Lecture Notes

Quench dynamics in the Transverse Field Ising Model

Many-body Quantum Dynamics @ TU Dresden

Contents

1 Selected bibliography

- These notes follow the structure from Ref. [1]: F. H. L. Essler and M. Fagotti, "Quench dynamics and relaxation in isolated integrable quantum spin chains," J. Stat. Mech. **2016**, 064002 (2016)

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2 Introduction

In the previous lectures, we have argued that generic (i.e. chaotic) quantum many-body systems relax locally, in such a way that the expectation values of local observables can be described using the thermal Gibbs distribution. Underlying this result was the Eigenstate Thermalization Hypothesis, for which a plethora of numerical evidence exists. However, thermalization can not be rigorously proven in general. In the following, we consider a model for which the dynamics can be solved exactly, the *Transverse Field Ising Model*. This model has the special property that the dynamics of a spin chain can be mapped to the dynamics of a set of noninteracting fermions, for which the dynamics remain tractable. In analogy with classical systems for which the equations of motion can be exactly integrated, such models are also referred to as *integrable systems*. Integrable models have served as the testbed for many ideas of many-body quantum dynamics (see also the lectures on Floquet dynamics and nonadiabatic response).

The mapping to noninteracting fermions is not just a mathematical curiosity, but is reflected physically in the appearance of an extensive set of local conserved charges. In this model, and more generally in integrable models, it is not just the energy that is conserved, but we can rather find as many (local) conserved charges as there are spin degrees of freedom. The existence of additional (local) conservation laws precludes relaxation to a thermal ensemble: suppose the Hamiltonian has an additional conservation law $[\hat{H},\hat{I}]=0$. The expectation value of this conserved quantity is constant in time,

$$\langle\psi(t)|\hat{I}|\psi(t)\rangle = \langle\psi(0)|\hat{I}|\psi(0)\rangle\,, \tag{1}$$

such that any steady-state reduced density matrix should reproduce the initial expectation value. However, if this steady-state density matrix is the thermal density matrix, it only depends on the initial energy – and two initial states with the same energy would give rise to the same steady state and hence the same steady-state expectation value of $\hat{I}$, irrespective of its initial expectation value.

Taken this into account, we need to modify the notion of a Gibbs ensemble to a *Generalized Gibbs Ensemble (GGE)*. If the Hamiltonian has a number of conserved charges $I^{(n)}$ satisfying $[H,I^{(n)}]=0$ where typically also $[I^{(m)},I^{(n)}]=0$, the GGE is defined as

$$\rho_{\mathrm{GGE}} = \frac{e^{-\sum_n \lambda_n \hat{I}^{(n)}}}{\mathrm{Tr}\left[e^{-\sum_n \lambda_n \hat{I}^{(n)}}\right]}\,. \tag{2}$$

Here we have implicitly included the Hamiltonian as one of the conserved charges $I^{(n)}$. Rather than the thermal ensemble, which only depends on the Hamiltonian and a corresponding inverse temperature set by the initial energy, here the steady state is determined by all conservation laws, where each conservation law $I^{(n)}$ also has a corresponding Lagrange multiplier λ_n. These Lagrange multipliers are again uniquely determines by the expectation values of the conserved charges in the initial state, since the conservation laws imply

$$\langle\psi(t=0)|\hat{I}^{(n)}|\psi(t=0)\rangle = \langle\psi(t)|\hat{I}^{(n)}|\psi(t)\rangle = \mathrm{Tr}[\rho_{\mathrm{GGE}}\hat{I}^{(n)}]\,. \tag{3}$$

3 Aspects of fermionic operators

In the first part of these lectures, we will focus on (spinless) *fermionic models*, since the results for the dynamics in these models is more transparent. These lattice models are easiest to formulate in the language of *second quantization*.

Consider lattice models where each lattice site is either empty or occupied by a single fermion. Denoting these states as $|0\rangle_j$ and $|1\rangle_j$ for site j, we can introduce fermionic creation/annihilation operators in the same way as spin raising/lowering operators

$$\hat{c}_j^\dagger\,|0\rangle_j = |1\rangle_j\,, \qquad \hat{c}_j\,|1\rangle_j = |0\rangle_j\,. \tag{4}$$

Fermionic operators have the property that they square to zero $(\hat{c}_j^\dagger)^2 = (\hat{c}_j)^2 = 0$, as implied by the Pauli exclusion principle, such that we have

$$\hat{c}_j^\dagger\,|1\rangle_j = 0, \qquad \hat{c}_j\,|0\rangle_j = 0\,. \tag{5}$$

The operator $\hat{n}_j \equiv \hat{c}_j^\dagger \hat{c}_j$ acts as the number operator, since

$$\hat{n}_j\,|0\rangle_j = 0, \qquad \hat{n}_j\,|1\rangle_j = |1\rangle_j \qquad \Rightarrow \qquad \hat{n}_j\,|n\rangle_j = n\,|n\rangle_j\,. \tag{6}$$

Many-body states can be defined starting from the vacuum state $|0\rangle = \otimes_j |0\rangle_j$, where e.g. the state with a fermion on site j and site l can be obtained by first creating a fermion on site l and then a fermion on site j:

$$\hat{c}_j^\dagger \hat{c}_l^\dagger\,|0\rangle\,. \tag{7}$$

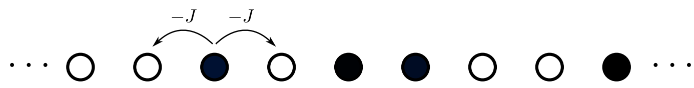

Figure 1: Illustration of a one-dimensional lattice on which fermions can hop with amplitude $-J$. Note that hopping is constrained because no two fermions can occupy the same lattice site due to the Pauli exclusion principle.

Alternatively, we could have created this state as $\hat{c}_l^\dagger \hat{c}_j^\dagger |0\rangle$, which can also be obtained by exchanging j and l. Crucially, the fermionic antisymmetry implies that such an exchange should lead to an overall minus sign, indicating that fermionic raising/annihilation operators on different lattice sites anticommute:

$$\hat{c}_j^\dagger \hat{c}_l^\dagger = -\hat{c}_l^\dagger \hat{c}_j^\dagger \,. \tag{8}$$

More generally, fermions satisfy the anticommutation relations, with $\{A, B\}_+ = AB + BA$ the anticommutator,

$$\{\hat{c}_j, \hat{c}_l\}_+ = 0, \qquad \{\hat{c}_j^\dagger, \hat{c}_l^\dagger\}_+ = 0, \qquad \{\hat{c}_j^\dagger, \hat{c}_l\}_+ = \delta_{jl} \,. \tag{9}$$

These can be used to calculate e.g. expectation values of the form

$$\begin{aligned} \langle 0|\hat{c}_j \hat{c}_l \hat{c}_m^\dagger \hat{c}_n^\dagger|0\rangle &= \langle 0|\hat{c}_j \{\hat{c}_l, \hat{c}_m^\dagger\}_+ \hat{c}_n^\dagger|0\rangle - \langle 0|\hat{c}_j \hat{c}_m^\dagger \hat{c}_l \hat{c}_n^\dagger|0\rangle \\ &= \delta_{lm} \langle 0|\hat{c}_j \hat{c}_n^\dagger|0\rangle - \langle 0|\hat{c}_j \hat{c}_m^\dagger \{\hat{c}_l, \hat{c}_n^\dagger\}_+|0\rangle + \langle 0|\hat{c}_j \hat{c}_m^\dagger \hat{c}_n^\dagger \hat{c}_l|0\rangle \\ &= \delta_{lm} \langle 0|\hat{c}_j \hat{c}_n^\dagger|0\rangle - \delta_{ln} \langle 0|\hat{c}_j \hat{c}_m^\dagger|0\rangle \,, \end{aligned} \tag{10}$$

where we have only used the fermionic anticommutation relation and that the vacuum is annihilated by all annihilation operators. The final expression then evaluates to

$$\begin{aligned} \langle 0|\hat{c}_j \hat{c}_l \hat{c}_m^\dagger \hat{c}_n^\dagger|0\rangle &= \delta_{lm} \langle 0|\{\hat{c}_j, \hat{c}_n^\dagger\}_+|0\rangle - \delta_{lm} \langle 0|\hat{c}_n^\dagger \hat{c}_j|0\rangle - \delta_{ln} \langle 0|\{\hat{c}_j, \hat{c}_m^\dagger\}_+|0\rangle + \delta_{ln} \langle 0|\hat{c}_m^\dagger \hat{c}_j|0\rangle \\ &= \delta_{lm}\delta_{jn} - \delta_{ln}\delta_{jm} \,. \end{aligned} \tag{11}$$

When calculating expectation values w.r.t. the vacuum, the only nonvanishing terms are those where fermions are first created and subsequently annihilated, with appropriate signs depending on the order of the operators.

4 Noninteracting fermions

Consider a one-dimensional lattice of L sites, on which fermions can hop with hopping strength $-J$ in the presence of a chemical potential with strength μ. The Hamiltonian reads

$$\hat{H} = -J \sum_{j=1}^{L} \left(\hat{c}_j^\dagger \hat{c}_{j+1} + \hat{c}_{j+1}^\dagger \hat{c}_j \right) - \mu \sum_{j=1}^{L} \hat{c}_j^\dagger \hat{c}_j, \tag{12}$$

where we again assume periodic boundary conditions, $\hat{c}_{L+1} \equiv \hat{c}_1$. This model is translationally invariant, such that (discrete) momentum should be conserved. We can introduce fermionic operators in momentum space

$$\hat{c}(k) = \frac{1}{\sqrt{L}} \sum_{j=1}^{L} e^{ikj} \hat{c}_j \,, \tag{13}$$

where periodic boundary conditions imply that $e^{ikL}=1$ such that k can only take the quantized values $2\pi n/L$, $n=0,1,\dots,L-1$. Any summation over k in these notes should be read as a summation over these L quantized values, e.g. in

$$\hat{c}_j=\frac{1}{\sqrt{L}}\sum_k e^{-ikj}\hat{c}(k)\,. \tag{14}$$

Crucially, these operators satisfy the same anticommutation relations as the original fermionic raising/lowering operators. Introducing these operators in the Hamiltonian (12), it is a straightforward exercise to show that

$$\hat{H}=\sum_k \epsilon_0(k)\hat{c}^\dagger(k)\hat{c}(k)=\sum_k \epsilon_0(k)\hat{n}(k) \qquad \text{with} \qquad \epsilon_0(k)=-2J\cos(k)-\mu\,. \tag{15}$$

That Hamiltonian decomposes in a sum of noninteracting modes: $\hat{n}(k)$ is the *mode occupation number* of the fermion with momentum k, which can be either zero or one, and each mode has an excitation energy $\epsilon_0(k)$. The eigenvalues immediately follow as

$$E(\{n_k\})=\sum_k \epsilon_0(k)n_k, \qquad n_k\in\{0,1\}\,. \tag{16}$$

These mode occupation numbers additionally act as conserved charges, satisfying

$$[\hat{H},\hat{n}(k)]=[\hat{n}(k),\hat{n}(q)]=0, \qquad \forall k,q\,. \tag{17}$$

While these are conserved quantities, these are also completely nonlocal: if we rewrite these operators in terms of the original fermionic operators they contain interaction terms between all possible lattice sites,

$$\hat{n}(k)=\frac{1}{L}\sum_{j,\ell} e^{-ik\ell}\hat{c}^\dagger_{j+\ell}\hat{c}_j\,. \tag{18}$$

In our discussion of ETH we have already highlighed the importance of locality in the dynamics, such that it is not obvious how these conserved charges should constrain the dynamics. Crucially, it is possible to construct an equivalent set of *local* conserved quantities for the Hamiltonian (12). Consider linear combinations of the form

$$\hat{I}^{(n,+)}=2J\sum_k \cos(nk)\,\hat{c}(k)^\dagger\hat{c}(k)=J\sum_{j=1}^{L}\left(\hat{c}^\dagger_j\hat{c}_{j+n}+\hat{c}^\dagger_{j+n}\hat{c}_j\right), \tag{19}$$

$$\hat{I}^{(n,-)}=2J\sum_k \sin(nk)\,\hat{c}(k)^\dagger\hat{c}(k)=iJ\sum_{j=1}^{L}\left(\hat{c}^\dagger_j\hat{c}_{j+n}-\hat{c}^\dagger_{j+n}\hat{c}_j\right). \tag{20}$$

Since these are linear combinations of the mode occupation numbers, these again mutually commute and commute with the Hamiltonian, $[\hat{I}^{(n,\sigma)},\hat{I}^{(m,\tau)}]=0$, and these operators are now well-behaved local operators.

We can write the GGE in two equivalent ways,

$$\rho_{\text{GGE}}=\frac{e^{-\sum_{n,\sigma}\lambda_{n,\sigma}\hat{I}^{(n,\sigma)}}}{\text{Tr}\left[e^{-\sum_{n,\sigma}\lambda_{n,\sigma}\hat{I}^{(n,\sigma)}}\right]}=\frac{e^{-\sum_k \mu_k\hat{n}(k)}}{\text{Tr}\left[e^{-\sum_k\mu_k\hat{n}(k)}\right]}, \tag{21}$$

where the first is the 'physical' GGE expressed in terms of local conserved quantities, whereas the second is often more convenient mathematically and expressed in terms of the mode occupation numbers. Because of the linear transformation between mode occupation numbers and local conserved charges, these two expressions are fully equivalent. Crucially, the locality of the conserved quantities guarantees that we will obtain a nontrivial reduced density matrix when tracing out an extensive part of the system.

5 Quantum quenches

Relaxation is most often studied in so-called *quantum quenches*: the system is prepared in the ground state of some (local) Hamiltonian, and then a parameter in the Hamiltonian is abruptly changed and the system is left the evolve under the new Hamiltonian. In such a setup the quench injects an extensive amount of energy in the system, and the initial state has overlap with an exponentially large amount of eigenstates of the Hamiltonian governing the dynamics.

In order to illustrate this using the previous Hamiltonian, we introduce a new parameter Δ (whose interpretation will be made explicit later) in the fermion chain,

$$\hat{H}(\Delta) = -J\sum_j \left(\hat{c}_j^\dagger \hat{c}_{j+1} + \hat{c}_{j+1}^\dagger \hat{c}_j\right) - \mu \sum_j \hat{c}_j^\dagger \hat{c}_j + \Delta \sum_j \left(\hat{c}_j^\dagger \hat{c}_{j+1}^\dagger + \hat{c}_{j+1}\hat{c}_j\right). \tag{22}$$

This Hamiltonian is also known as the *Kitaev chain*. In momentum space this term has the effect of coupling the mode with momentum k to the mode with momentum $-k$, since the Hamiltonian reads

$$\hat{H}(\Delta) = \sum_k \epsilon_0(k)\hat{c}(k)^\dagger \hat{c}(k) - i\Delta \sin(k)\left(\hat{c}(k)^\dagger \hat{c}(-k)^\dagger - \hat{c}(-k)\hat{c}(k)\right). \tag{23}$$

The eigenstates and eigenvalues of this Hamiltonian can again be analytically obtained. We can write

$$\begin{aligned}\hat{H}(\Delta) = \sum_{k>0} \begin{bmatrix} \hat{c}(k)^\dagger & \hat{c}(-k) \end{bmatrix} \begin{bmatrix} \epsilon_0(k) & -2i\Delta\sin(k) \\ 2i\Delta \sin(k) & -\epsilon_0(-k) \end{bmatrix} \begin{bmatrix} \hat{c}(k) \\ \hat{c}(-k)^\dagger \end{bmatrix} \\ + \epsilon_0(0)\hat{c}(0)^\dagger\hat{c}(0) + \text{Cst.},\end{aligned} \tag{24}$$

where we restrict the summation over k to the quantized values $2\pi n/L$, $n = 1, \ldots, (L-1)/2$, assuming L odd, and denote these as $k > 0$ (since the other values of k evaluate to minus these values by subtracting 2π).

Bogoliubov transformation. Diagonalizing the 2×2 matrices, we can re-express this Hamiltonian in terms of mode occupation numbers of new fermionic operators. These matrices can be diagonalized by defining

$$\begin{bmatrix} \alpha(k) \\ \alpha(-k)^\dagger \end{bmatrix} = \begin{bmatrix} \cos(\theta_k/2) & -i\sin(\theta_k/2) \\ -i\sin(\theta_k/2) & \cos(\theta_k/2) \end{bmatrix} \begin{bmatrix} \hat{c}(k) \\ \hat{c}(-k)^\dagger \end{bmatrix} \tag{25}$$

with

$$\epsilon(k) = \sqrt{\epsilon_0(k)^2 + 4\Delta^2 \sin^2(k)} \qquad \text{and} \qquad e^{i\theta_k} = \frac{\epsilon_0(k) + 2i\Delta\sin(k)}{\epsilon(k)}. \tag{26}$$

This transformation is also known as a *Bogoliubov* transformation. It is a straightforward exercise to check that the operators $\alpha(k)$ again satisfy the expected fermionic anticommutation relations. In terms of these new fermions, the nontrivial part of the Hamiltonian reads

$$\begin{aligned}&\begin{bmatrix} \hat{c}(k)^\dagger & \hat{c}(-k) \end{bmatrix} \begin{bmatrix} \epsilon_0(k) & -2i\Delta\sin(k) \\ 2i\Delta\sin(k) & -\epsilon_0(-k) \end{bmatrix} \begin{bmatrix} \hat{c}(k) \\ \hat{c}(-k)^\dagger \end{bmatrix} \\ &\qquad = \begin{bmatrix} \hat{\alpha}(k)^\dagger & \hat{\alpha}(-k) \end{bmatrix} \begin{bmatrix} \epsilon(k) & 0 \\ 0 & -\epsilon(-k) \end{bmatrix} \begin{bmatrix} \hat{\alpha}(k) \\ \hat{\alpha}^\dagger(-k) \end{bmatrix}.\end{aligned} \tag{27}$$

The total Hamiltonian then reduces to

$$\hat{H}(\Delta) = \sum_k \epsilon(k)\,\hat{\alpha}(k)^\dagger \hat{\alpha}(k) + \text{Cst.}, \tag{28}$$

where, for convenience, we have defined $\hat{\alpha}(0) = \hat{c}(0)$. The eigenstates of the Hamiltonian hence correspond to states with fixed mode occupation number of the Bogoliubov fermions.

The ground state of this Hamiltonian is the Bogoliubov vacuum, i.e. the state annihilated by all Bogoliubov annihilation operators,

$$|\psi_0\rangle = |0\rangle \qquad \text{with} \qquad \hat{\alpha}(k)\,|0\rangle = 0, \forall k. \tag{29}$$

Operator dynamics. We are now in a position to analyze quantum quenches in this Hamiltonian. For simplicity we will focus on the situation where we prepare the system in the ground state of the Hamiltonian (22) with a nonzero Δ and then quench the Hamiltonian to $\Delta = 0$. In this scenario we would expect the emergence of the GGE (21) at late times.

Because of the free-fermionic nature of the Hamiltonian, it is possible to directly consider the operator dynamics

$$\frac{d}{dt}\hat{c}(k;t) = i\left[\hat{H}, \hat{c}(k;t)\right] = -i\epsilon_0(k)\hat{c}(k;t), \tag{30}$$

such that we find that

$$\hat{c}(k;t) = e^{-i\epsilon_0(k)t}\hat{c}(k) = e^{-i\epsilon_0(k)t}\left[\cos(\theta_k/2)\alpha(k) + i\sin(\theta_k/2)\alpha^\dagger(-k)\right]. \tag{31}$$

The dynamics for $\hat{c}(k)^\dagger$ follows directly as the hermitian conjugate of the above equation. Focusing on two-point function, i.e. expectation values of observables of the form

$$\langle\psi_0|\hat{c}(k;t)^\dagger\hat{c}(q;t)|\psi_0\rangle = \delta_{k,q}\sin^2(\theta_k/2) \tag{32}$$

$$\langle\psi_0|\hat{c}(k;t)\hat{c}(q;t)|\psi_0\rangle = -\frac{i}{2}\delta_{k,-q}\sin(\theta_k)\,e^{-2i\epsilon_0(k)t}. \tag{33}$$

In evaluating these expressions, we have made use of the fact that the initial state was annihilated by the Bogoliubov fermions and only expectation values of the form $\langle\psi_0|\alpha(k)\alpha(q)^\dagger|\psi_0\rangle = \delta_{k,q}$ are nonvanishing.

If we want to relate the dynamics in momentum space back to the dynamics of local operators, we can undo the Fourier transformation to write

$$\langle\psi(t)|\hat{c}^\dagger_{j+\ell}\hat{c}_j|\psi(t)\rangle = \frac{1}{L}\sum_k e^{ik\ell}\sin^2(\theta_k/2) \equiv f(\ell) \tag{34}$$

$$\langle\psi(t)|\hat{c}_{j+\ell}\hat{c}_j|\psi(t)\rangle = -\frac{i}{2L}\sum_k e^{-2i\epsilon_0(k)t}e^{-ik\ell}\sin(\theta_k) \equiv g(\ell,t) \tag{35}$$

We can directly take the thermodynamic limit of an infinite system size, i.e. $L \to \infty$, which turns these summations into integrals

$$f(\ell) = \int_0^{2\pi}\frac{dk}{2\pi}e^{ik\ell}\sin^2(\theta_k/2), \qquad g(\ell,t) = -\frac{i}{2}\int_0^{2\pi}\frac{dk}{2\pi}e^{-2i\epsilon_0(k)t}e^{-ik\ell}\sin(\theta_k). \tag{36}$$

And that's it! By knowing the dynamics of these sets of observables, we can fully analyze the dynamics of any arbitrary observable in the thermodynamic limit, again due to the free-fermionic nature of the model and due to our choice of initial state. Namely, since the initial state is a vacuum state, we can express any expectation value of any product of creation/annihilation operators in terms of these two-point functions. This is a direct consequence of *Wick's theorem*.

The basic idea is simple: since we are taking expectation values w.r.t. a vacuum state, the only nontrivial contributions can be those were particle are initially create before being annihilated, such that the only nonvanishing contributions to any expectation value will

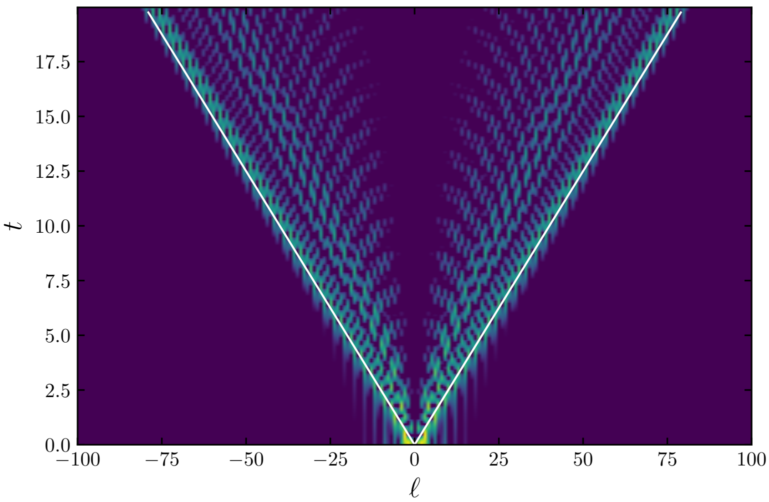

Figure 2: Time-dependent part of the connected density-density correlator (37) after a quantum quench where the system is initialized in the ground state of the Hamiltonian with $\Delta = 2J$ and $\mu = J$, and time evolved with the corresponding Hamiltonian with $\Delta = 0$. The correlations spread within a causal light cone, with maximal velocity $v_{\max} = 2J$.

be of the form $\langle \alpha_{k_1} \alpha^\dagger_{k_1} \alpha_{k_2} \alpha^\dagger_{k_2} \dots \rangle$. Any operators can always be brought in this ordering by making use of the fermionic anticommutation relations. Furthermore, expectation values of this form always factorize in expectation values over different momentum sectors: $\langle \alpha_{k_1} \alpha^\dagger_{k_1} \alpha_{k_2} \alpha^\dagger_{k_2} \rangle = \langle \alpha_{k_1} \alpha^\dagger_{k_1} \rangle \langle \alpha_{k_2} \alpha^\dagger_{k_2} \rangle$ if $k_1 \neq k_2$ and if $k_1 = k_2$ we can write $\langle \alpha_{k_1} \alpha^\dagger_{k_1} \alpha_{k_1} \alpha^\dagger_{k_1} \rangle = \langle \alpha_{k_1} \alpha^\dagger_{k_1} \rangle$.

Rather than proving this theorem in full generality, we can e.g. consider an example and verify that

$$\langle \psi(t) | \hat{c}^\dagger_{j+\ell} \hat{c}_{j+\ell} \hat{c}^\dagger_j \hat{c}_j | \psi(t) \rangle - \langle \psi(t) | \hat{c}^\dagger_{j+\ell} \hat{c}_{j+\ell} | \psi(t) \rangle \langle \psi(t) | \hat{c}^\dagger_j \hat{c}_j | \psi(t) \rangle = |g(-\ell, t)|^2 - |f(\ell)|^2 . \tag{37}$$

The above equality can be derived in a straightforward but slightly tedious way. This object characterizes the density-density correlations, and its dynamical part $|g(-\ell, t)|^2$ is shown in Fig. 2. At long times this dynamical part vanishes, indicating that the correlations have reached a steady state. The steady-state value of the density-density correlator is shown in Fig. 4.

We also clearly see that these correlations spread out with a maximal velocity, being exponentially small outside a causal lightcone $\ell = 2v_{\max} t$ with $v_{\max} = 2J$. The time $t = \ell/(2v_{\max})$ is also known as the *Fermi time*. The appearance of a causal lightcone is a universal property in dynamics governed by local Hamiltonians, as quantified in e.g. Lieb-Robinson bounds (see e.g. Ref. [2])

The maximal velocity can be understood by considering the behavior of $g(\ell = 2vt, t)$ for arbitrary velocities v, i.e. considering the dynamics on so-called 'light rays', at long times. In this case the integral reads

$$g(\ell = 2vt, t) = -\frac{i}{2} \int_0^{2\pi} \frac{dk}{2\pi} \sin(\theta_k) e^{-2it[kv + \epsilon_0(k)]} . \tag{38}$$

At late times we can approximate the integral using a stationary phase approximation, which

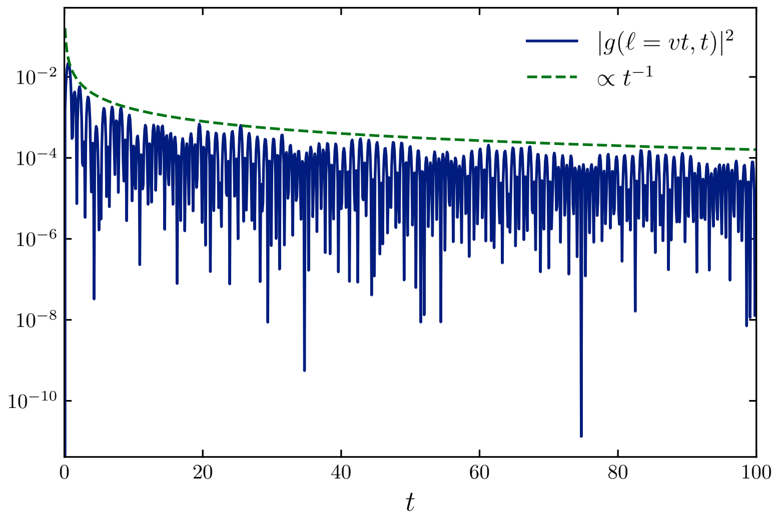

Figure 3: Decay of the function $|g(\ell = vt, t)|$ for a quantum quench where the system is initialized in the ground state of the Hamiltonian with $\Delta = 2J$ and $\mu = J$, and time evolved with the corresponding Hamiltonian with $\Delta = 0$. The correlations are evaluated on a light ray with velocity $v = 1 < v_{\max} = 2J$.

boils down to finding the values where the phase becomes extremal:

$$\frac{\partial}{\partial k}[kv + \epsilon_0(k)] = v + \partial_k \epsilon_0(k) = v + 2J\sin(k). \tag{39}$$

For $|v| > 2J$, this equation does now allow a solution, indicating that the integral rapidly dephases at all values of k, which leads to an exponential suppression of the correlations. For $|v| < 2J$, this equation can be solved and a stationary phase approximation results in expressions of the form

$$\int dk\, e^{-i\epsilon_0''(k)k^2 t} \propto t^{-1/2}, \tag{40}$$

leading to a powerlaw decay $t^{-1/2}$ of correlations along this light ray. This decay is illustrated in Fig. 3. Intuitively, the maximal velocity can be understood by noting that the dynamics decomposes into the dynamics of noninteracting modes with momentum k and energy $\epsilon_0(k)$. Interpreting these as a wave packet, the group velocity is given by

$$\frac{\partial \epsilon_0(k)}{\partial k} = 2J\sin(k), \tag{41}$$

which takes a maximal value of $2J$.

Agreement with GGE. In order to reproduce the correct stationary observables, the GGE should satisfy

$$\mathrm{Tr}\left[\hat{c}^\dagger_{j+\ell}\hat{c}_j \rho_{\mathrm{GGE}}\right] = f(\ell), \tag{42}$$

$$\mathrm{Tr}\left[\hat{c}_{j+\ell}\hat{c}_j \rho_{\mathrm{GGE}}\right] = 0. \tag{43}$$

We will consider the GGE expressed in terms of mode occupation numbers,

$$\rho_{\mathrm{GGE}} = \frac{e^{-\sum_k \mu_k \hat{c}(k)^\dagger \hat{c}(k)}}{\mathrm{Tr}\left[e^{-\sum_k \mu_k \hat{c}(k)^\dagger \hat{c}(k)}\right]}. \tag{44}$$

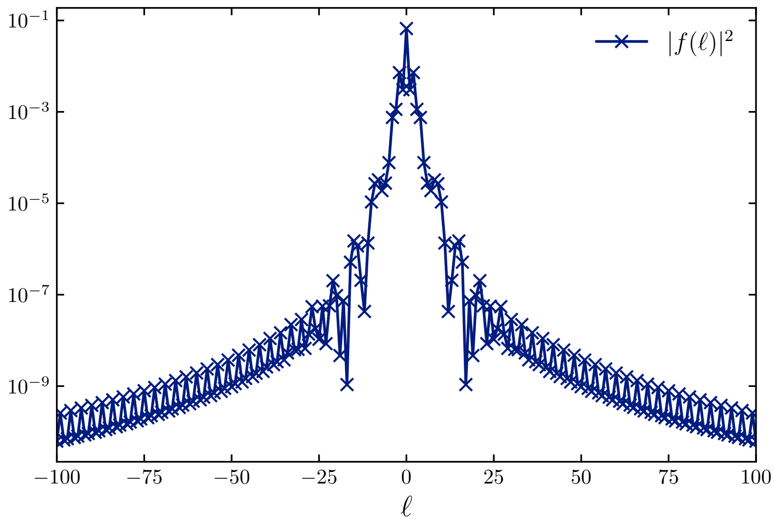

Figure 4: Steady-state density-density correlation functions $|f(\ell)|^2$ for a quantum quench where the system is initialized in the ground state of the Hamiltonian with $\Delta = 2J$ and $\mu = J$, and time evolved with the corresponding Hamiltonian with $\Delta = 0$.

The chemical potentials are fixed by setting

$$\langle\psi_0|\hat{c}(k)^\dagger\hat{c}(k)|\psi_0\rangle = \sin^2(\theta_k/2) = \mathrm{Tr}\left[\hat{c}(k)^\dagger\hat{c}(k)\rho_{\mathrm{GGE}}\right] = \frac{e^{-\mu_k}}{1+e^{-\mu_k}}. \tag{45}$$

This equation can be directly solved to return

$$\mu_k = 2\mathrm{arctanh}(\cos\theta_k)\,. \tag{46}$$

We can already make some initial observation about the GGE: since it is expressed purely in terms of occupation numbers, it has a fixed number of particles, such that any expectation value of operators that change the particle number will vanish. Specifically, we observe that this immediately implies that

$$\mathrm{Tr}\left[\hat{c}_{j+\ell}\hat{c}_j\,\rho_{\mathrm{GGE}}\right] = 0, \tag{47}$$

and the second part of Eq. (42) is satisfied. For the first part, we find that

$$\begin{aligned}\mathrm{Tr}\left[\hat{c}^\dagger_{j+\ell}\hat{c}_j\,\rho_{\mathrm{GGE}}\right] &= \frac{1}{L}\sum_{k,q} e^{ik(j+\ell)}e^{-iqj}\mathrm{Tr}\left[\hat{c}(k)^\dagger\hat{c}(q)\,\rho_{\mathrm{GGE}}\right] \\ &= \frac{1}{L}\sum_{k} e^{ik(j+\ell)}e^{-iqj}\delta_{k,q}\sin^2(\theta_k/2) = \frac{1}{L}\sum_k e^{ik\ell}\sin^2(\theta_k/2) = f(\ell)\,. \end{aligned} \tag{48}$$

Again, the GGE reproduces the correct two-point functions, where we have made use of the 'diagonal' form of the GGE and the fact that it reproduces the correct expectation values of the mode occupation numbers. For the specific case of the GGE, it is again possible to show that all higher-point functions can be reduced to two-point functions in the same way as for the vacuum state. In this way, the GGE is proven to reproduce all the correct steady-state expectation values.

6 The Transverse-Field Ising Model

Studying the dynamics of noninteracting fermions already allowed us to show generalized thermalization, the spreading of local correlations, and the approach to a GGE at late times. However, much of the focus of this course will be on the dynamics of spin models.

Remarkably, there exists a close connection between spin and fermion models on one-dimensional lattices, known as the *Jordan-Wigner transformation*. This transformation presents a one-to-one correspondence between spin and fermion operators. Crucially, this transformation is nonlocal, such that it will also serve to highlight the importance of locality in e.g. the construction of the GGE for spin models. We first outline this transformation (see Ref. [3] for more details), and then apply it to analyze the dynamics of the Transverse-Field Ising Model (TFIM), with Hamiltonian

$$\hat{H} = -J \sum_{j=1}^{L} \hat{\sigma}_j^x \hat{\sigma}_{j+1}^x + h \sum_{j=1}^{L} \hat{\sigma}_j^z . \tag{49}$$

The Jordan-Wigner transformation. Let us first observe that we can define (spin) raising and lowering operators $\sigma^{\pm}$ from the Pauli matrices as

$$\hat{\sigma}^+ = \frac{\hat{\sigma}^x + i\hat{\sigma}^y}{2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \hat{\sigma}^- = \frac{\hat{\sigma}^x - i\hat{\sigma}^y}{2} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{50}$$

which have highly similar properties to the fermionic creation and annihilation operators:

$$(\hat{\sigma}^+)^2 = (\hat{\sigma}^-)^2 = 0, \qquad \hat{\sigma}^+\hat{\sigma}^- + \hat{\sigma}^-\hat{\sigma}^+ = \mathbb{1}, \tag{51}$$

which looks identical to $(\hat{c}^\dagger)^2 = \hat{c}^2 = 0$ and $\hat{c}\hat{c}^\dagger + \hat{c}^\dagger\hat{c} = \mathbb{1}$. These operators act on a two-dimensional space $\{|\downarrow\rangle, |\uparrow\rangle\}$, same as the fermionic states $\{|0\rangle, |1\rangle\}$. It is tempting to associate $\hat{c}^\dagger$ with $\hat{\sigma}_+$ and $\hat{c}$ with $\hat{\sigma}_-$, such that the number operator $\hat{n} = \hat{c}^\dagger\hat{c}$ would be analogous to

$$\hat{\sigma}^+\hat{\sigma}^- = (\hat{\sigma}^x)^2 + (\hat{\sigma}^y)^2 + i[\hat{\sigma}^y, \hat{\sigma}^x] = \frac{1+\hat{\sigma}^z}{2}. \tag{52}$$

Indeed, this operator has eigenvalue 0 for the $|\downarrow\rangle$ state and eigenvalue 1 for the $|\uparrow\rangle$ state. Unfortunately, this identification fails if we consider multiple sites, where particle statistics become important: the Pauli matrices mutually commute while the fermionic operators anticommute. It turns out, however, that there is an easy fix in one dimension, leading to the Jordan-Wigner transformation. Let us assume that we can label our fermionic operators by a label $j = 0, 1, 2, \dots$. Then we can construct the correct anticommuting fermionic operators by manually reintroducing different phases as

$$\hat{\sigma}_j^+ = \left(e^{-i\pi\sum_{\ell=1}^{j-1}\hat{c}_\ell^\dagger\hat{c}_\ell}\right)\hat{c}_j^\dagger = \prod_{\ell=1}^{j-1}(-1)^{\hat{n}_\ell}\,\hat{c}_j^\dagger \tag{53}$$

$$\hat{\sigma}_j^- = \left(e^{i\pi\sum_{\ell=1}^{j-1}\hat{c}_\ell^\dagger\hat{c}_\ell}\right)\hat{c}_j^\dagger = \prod_{\ell=1}^{j-1}(-1)^{\hat{n}_\ell}\,\hat{c}_j \tag{54}$$

$$\hat{\sigma}_j^z = 2\hat{c}_j^\dagger\hat{c}_j - \mathbb{1}. \tag{55}$$

The products of phases appearing in these terms are known as *Jordan-Wigner strings*, and this transformation is known as the *Jordan-Wigner transformation*. We are essentially mapping fermionic operators to simple single-site Pauli operators, with attached strings of phases. These strings run over all previous sites, which makes it clear why one-dimensional systems are the most natural for this construction – in higher-dimensional systems there exists no unique or straightforward ordering of the different sites into a single line (necessary to define the notion of $\ell < j$).

We first note that these strings do not affect the single-site commutation relations, since for each i we have $(-1)^{2\hat{n}_\ell} = 1$ such that the square of any string equals the identity operator. On different sites these phases allow us to express the commuting spin operators in terms of the anticommuting fermionic operators, since the action of a creation/annihilation operator on a site included in this Jordan-Wigner string changes the sign. Choosing $j < \ell$ without loss of generality, we can write

$$\begin{aligned}\hat{\sigma}_j^+ \sigma_\ell^- &= \prod_{i=1}^{j-1}(-1)^{\hat{n}_i}\, \hat{c}_j^\dagger \prod_{l=1}^{\ell-1}(-1)^{\hat{n}_l}\, \hat{c}_\ell = \hat{c}_j^\dagger \prod_{i=j}^{\ell-1}(-1)^{\hat{n}_i}\hat{c}_\ell \\ &= -\prod_{i=j}^{\ell-1}(-1)^{\hat{n}_i}\hat{c}_j^\dagger\hat{c}_\ell = \prod_{i=j}^{\ell-1}(-1)^{\hat{n}_i}\hat{c}_\ell\hat{c}_j^\dagger = \sigma_\ell^-\hat{\sigma}_j^+ \end{aligned} \tag{56}$$

In the first equality we have used that all phases that are included twice square to zero, in the second that $\hat{c}_j^\dagger(-1)^{\hat{n}_j} = -(-1)^{\hat{n}_j}\hat{c}_j^\dagger$ since in the left-hand side $n_j = 0$ and in the right-hand side $n_j = 1$, and in the third equality we have used the anticommutation relations of the fermionic operators.

We can similarly verify that all other commutation relations are automatically satisfied. This transformation can also be inverted, writing fermionic operators in terms of spin operators, as

$$\hat{c}_j^\dagger = \left(\prod_{i=0}^{j-1}\hat{\sigma}_i^z\right)\hat{\sigma}_j^+, \qquad \hat{c}_j = \left(\prod_{i=0}^{j-1}\hat{\sigma}_i^z\right)\hat{\sigma}_j^-. \tag{57}$$

The Jordan-Wigner transformation highlights the important fact that the construction of anticommuting objects that are separated in space requires the introduction of non-local objects. If we have a fermionic particle on a single site and add a fermionic particle to another site some distance away, the resulting antisymmetry of the total wave function requires that a description in terms of commuting operators includes nonlocal effects between these spatially-separated sites in order to acquire the correct signs.

In the specific case of the Transverse Field Ising Model (49), we can use the Jordan-Wigner transformation to map the spin Hamiltonian to a quadratic fermionic Hamiltonian. Crucially, all nonlocal terms vanish and the local spin Hamiltonian gets mapped to a local fermionic Hamiltonian. We can write

$$\begin{aligned}\hat{H} &= -J\sum_j \hat{\sigma}_j^x\hat{\sigma}_{j+1}^x + h\sum_j \hat{\sigma}_j^z \\ &= -J\sum_j(\hat{c}_j + \hat{c}_j^\dagger)(-1)^{\hat{n}_j}(\hat{c}_{j+1}+\hat{c}_{j+1}^\dagger) + h\sum_j(1-2\hat{c}_j^\dagger\hat{c}_j) \qquad (58)\\ &= -J\sum_j(-\hat{c}_j + \hat{c}_j^\dagger)(\hat{c}_{j+1}+\hat{c}_{j+1}^\dagger) - 2h\sum_j \hat{c}_j^\dagger\hat{c}_j + hL. \qquad (59)\end{aligned}$$

Here we have used that in the quadratic term the Jordan-Wigner strings almost fully cancel, apart from the contribution at j, which then simply result in a sign change since

$$\hat{c}_j^\dagger(-1)^{\hat{n}_j} = \hat{c}_j^\dagger \qquad \text{and} \qquad \hat{c}_j(-1)^{\hat{n}_j} = -\hat{c}_j, \tag{60}$$

i.e. the raising operator can only act nontrivially on the state with no fermions, and the lowering operator can only act nontrivially on the state with a single fermion. Making use of the fermionic anticommutation relations, we can rewrite this Hamiltonian as

$$\hat{H} = -J \sum_j \left[\hat{c}_j^\dagger \hat{c}_{j+1} + \hat{c}_{j+1}^\dagger \hat{c}_j + (\hat{c}_j^\dagger \hat{c}_{j+1}^\dagger - \hat{c}_j \hat{c}_{j+1}) + \frac{2h}{J} \hat{c}_j^\dagger \hat{c}_j \right] + \text{Cst.} \tag{61}$$

This equation is exactly of the form of the previously analyzed Hamiltonian (22), such that our results on quench dynamics (almost) directly extend to this setup.

Caveat: One subtlety is that periodic boundary conditions in the spin language do not directly translate to periodic boundary conditions in the fermionic language. The Hamiltonian preserves fermion parity $(-1)^N$ with $N = \sum_j \hat{c}_j^\dagger \hat{c}_j$, and for even parity the periodic boundary conditions in the spin language translate to antiperiodic boundary conditions for the fermions, whereas for odd parity these again result in periodic boundary conditions for the fermions. These different boundary conditions do not pose an issue for any of the presented derivations, they only modify the values that the discrete momentum k can take;

$$k = \frac{2\pi n}{L}, n = 0 \dots L-1 \qquad (N \text{ odd}), \tag{62}$$

$$k = \frac{2\pi(n+1/2)}{L}, n = 0 \dots L-1 \qquad (N \text{ even}), \tag{63}$$

The dynamics of any parity-invariant operators will not mix these subspaces, such that we can focus on a single subspace in general. For the context of quantum quenches, we find that the ground state generally has even parity, such that we need to work with the second set of quantized momentum values.

Bogoliubov transformation. Comparing this expression with the Kitaev chain, we find that the TFIM corresponds to the Kitaev chain with $J = J$, $\mu = -2h$ and $\Delta = -J$. As such, the Bogoliubov transformation that transforms the Jordan-Wigner fermions in momentum space is given by

$$\begin{bmatrix} \hat{c}(k) \\ \hat{c}(-k)^\dagger \end{bmatrix} = \begin{bmatrix} \cos(\theta_h(k)/2) & i\sin(\theta_h(k)/2) \\ i\sin(\theta_h(k)/2) & \cos(\theta_h(k)/2) \end{bmatrix} \begin{bmatrix} \alpha_h(k) \\ \alpha_h(-k)^\dagger \end{bmatrix}, \tag{64}$$

in which we have made the dependence on h explicit and where the rotation angle is defined using

$$\epsilon_h(k) = \sqrt{(2J\cos k - 2h)^2 + 4J^2 \sin^2 k} = 2\sqrt{J^2 + h^2 - 2Jh\cos k} \tag{65}$$

as

$$e^{i\theta_h(k)} = \frac{-2J\cos k + 2h - 2iJ\sin k}{\epsilon(k)} = \frac{h - Je^{ik}}{\sqrt{J^2 + h^2 - 2Jh\cos k}}. \tag{66}$$

In these Bogoliubov fermions, the Hamiltonian reads

$$H = \sum_k \epsilon_h(k) \left[\alpha_h(k)^\dagger \alpha_h(k) - \frac{1}{2} \right]. \tag{67}$$

The eigenstates of the TFIM are states where each Bogoliubov fermion has a fixed mode occupation number, $n_k = 0$ or 1. As such, we find that the eigenspectrum of the transverse-field Ising model can be obtained as

$$E = \sum_k \epsilon_h(k)(n_k - 1/2) = \sum_k \sqrt{J^2 + h^2 - 2Jh\cos k}\,(2n_k - 1), \tag{68}$$

where k is quantized according to Eq. (63) with $N = \sum_k n_k$.

Conserved charges. In the same way as for the Kitaev chain, we can construct mutually commuting local conserved charges from the mode occupation numbers, which here read

$$I^{(n,+)} = -J\sum_j (S^{xx}_{j:j+n} + S^{yy}_{j:j+n-2}) + h\sum_j (S^{xx}_{j:j+n-1} + S^{yy}_{j:j+n-1}), \tag{69}$$

$$I^{(n,-)} = iJ\sum_j (S^{xy}_{j:j+n} - S^{yx}_{j:j+n}), \tag{70}$$

where we have defined operators

$$S^{\alpha\beta}_{j:\ell} = \sigma^{\alpha}_j \left(\prod_{k=1}^{\ell-1} \sigma^z_{j+k}\right) \sigma^{\beta}_{j+\ell}\,. \tag{71}$$

These are again defined as linear combinations from the mode occupation numbers [4]

$$I^{(n,+)} \propto \sum_k \cos(nk)\epsilon_h(k)\alpha_h(k)^\dagger\alpha_h(k), \qquad I^{(n,-)} \propto \sum_k \sin(nk)\alpha_h(k)^\dagger\alpha_h(k) \tag{72}$$

Quantum quenches. Suppose now that we prepare the system in the ground state of the TFIM at transverse field strength h_0 and subsequently change the field strength to h. The ground state of the TFIM is ferromagnetic for $h < 1$ and we need to consider spontaneous symmetry breaking, whereas in the paramagnetic phase $h > 1$ the ground state is the Bogoliubov vacuum.

In order to avoid having to deal with spontaneous symmetry breaking, we hence consider $h > 1$, where the Hamiltonian has a unique ground state. The dynamics can be analyzed in the exact same way as for the Kitaev chain. However, we need to be introduce an additional transformation, since the Bogoliubov fermions for the Hamiltonian governing the dynamics no longer correspond to the original fermions. Labelling the Bogoliubov fermions at the new interaction strength as γ and the Bogoliubov fermions at the original interaction strength as α, we have that

$$\begin{bmatrix} \hat{c}(k) \\ \hat{c}(-k)^\dagger \end{bmatrix} = M(h_0)\begin{bmatrix} \alpha(k) \\ \alpha(-k)^\dagger \end{bmatrix} = M(h)\begin{bmatrix} \gamma(k) \\ \gamma(-k)^\dagger \end{bmatrix}, \tag{73}$$

with

$$M(h) = \begin{bmatrix} \cos(\theta_h(k)/2) & i\sin(\theta_h(k)/2) \\ i\sin(\theta_h(k)/2) & \cos(\theta_h(k)/2) \end{bmatrix}\begin{bmatrix} \alpha_h(k) \\ \alpha_h(-k)^\dagger \end{bmatrix}. \tag{74}$$

We can hence express the γ fermions (for which the dynamics is easy) in terms of the α fermions (for which the matrix elements in the initial state are easy) as

$$\begin{bmatrix} \gamma(k) \\ \gamma(-k)^\dagger \end{bmatrix} = M^{-1}(h)M(h_0)\begin{bmatrix} \alpha(k) \\ \alpha(-k)^\dagger \end{bmatrix}, \tag{75}$$

The matrix $\tilde{M}(h_0, h) = M^{-1}(h)M(h_0)$ takes a particularly simple form

$$\tilde{M}(h_0, h) = \begin{bmatrix} \cos(\Delta_k/2) & -i\sin(\Delta_k/2) \\ -i\sin(\Delta_k/2) & \cos(\Delta_k/2) \end{bmatrix} \qquad \text{with} \qquad \Delta_k = \theta_k(h) - \theta_k(h_0)\,. \tag{76}$$

We can now express the dynamics of the original momentum fermion operator $\hat{c}(k)$ by first expressing these in terms of the γ fermions, which gain a dynamical phase,

$$\hat{c}(k) = \cos(\theta_h(k)/2)\gamma(k) + i\sin(\theta_h(k)/2)\gamma(-k)^\dagger, \tag{77}$$

such that

$$\hat{c}(k;t) = \cos(\theta_h(k)/2)\, e^{-i\epsilon_h(k)t}\, \gamma(k) + i\sin(\theta_h(k)/2)\, e^{i\epsilon_h(k)t}\, \gamma(-k)^\dagger, \tag{78}$$

where we have used that $\epsilon_h(k) = \epsilon_h(-k)$. Since we're ultimately interested in matrix elements w.r.t. the Bogoliubov vacuum at h_0 we can introduce the transformation to $\alpha(k)$ fermions as

$$\begin{aligned}\hat{c}(k;t) =& \cos(\theta_h(k)/2)\left[\cos(\Delta_k/2)\alpha(k) - i\sin(\Delta_k/2)\alpha(-k)^\dagger\right] e^{-i\epsilon_h(k)t} \\ &+ i\sin(\theta_h(k)/2)\left[-i\sin(\Delta_k/2)\alpha(k) + \cos(\Delta_k/2)\alpha(-k)^\dagger\right] e^{i\epsilon_h(k)t}.\end{aligned} \tag{79}$$

We can again hide the complexity by defining functions

$$A(k,t) = \cos(\theta_h(k)/2)\cos(\Delta_k/2)\, e^{-i\epsilon_h(k)t} + \sin(\theta_h(k)/2)\sin(\Delta_k/2)\, e^{i\epsilon_h(k)t}, \tag{80}$$

$$B(k,t) = -i\cos(\theta_h(k)/2)\sin(\Delta_k/2)\, e^{-i\epsilon_h(k)t} + i\sin(\theta_h(k)/2)\cos(\Delta_k/2)\, e^{i\epsilon_h(k)t}, \tag{81}$$

such that we can write

$$c(k;t) = A(k,t)\alpha(k) + B(k,t)\alpha(-k)^\dagger. \tag{82}$$

The two-point functions w.r.t. the initial state now directly follow as

$$\langle\psi(t)|\hat{c}(k)\hat{c}(-q)|\psi(t)\rangle = \delta_{k,q} A(k,t)B(-k,t), \tag{83}$$

$$\langle\psi(t)|\hat{c}^\dagger(k)\hat{c}(q)|\psi(t)\rangle = \delta_{k,q}\, |B(k,t)|^2. \tag{84}$$

While these equations are more complicated than those for the Kitaev chain, all calculations can be performed in a similar way, to show that at long times all expectation values of local operators are reproduced by the GGE in terms of the conserved charges from Eq. (69):

$$\rho_{\text{GGE}} = \frac{e^{-\sum_{n,\sigma}\lambda_{n,\sigma}\hat{I}^{(n,\sigma)}}}{\text{Tr}\left[e^{-\sum_{n,\sigma}\lambda_{n,\sigma}\hat{I}^{(n,\sigma)}}\right]}. \tag{85}$$

Note that for subsystems of a fixed number of sites it is typically not necessary to know the expectation values of all conservation laws: rather, conservation laws with increasing support become increasingly less important as the support is increased beyond the subsystem size, such that the GGE can be truncated at some finite value set by the subsystem size.

As one nontrivial example, we can consider the dynamics of the average transverse magnetization, i.e. choosing

$$\hat{O} = \frac{1}{L}\sum_{j=1}^{L}\hat{\sigma}_j^z = \frac{2}{L}\sum_{j=1}^{L}\hat{c}_j^\dagger\hat{c}_j - 1 = \frac{2}{L}\sum_k \hat{c}(k)^\dagger\hat{c}(k) - 1. \tag{86}$$

The dynamical expectation value in the thermodynamic limit directly follows as

$$\langle\hat{O}(t)\rangle = \frac{1}{\pi}\int_0^{2\pi} dk\, |B(k,t)|^2 - 1. \tag{87}$$

Using a similar stationary phase approximation as before, it follows that the transverse magnetization decays to a steady-state value as $t^{-3/2}$. Note that, while the appearance of power-law decay is a general feature of free-fermionic system, the specific exponent will depend on both the observable and the Hamiltonian under study.

The dynamics is shown in Fig. 5 for a quench from $h_0 = 5$ to $h = 2$. The short-time dynamics is identical for different system sizes. However, we also observe the presence of revivals for (relatively large) system sizes, where the revival time increases linearly as system size increases, until there are no revivals in the thermodynamic limit $L \to \infty$. Such revivals would take much longer in ergodic many-body dynamics, but in free-fermionic models all relaxation is purely due to the dephasing between the L different frequencies. This should be contrasted with both the exponential amount of frequencies in the generic case and the added randomness due to ETH.

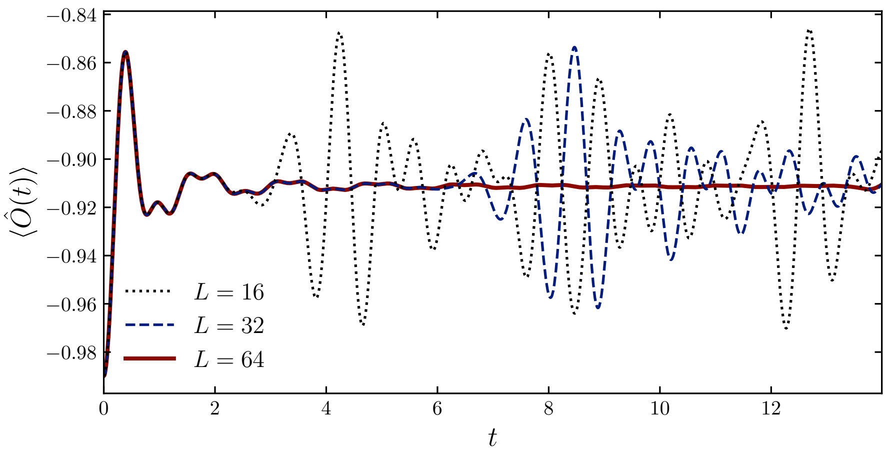

Figure 5: Dynamics of the average transverse magnetization for a quench from $h_0 = 5$ to $h = 2$ for different system sizes L.

Failure of ETH

The existence of additional conservation laws and the absence of thermalization can also be observed numerically. The eigenspectrum of the TFIM does not exhibit the GOE statistics expected from a random matrix. The eigenvalues rather satisfy Poissonian statistics, as generally expected for integrable models. The Poissonian statistics indicates the absence of level repulsion, as can also be observed in Fig. 6 by showing (part of) the eigenspectrum as the transverse field strength is varied. The eigenvalues change smoothly as the transverse field is changed, as should be clear from our explicit solution: all excitation energies vary smoothly as h is changed. As such, these eigenvalues exhibit an abundance of exact level crossings and corresponding accidental degeneracies. The other predictions from ETH are similarly not satisfied.

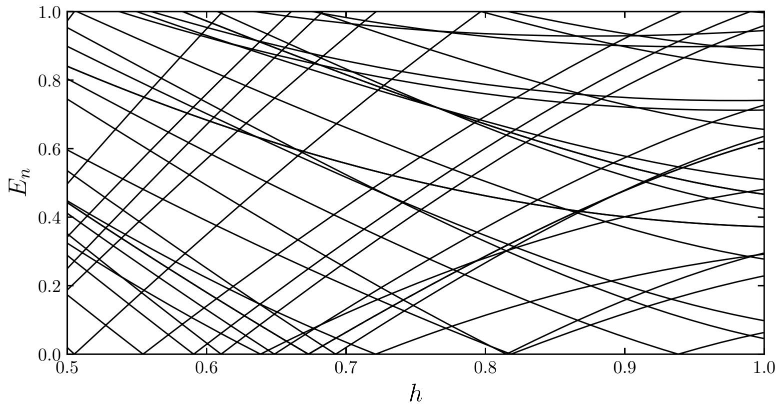

Figure 6: Part of the energy spectrum of the transverse-field Ising Hamiltonian (49) with $J = 1$ as h is varied for $L = 14$. As the Hamiltonian is changed the energy levels undergo exact level crossings and show no sign of level repulsion.

In Fig. 7 we show the expectation values of an observable in the eigenstates of the TFIM at different system sizes as function of the energy density. If ETH holds these expectation values are expected to approach a smooth function as the system size is increased. However, for the

TFIM these expectation values clearly exhibit large fluctuations that are not suppressed as the system size is increased.

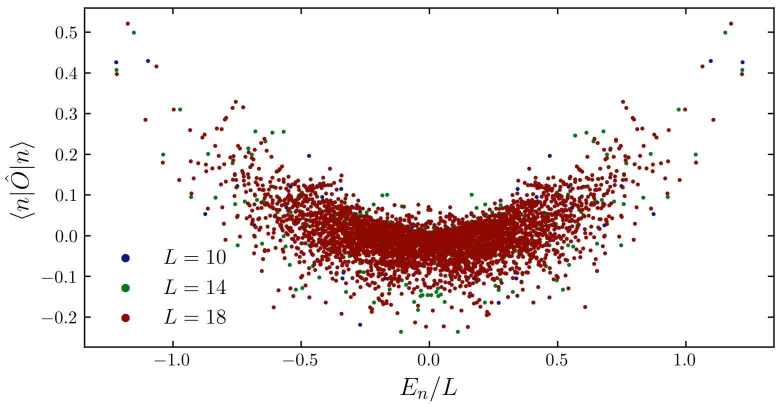

Figure 7: Expectation values of $\hat{O} = \frac{1}{L}\sum_{j=1}^{L} \sigma_j^z \sigma_{j+1}^z$ in the eigenstates of the transverse-field Ising Hamiltonian (49) as a function of the energy density of the corresponding eigenstate.

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