# Floquet engineering

Q: how should we design the Hamiltonian $H(t)$, so that we can prescribe desired properties of $H_F$ ?

## Example 1: dynamical stabilization

→ rotating saddle / Paul trap (Wolfgang Paul, Nobel prize '89)

→ Kapitza oscillator (Piotr Kapitza, '51)

– Kapitza pendulum using FM expansion:

$$H(t) = \frac{p_\theta^2}{2m} - m\left(\omega_0^2 + A\omega\cos\omega t\right)\cos\theta$$

$$= H_{kin} + H_{pot} + \omega f(t) H_{drive}$$

where: $H_{kin} = p_\theta^2/2m$

$H_{pot} = -m\omega_0^2\cos\theta$

$H_{drive} = -m\cos\theta \quad ; \quad f(t) = A\cos\omega t$

– due to scaling of drive amplitude w/ frequency $\omega$, we cannot simply take the time-average in lab frame

→ limit $\omega\to\infty$ not just given by time-ave

→ missing higher order contributions: $\frac{1}{\cancel{\omega}}\left[H_{kin}+H_{pot}, \cancel{\omega} H_{drive}\right]$

a) n'th order term contains n-nested commutators

b) & n-fold time-ordered integral $\sim 1/\omega^{n-1}$

Q: is there a more efficient way to compute $H_{FM}$ ?

yes! → in the rotating frame:

$$H(t) = \frac{p^2}{2m} - m\left(\omega_0^2 + A\omega\cos\omega t\right)\cos\theta$$

"problematic" scaling, messes up the power counting of $\omega$ !

recall: $H_{rot}(t) = V^\dagger(t)\, H(t)\, V(t) - i V^\dagger(t)\,\partial_t V(t)$

idea: use Galilean term to cancel term $\propto \omega$ in $H_{rot}(t)$ by choosing $V(t)$ suitably

e.g. $$V(t) = \exp\Big(-i \underbrace{\int_0^t (-m A\omega \cos\omega s)\, ds}_{=:\,\Delta(t)\,=\,-mA\sin\omega t} \times \cos\theta\Big)$$

$$= e^{-i\Delta(t)\cos\theta}$$

<u>then:</u> $$i V^\dagger \partial_t V = -m A\omega \cos\omega t \cos\theta = +\omega f(t) H_{drive}$$

compute:
$$V^\dagger(t) H(t) V(t) = e^{-i\Delta\cos\theta}\big(H_{kin} + H_{pot} + \omega f(t) H_{drive}\big) e^{+i\Delta\cos\theta}$$

$$= e^{-i\Delta\cos\theta} H_{kin}\, e^{+i\Delta\cos\theta} + H_{pot} + \omega f(t) H_{drive}$$

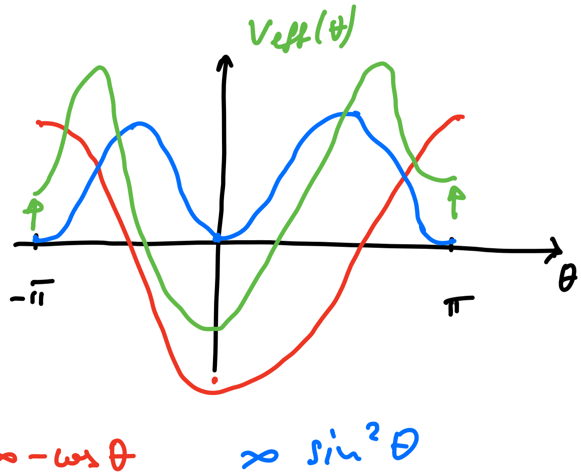
$$e^{-i\Delta\cos\theta}\frac{p^2}{2m} e^{+i\Delta\cos\theta} = \frac{p^2}{2m} + \frac{\Delta^2(t)}{2m}\sin^2\theta + \frac{\Delta(t)}{2m}\{\sin\theta, p\}_+$$

$$\Rightarrow H_{rot}(t) = \frac{p^2}{2m} - m\omega_0^2\cos\theta + \frac{\Delta^2(t)}{2m}\sin^2\theta + \frac{\Delta(t)}{2m}\{\sin\theta, p\}_+$$

since $\Delta \propto \mathcal{O}(\omega^0)$, we can apply expansion in rot frame

$$H_F^{(0)} = \frac{1}{T}\int_0^T dt\, H_{rot}(t)$$

$$= \frac{p^2}{2m} \underbrace{- m\omega_0^2\cos\theta + \frac{A^2 m}{4}\sin^2\theta}_{=:\,V_{eff}(\theta)\ \text{effective potential}}$$




$$\partial_\theta^2 V_{eff} = m\omega_0^2\cos\theta + \frac{A^2}{2} m\cos 2\theta$$

$$\underset{\theta=\pi}{=} -m\omega_0^2 + \frac{A^2}{2}m \overset{!}{>} 0$$

stable for $A > \sqrt{2}\,\omega_0$ at inverted equilibrium at $\theta = \pi$

$\rightarrow$ dynamical stabilization

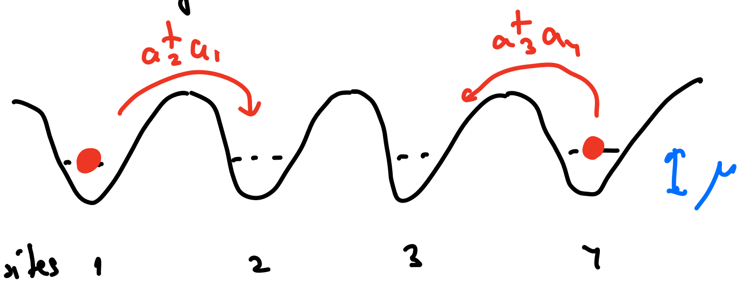
→ can identify correct $1/\omega$ corrections

- going to rotating frame leads to a resummation of a entire subseries of the FM expansion
  ↳ non-perturbative effects

<u>Example 2</u> : <u>dynamical localization</u>

$$H_0 = \sum_j -J\,(a^\dagger_{j+1} a_j + h.c.) - \mu \underbrace{a^\dagger_j a_j}_{=:\, n_j}$$

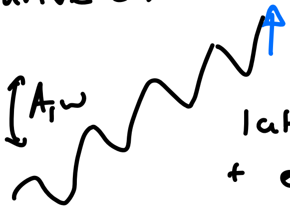
$=: n_j$ density at site $j$




free particle hopping on lattice
→ hopping delocalizes wavefn.

<u>want</u>: localize particles, i.e. suppress tunneling $J$

$$H_{drive}(t) = A\omega \cos\omega t \sum j\, n_j$$ oscillating electrostatic potential




lattice
+ el. pot. $\phi(r) \sim r$

total Hamiltonian

eliminate by going to rot. frame

$$H(t) = \sum_j \{ -J\,(a^\dagger_{j+1} a_j + h.c.) - \mu n_j + A\omega \cos\omega t\, j\, n_j \}$$

$$V(t) := e^{-iA\sin\omega t \sum_j j n_j} = \prod_j e^{-iA\sin\omega t\, j\, n_j}$$

$$H_{rot}(t) = \sum_j -J\, \underbrace{V^\dagger(t)\, a^\dagger_{j+1} a_j\, V(t)}_{= V^\dagger a^\dagger_{j+1} V V^\dagger a_j V} + h.c. - \mu n_j$$

$$V^\dagger(t)\, a_j\, V(t) = \prod_k e^{+iA\sin\omega t\, k n_k}\, a_j \prod_{k'} e^{-iA\sin\omega t\, k' n_{k'}}$$

$$= e^{iA\sin\omega t\, j n_j}\, a_j\, e^{-iA\sin\omega t\, j n_j}$$

need: $e^{i\alpha n}\, a\, e^{-i\alpha n} =: F(\alpha) \quad (*)$ , $\alpha = A j \sin\omega t$

$$\partial_\alpha F = e^{i\alpha n} \underbrace{i[n,a]}_{=-ia} e^{-i\alpha n} = -i F$$

$$\Rightarrow F(\alpha) = F(0)\, e^{-i\alpha} \overset{(*)}{=} a\, e^{-i\alpha}$$

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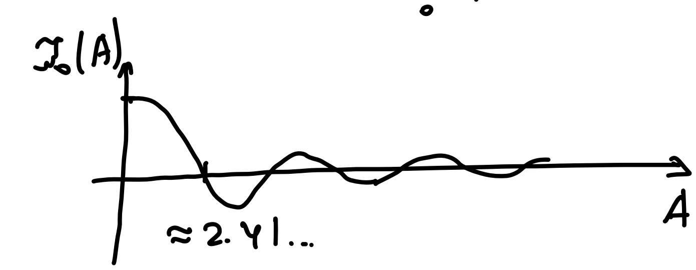
$$H_{rot}(t) = \sum_j -J e^{+iA\sin\omega t (j+1)} a^\dagger_{j+1} e^{-iA\sin\omega t\, j} a_j + h.c. - \mu n_j$$

$$= \sum_j -J\left(e^{+iA\sin\omega t} a^\dagger_{j+1} a_j + h.c.\right) - \mu n_j$$

- apply IFE :

$$H_F^{(0)} = \frac{1}{T}\int_0^T dt\, H_{rot}(t) = \sum_j -J_{eff}(A)\left(a^\dagger_{j+1} a_j + h.c.\right) - \mu n_j$$

where $J_{eff}(A) = \int_0^T \frac{dt}{T} e^{-iA\sin\omega t} = \mathcal{J}_0(A)$ zeroth Bessel fn of $1^{st}$ kind



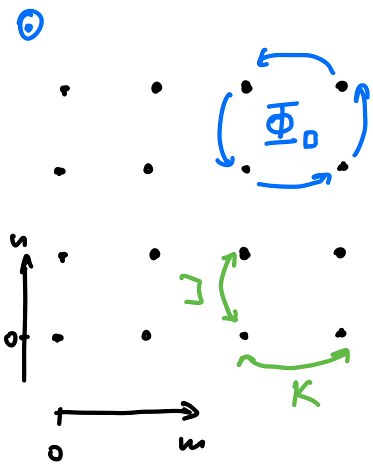

at $A_* \approx 2.41$

$J_{eff}(A_*) \approx 0 \rightarrow$ suppression of coherent tunneling

$\rightarrow$ dynamical localization

<u>Example</u> 3 : artificial magnetic fields

consider the Harper-Hofstadter model: PRB 14 2239, 1976




$$H_{HH} = -K\sum_{m,n} e^{i\varphi_{mn}} a^\dagger_{m+1,n} a_{mn} + h.c.$$

$$- J\sum_{m,n} a^\dagger_{m,n+1} a_{mn} + h.c.$$

$$\varphi_{mn} = \Phi_\square (m+n)$$

$n_{mn} := a^\dagger_{mn} a_{mn}$ particle op.

- $\Phi_\square$ magnetic flux per plaquette
  $\rightarrow$ breaks time-reversal, cannot be gauged away

- in materials : $a^\dagger_{mn}$ $e^-$ operator
  $\rightarrow$ $e^-$ are charged , couple to EM field

<u>issue</u> : $\Phi_\Box$ is limited by strength of B-field ($\rightarrow$ technical challenge)

- quantum simulator
  - <u>neutral</u> atoms : do not couple to B-field

<u>idea</u> : use Floquet engineering

$$H(t) = H_0 + H_{drive}(t)$$

$$H_0 = -\sum_{m,n} J_x \left( a^\dagger_{m+1,n} a_{m,n} + h.c. \right) + J_y \left( a^\dagger_{m,n+1} a_{mn} + h.c. \right)$$

$$H_{drive}(t) = \omega \sum_{m,n} \left[ \frac{A}{2} \sin\left(\omega t - \varphi_{mn} + \frac{\Phi_\Box}{2}\right) + m \right] n_{mn}$$

$\varphi_{mn} = \Phi_\Box (m+n)$
spatially inhom.
phase of drive

gradient only along x

$\rightarrow$ breaks time reversal !

$\rightarrow$ go to rot. frame to eliminate term $\propto \omega$

$$V(t) = e^{-i \int_0^t ds\, H_{drive}(s)}$$

$$H_{rot}(t) = G(t) + G^\dagger(t)$$

$$G(t) = -\sum_{m,n} J_x \, e^{-i \zeta_\Phi \sin(\omega t - \varphi_{mn}) + i\omega t} a^\dagger_{m+1,n} a_{mn} + h.c.$$

$$+ J_y \, e^{-i \zeta_\Phi \sin(\omega t - \varphi_{mn})} a^\dagger_{m,n+1} a_{mn} + h.c.$$

where $\zeta_\Phi = A \sin \frac{\Phi_\Box}{2}$

- effective Hamiltonian :

use $e^{i\alpha \sin(\omega t - \varphi)} \overset{\text{Fourier exp.}}{=} \sum_{l \in \mathbb{Z}} \mathcal{J}_l(\alpha) e^{-il(\omega t - \varphi)}$

$$\frac{1}{T}\int_0^T dt\, e^{-i\zeta_\Phi \sin(\omega t - \varphi_{mn}) + i\omega t} = \sum_l \mathcal{J}_l(\zeta_\Phi) \underbrace{\int_0^T \frac{dt}{T} e^{-il(\omega t - \varphi_{mn})} e^{i\omega t}}_{= e^{+i\varphi_{mn}} \delta_{l,1}}$$

$$= \mathcal{J}_1(\zeta_\Phi) e^{i\varphi_{mn}}$$

$$H_F^{(0)} = -\sum_{m,n} K e^{i\varphi_{mn}} a^\dagger_{m+1,n} a_{mn} + h.c.$$

$$J a^\dagger_{m,n+1} a_{mn} + h.c.$$

HH Hamiltonian w/ flux $\Phi_\square$ → artifical magnetic field

where $K = J_x \mathcal{J}_1(\zeta_\Phi)$ & $J = J_y \mathcal{J}_0(\zeta_\Phi)$

note: flux $\Phi_\square$ come from phase of drive
⇒ can simulate arbitrary fluxes

- Floquet engineering is limited by
  1) laws of physics
  2) your own creativity!
- all of the above examples generalize to interacting systems (density-density interactions remain the same in rot frame)

→ but: system may (& in general, will) absorb energy from drive → heating, prethermalization