_{N\to\infty} \prod_{n=1}^{N} e^{-i\frac{t-t_0}{N} H\left(n\frac{t-t_0}{N}\right)}$$

↑ earlier times come first!

analytical def.

$$U(t,t_0) \overset{\downarrow}{=} \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n\; \mathcal{T}\big(H(t_1)H(t_2)\cdots H(t_n)\big)$$

$$= \sum_{n=0}^{\infty} (-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \cdots \int_{t_0}^{t_{n-1}} dt_n\; H(t_1)H(t_2)\cdots H(t_n)$$

$$= \mathbb{1} - i\int_{t_0}^{t} dt_1\, H(t_1) - \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\; H(t_1)H(t_2) + \cdots$$

- difficult to compute for arbitrary t-dep.
  - → pert. theory (truncate to first few orders)
    - only valid at "short" times

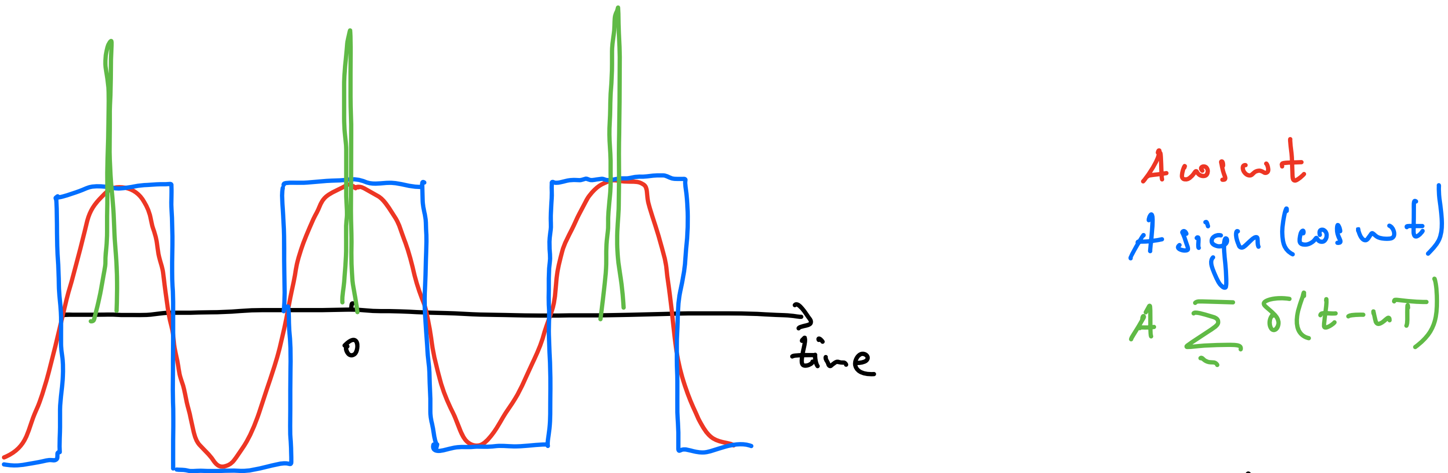
– periodic time dependence : $H(t) = H(t+T)$

→ relevant params.:

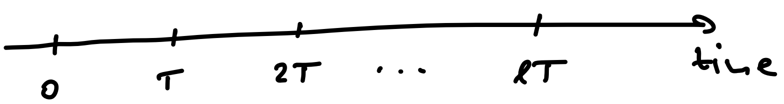
- amplitude A
- frequency $\omega = 2\pi/T$
- phase of drive φ / starting time t_0 cos vs sin

– examples of periodic drives:

1) continuous drives : $H(t) = H_0 + A \cos\omega t \, H_1$
2) step/square drives: $H(t) = H_0 + A\,\mathrm{sign}(\cos\omega t)\, H_1$
3) kicked drives : $H(t) = H_0 + A \sum_{n=-\infty}^{\infty} \delta(t-nT)\, H_1$



– integer multiples of drive period , ℓT , $\ell \in \mathbb{N}$



stroboscopic times

– some intuitive limits:

1) weak coupling limit : $A \ll \omega_0 \neq \omega$
natural energy scale of H_0
→ apply pert. theory
→ captures short time dynamics

2) high-freq. limit : $\omega \gg \omega_0$
→ system "sees" time-averaged Hamiltonian

$$H_{ave} = \frac{1}{T}\int_0^T dt\, H(t)$$

→ apply inverse frequency expansion (↪ next time)

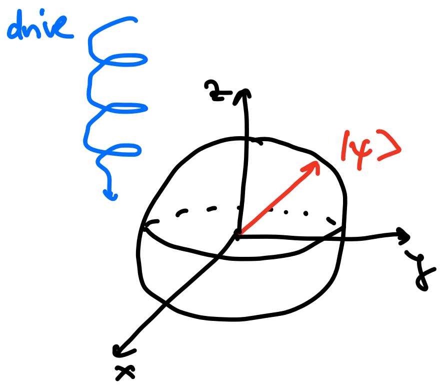
3) slow-freq. / adiabatic limit

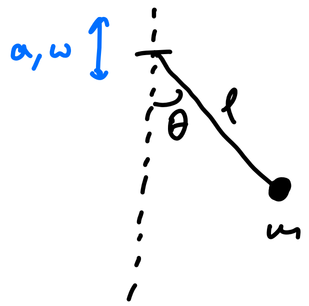
→ system follows drive instantaneously (see prev. lectures)

- examples of periodically driven systems

1) 2LS in circularly polarized "light"

$$H(t) = B_z \sigma^z + B_{\parallel} (\sigma^x \cos\omega t + \sigma^y \sin\omega t)$$





$$\omega_0 = \sqrt{g/\ell}$$

$$x(t) = \ell \sin\theta(t)$$

$$y(t) = \ell \cos\theta(t) + a\cos\omega t$$

2) Kapitza pendulum: $H(t) = \frac{1}{2m\ell^2} p_\theta^2 - m\ell^2\left(\omega_0^2 + \frac{a\omega}{\ell}\,\omega\cos\omega t\right)\cos\theta$

re-define: $m\ell^2 \to m$

$$\frac{a\omega}{\ell} \to A$$

Hamiltonian for Kapitza pendulum:

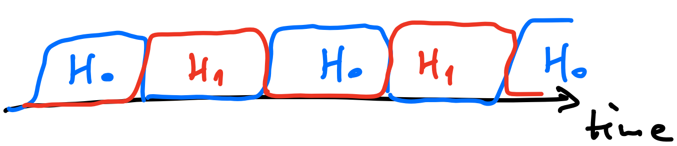
$$H(t) = \frac{p_\theta^2}{2m} - m\left(\omega_0^2 + A\omega\cos\omega t\right)\cos\theta$$

- recall video

→ stabilization of inverted equilibrium position at high enough freq. ω; how high?

3) periodically kicked spin chain:

$$H(t) = \begin{cases} H_0 = \sum_j J\,\sigma^z_{j+1}\sigma^z_j + h_z \sigma^z_j, & t \in [0, T/2) \mod T \\ H_1 = \sum_j h_x \sigma^x_j, & t \in [T/2, T) \mod T \end{cases}$$



$$U(T,0) = \mathcal{T} e^{-i\int_0^T dt\, H(t)} \underset{\text{comp. def.}}{=} e^{-i\frac{T}{2}H_1}\, e^{-i\frac{T}{2}H_0}$$

- if $J = J_{ij}$ random numbers / disordered ($\to$ MBL)

  $\to$ Floquet time crystal ($\to$ later)

<u>recall</u>:

- static unitary changes the basis (observable expectations remain unchanged)
- time-dep. unitary changes reference frame (expect. values of obs. may change: e.g. energy, currents, etc.)

$$\begin{tikzcd}[column sep=huge] \underline{\text{lab frame}} \arrow[r, bend right, "V^\dagger(t)"'] & \underline{\text{rot. frame}} \arrow[l, bend right, "V(t)"'] \end{tikzcd}$$

$H_{lab}(t)$	$H_{rot}(t)$
$U_{lab}(t_2, t_1)$	$U_{rot}(t_2, t_1)$
$\|\psi_{lab}(t)\rangle$	$\|\psi_{rot}(t)\rangle$

$\to$ relation b/w states: $|\psi_{rot}(t)\rangle = V^\dagger(t)\,|\psi_{lab}(t)\rangle$

$\to$ — // — evo. operators:

$$U_{lab}(t_2, t_1) = V(t_2)\, U_{rot}(t_2, t_1)\, V^\dagger(t_1)$$

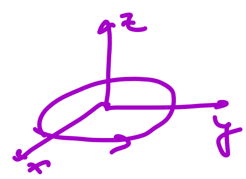
$$U_{rot}(t_2, t_1) = T\exp\left(-i\int_{t_1}^{t_2} dt\, H_{rot}(t)\right)$$

$\to$ relation b/w Hamiltonians:

$$H_{rot}(t) = V^\dagger(t)\, H_{lab}(t)\, V(t) - \underbrace{V^\dagger(t)\, i\partial_t V(t)}_{\substack{\text{Galilean term} \\ \text{(e.g. centrifugal force, etc.)}}}$$

<u>Ex. 1</u>: solution to 2LS: $H(t) = B_z\sigma^z + B_{\parallel}(\cos\omega t\,\sigma^x + \sin\omega t\,\sigma^y)$

want to compute dynamics under $H(t)$, i.e.



$$|\psi(t)\rangle = U(t,0)\,|\psi(0)\rangle$$

<u>trick</u>: let's transform to co-rotating frame

use: $V(t) = e^{-i\omega t \frac{\sigma^z}{2}}$

$\Rightarrow H_{rot} = B_z \sigma^z + B_{\parallel} \sigma^x - \frac{\omega}{2} \sigma^z$ time-indep!

$\Rightarrow U_{rot}(t_2, t_1) = \exp\left(-i(t_2 - t_1) H_{rot}\right)$ easy in rot. frame!

- rotating frame for a generic problem is difficult/impossible to identify

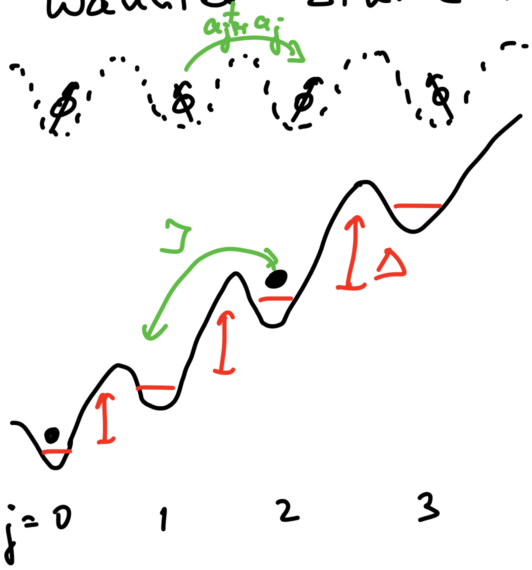
Ex. 2: static systems viewed as periodically driven in rot frame

Wannier-Stark ladder

realization: e^- in external el. field

$$H = \sum_j -J\left(a_{j+1}^\dagger a_j + h.c.\right) + j \Delta n_j$$

$$n_j = a_j^\dagger a_j$$



strong field: $\Delta \gg J$

$\rightarrow$ particles cannot hop b/c energy released Δ cannot be transformed into kinetic energy J

- alternative way to see this:

rot frame: $V(t) = \exp\left(-it\Delta \sum_j j\, n_j\right)$

$$H_{rot}(t) = \sum_j -J\left(e^{it\Delta} a_{j+1}^\dagger a_j + h.c.\right) \xrightarrow[\int_0^T dt\, e^{i\Delta t} = 0]{\text{high-freq.}\ \Delta \gg J} 0$$

Q: can we exploit the time-periodic structure of $H(t)$?

Thm (Gaston Floquet, 1883) (theory ODE's)

let $H: \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{C}^{n\times n} \\ t \longmapsto H(t) \end{array}$ be continuous, matrix-valued fn w/ period T: $H(t+T) = H(t)$

(Hamiltonian)

let $U(t)$ be the fundamental matrix (time-evo op.)

to the first-order linear ODE

$i\partial_t \psi(t) = H(t)\psi(t)$; $i\partial_t U(t) = H(t)\,U(t)$

$U(0) = \mathbb{1}$

Then:

1) $U(t+T)$ is also a fundamental matrix

2) there exists a non-singular, continuously diff'ble matrix valued fn:

$$P : \begin{array}{l} \mathbb{R} \to \mathbb{C}^{n\times n} \\ t \mapsto P(t) \end{array} \quad \text{with period } T: \; P(t+T) = P(t)$$

and a time-indep. matrix $H_F \in \mathbb{C}^{n\times n}$, s.t.

$$\boxed{U(t) = P(t)\, e^{-itH_F}}$$

Corrollary: stroboscopically, i.e. at $t = \ell T$

$$U(\ell T) = e^{-i\ell T H_F}$$

Proof:

1) $i\partial_t U(t+T) = i\dot{U}(t+T) \overset{\text{def. of } U(t)}{=} H(t+T)\,U(t+T) \overset{H(t)=H(t+T)}{=} H(t)\,U(t+T)$ ✓

2) $U(t)$ & $U(t+T)$ are both fundamental matrices

$\Rightarrow$ there is a static linear transformation that relates them

$\Rightarrow \; U(t+T) = U(t)\,U_F \quad (**)$

$$U_F : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$$

by the existence of matrix log, define H_F via:

$$U_F = e^{-iTH_F}$$

set: $P(t) := U(t)\, e^{+itH_F} \quad (*)$

check periodicity: $P(t+T) \overset{(*)}{=} U(t+T)\, e^{+i(t+T)H_F}$

$$\overset{(**)}{=} U(t) \underbrace{U_F\, e^{+itH_F}\, e^{+iTH_F}}_{=\mathbb{1}}$$

$= U(t)\, e^{+itH_F} = P(t)$ ✓

invert (*) : $U(t) = P(t)\, e^{-itH_F}$ ✓ ■

Rmks:

1) note: this requires a linear ODE

2) $H(t)$ need not be hermitian
   but if $H(t) = H^{\dagger}(t) \Rightarrow H_F = H_F^{\dagger}$

3) proof is not constructive