Equilibration and eigenstate thermalization

- An important motivation for the use of random matrix theory, originally due to Wigner, is that the eigenvalues of highly complicated many-body Hamiltonians exhibit *statistical features* of the eigenvalues of random matrices. Rather than trying to calculate individual eigenvalues, it is possible to make exact statements about the statistical properties of the eigenvalue spectrum. In Fig. 1 we show the level spacing distribution, i.e. the probability distribution for the distance between subsequent energy levels, $E_{n+1}-E_n$ with $E_{n+1} > E_n$, for the "Nuclear Data Ensemble", which comprises 1726 normalized level spacings for the eigenspectra of various heavy nuclei. This probability distribution can be calculated for random matrices drawn from the Gaussian Orthogonal Ensemble, and the agreement between the two is remarkable. This distribution is also contrasted with the Poisson distribution that would be obtained if the eigenvalues were statistically independent variables.

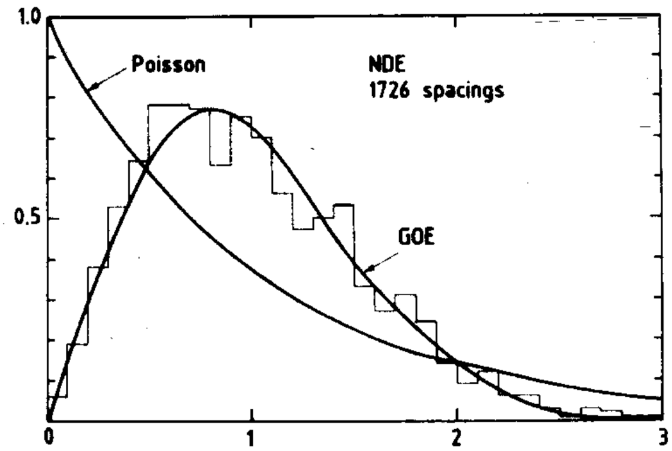

Figure 1: Nearest neighbor spacing distribution for the "Nuclear Data Ensemble" comprising 1726 spacings (histogram) versus $s = S/D$, where D is the mean level spacing and $S_n = E_{n+1} - E_n$ is the actual spacing. Lines represent the GOE and the Poisson distributions.

In this exercise, we will derive the distribution of the eigenvalue spacing of a random symmetric 2×2 matrix. Suppose we have a 2×2 random matrix

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{pmatrix}, \tag{1}$$

drawn from the Gaussian Orthogonal Ensemble, i.e.

$$P(M) \propto \exp\left[-\frac{1}{2}\left(M_{11}^2 + M_{22}^2 + 2M_{12}^2\right)\right]. \tag{2}$$

This matrix has two eigenvalues $E_\pm$ and eigenvalue spacing $s = E_+ - E_-$. Calculate the probability distribution of this spacing, also known as the *Wigner surmise*, and compare with Fig. 1. Hint: The necessary integral can be evaluated using polar coordinates.

How is the distribution modified for a Hermitian matrix for which the real and imaginary parts are independently distributed? Such matrices can be drawn from the Gaussian Unitary Ensemble:

$$P(M) \propto \exp\left[-\frac{1}{2}\left(M_{11}^2 + M_{22}^2 + 2|M_{12}|^2\right)\right]. \tag{3}$$

- The spectral function appearing in off-diagonal matrix elements in ETH naturally appears in linear response, underlying the important fluctuation-dissipation relation, and determines all nontrivial operator dynamics. Consider two-point autocorrelation functions at an inverse temperature β of the form

$$\kappa_2(t) = \langle \hat{O}(t)\hat{O}\rangle_\beta - \langle \hat{O}\rangle_\beta^2 \,. \tag{4}$$

Here the thermal expectation value is defined as $\langle \bullet \rangle_\beta = \mathrm{Tr}[\bullet\, e^{-\beta H}]/\mathcal{Z}$. Using ETH, argue that the autocorrelation function can be written as

$$\kappa_2(t) = \frac{1}{\mathcal{Z}} \int dE_m \int dE_n \, e^{-\beta E_n + S(E_m) + S(E_n) - S(E_{mn})} \, |f_O(E_{mn}, \omega_{mn})|^2 \, e^{-i\omega_{mn} t} \,. \tag{5}$$

This integration can be simplified by switching to so-called *Wigner variables* $E \equiv E_{mn}$ and $\omega \equiv \omega_{mn}$. Perform this change of variables and Taylor expand the entropies in ω. You can set $e^{S''(E)\omega^2/4}$ equal to 1 (argue why) and perform a saddle-point integration to return

$$\kappa_2(t) = \int d\omega \, e^{\beta\omega/2} \, |f_O(E_\beta, \omega)|^2 \, e^{-i\omega t}. \tag{6}$$

In this way we find that the spectral function can be interpreted as the Fourier transform of the autocorrelation function.

- Using a similar approach, evaluate the operator fluctuations in an eigenstate $|n\rangle$. Specifically, use ETH to express the eigenstate fluctuations

$$\delta O_n^2 = \langle n|O^2|n\rangle - \langle n|O|n\rangle^2 \tag{7}$$

in terms of the spectral function of O. Argue that these fluctuations are reproduced by the Gibbs state.